

MODULI OF SURFACES FIBERED IN (LOG) CALABI-YAU PAIRS II: ELLIPTIC SURFACES

GIOVANNI INCHIOSTRO AND JUNYAN ZHAO

ABSTRACT. This paper continues the study initiated in [ISZ25] on the moduli of surfaces admitting lc-trivial fibrations. Using the techniques developed in [ISZ25], we

- (1) provide a classification of the surfaces appearing on the boundary of the KSBA-moduli space of elliptic surfaces with a bisection;
- (2) recover the results of a series of papers on the moduli stacks of elliptic surfaces with a section [AB22, Inc20, Bru15].

Notably, our proof of (2) avoids the use of explicit steps of an MMP, such as the *La Nave flip* from [LN02], which plays a central role in [AB22, Inc20]. As an application, we compactify the moduli stack of hyperelliptic K3 surfaces.

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1. INTRODUCTION

In [ISZ25], we studied the KSBA moduli stack of fibered Calabi–Yau pairs. Elliptic surfaces with a fiberwise involution fit naturally into this framework after taking the quotient by the involution, which produces a fibration in log Calabi–Yau curves. In the present paper, we build on those results to analyze the KSBA compactification of certain moduli stacks of elliptic surfaces. Specifically, we consider the KSBA moduli of pairs $(X, \epsilon_1 R + \epsilon_2 F)$ where:

- (1) X is a projective surface which admits a fibration $f: X \rightarrow C$ whose fibers are integral genus-one nodal curves, with general fiber smooth and with singular fibers marked as F ;
- (2) X carries a fiberwise involution whose fixed locus is a horizontal divisor R ; and
- (3) the pair $(X, \epsilon_1 R + \epsilon_2 F)$ is KSBA-stable for $0 < \epsilon_1 \ll \epsilon_2 \ll 1$.

The key reduction of this paper is to take the quotient $X \rightarrow Y$ by the fiberwise involution, which allows us to transfer the KSBA analysis from elliptic surfaces to log Calabi–Yau fibrations. The resulting surface Y carries a natural divisor $R_Y \subseteq Y$ recording the ramification, and the induced fibration

$$f_Y: (Y, \tfrac{1}{2}R_Y) \rightarrow C$$

forms a fibered log Calabi–Yau pair. This allows us to apply the results of [ISZ25] to study the original pairs $(X, \epsilon_1 R + \epsilon_2 F)$.

We begin by analyzing such pairs in full generality, without imposing additional hypotheses. Our main results provide a classification of the surfaces that occur on the boundary of the KSBA moduli space that generically parametrizes pairs $(X, \epsilon_1 R + \epsilon_2 F)$ as above.

Theorem 1.1. *Let $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ be the closure of the locus of elliptic surfaces as above, and let $(X, \epsilon_1 R + \epsilon_2 F)$ be a pair appearing on the boundary of this moduli space. Then:*

- (1) *there is a morphism $X \rightarrow C$ to a nodal curve, with pure 1-dimensional fibers,*
- (2) *the possible singularities of $(X, \epsilon_1 R + \epsilon_2 F)$ over the smooth locus of C are classified in Theorems 3.2 and 3.3, and*
- (3) *the possible fibers of $(X, \epsilon_1 R + \epsilon_2 F) \rightarrow C$ are classified in Theorems 3.2, 3.3 and 3.4.*

Having an involution as above is closely related to the existence of a bisection for $f: X \rightarrow C$. Indeed, any elliptic surface X as above admits a bisection, obtained as the preimage of a generic section of the ruled surface $Y \rightarrow C$. Conversely, if $f: X \rightarrow C$ is an elliptic surface with integral nodal fibers and admitting a bisection $D \subseteq X$, then $f_* \mathcal{O}_X(D)$ is base-point free, and the induced morphism

$$X \longrightarrow Y := \text{Proj}_C(\text{Sym}(f_* \mathcal{O}_X(D)))$$

realizes X as a double cover of Y . In particular, X admits a fiberwise involution. In other words, the conditions above are equivalent to requiring that X carry a bisection.

Our second main result, therefore, concerns the special case in which $X \rightarrow C$ admits a *section* rather than merely a bisection. Let $0 < \epsilon \ll 1$ be a sufficiently small rational number, and let

$$\vec{a} = (a_1, \dots, a_n), \quad 0 < a_i \leq 1.$$

Consider a Weierstrass fibration

$$(X_\eta, \epsilon S_\eta + \vec{a} F_\eta) \rightarrow C_\eta$$

over the generic point η of the spectrum of a DVR A , whose fibers are at worst irreducible nodal curves, and such that the pair $(X_\eta, \epsilon S_\eta + \vec{a} F_\eta)$ is KSBA-stable. Here:

- S_η is a section of $X_\eta \rightarrow C_\eta$;
- $p_\eta = p_{\eta,1} + \dots + p_{\eta,n}$ is a sum of marked points on C_η , and

$$\vec{a} p_\eta := \sum_{i=1}^n a_i p_{\eta,i};$$

- $F_\eta = F_{\eta,1} + \dots + F_{\eta,n}$ is the corresponding sum of fibers, and

$$\vec{a} F_\eta := \sum_{i=1}^n a_i F_{\eta,i}.$$

Let \mathbf{M}_η denote the moduli part of the canonical bundle formula for $X_\eta \rightarrow C_\eta$.

Theorem 1.2. *The following statements hold:*

- (1) *After a finite base change of $\text{Spec } A$, the KSBA-stable extension $(X, \epsilon S + \vec{a} F)$ of $(X_\eta, \epsilon S_\eta + \vec{a} F_\eta)$ over $\text{Spec } A$ admits a fibration over a generalized pair*

$$(X, \epsilon S + \vec{a} F) \xrightarrow{\pi} (C, \vec{a} p + \mathbf{M}) \xrightarrow{g} \text{Spec } A,$$

with g a family of nodal pointed curves with $K_C + \vec{a} p + \mathbf{M}$ ample, and whose generic fiber is $(C_\eta, \vec{a} p_\eta + \mathbf{M}_\eta)$, and such that $f^(K_C + \mathbf{M} + \vec{a} p) \sim_{\mathbb{Q}} K_X + \vec{a} F$.*

- (2) The family $C \rightarrow \operatorname{Spec} A$ is the coarse space of a family of twisted curves $\mathcal{C} \rightarrow \operatorname{Spec} A$ equipped with a fibration $(\mathcal{X}, \mathcal{S}) \rightarrow \mathcal{C}$, whose geometric fibers $(\mathcal{X}_p, \mathcal{S}_p)$ are Weierstrass fibrations of the form

$$(y^2x = x^3 + axz^2 + bz^3, [0, 1, 0]),$$

and such that the induced morphism on coarse spaces is $(\mathcal{X}, \mathcal{S}) \rightarrow \mathcal{C}$ giving $(X, S) \rightarrow C$.

- (3) The fibers of $(X, S) \rightarrow C$ over codimension-one points of C have at worst nodal singularities.

The following diagram summarizes the situation:

$$\begin{array}{ccccc} (X_\eta, \epsilon S_\eta + \vec{a}F_\eta) & \longrightarrow & (X, \epsilon S + \vec{a}F) & \xleftarrow{\text{cms}} & (\mathcal{X}, \epsilon \mathcal{S} + \vec{a}\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \text{Weierstrass} \\ (C_\eta, \vec{a}p_\eta + \mathbf{M}) & \longrightarrow & (C, \vec{a}p + \mathbf{M}) & \xleftarrow{\text{cms}} & \mathcal{C} \\ \downarrow & & \downarrow & \nearrow & \\ \eta & \longrightarrow & \operatorname{Spec} A & & \end{array}$$

Moduli spaces of elliptic surfaces with a section have been extensively studied, ranging from the GIT approach [Mir81], to twisted stable maps [AV02], and to KSBA theory [LN02, AB19, Inc20, AB21, AB23, ABE22, BFH⁺24]. Our contribution is to provide an alternative—and in fact significantly streamlined—proof of certain results in [AB23, AB22, Inc20]. The stable reduction arguments in *loc. cit.* rely on an explicit MMP and, in particular, on an explicit flip—the *La Nave flip* [LN02]—which renders parts of the analysis rather technical. In contrast, the present work avoids the La Nave flip entirely by invoking general results from [ISZ25], especially [ISZ25, Theorem 1.2], together with the canonical bundle formula. In Section §4, we use the preceding theorem to give a different proof of [Inc20, Theorem 1.2] and [AB22, Theorem 1.1(a),(b)], as well as [Bru15, Theorem 9.1.4], where a related statement is established for generic \vec{a} .

Moduli of hyperelliptic K3 surfaces. A polarized K3 surface (X, L) of degree $(L^2) = 2g - 2$ is called *hyperelliptic* if L is globally generated but not very ample. By [May72], for a polarized K3 surface (X, L) , if L is not very ample, then

- (1) either (X, L) is hyperelliptic; or
- (2) (X, L) is *unigonal*, i.e. X admits an elliptic fibration $f : X \rightarrow \mathbb{P}^1$ with a section E , and $L \sim E + f^*\mathcal{O}_{\mathbb{P}^1}(g)$.

The hyperelliptic and unigonal K3 surfaces form two Noether-Lefschetz divisors in the moduli of polarized K3 surfaces, called the *hyperelliptic divisor* and *unigonal divisor* respectively. As an application, our main results compactify these two divisors, and describe the objects parametrized by the boundaries explicitly. As the moduli of unigonal K3 surfaces is studied in [Bru15, AB23], we only state the result concerning the hyperelliptic K3 surfaces; see Corollary 4.16.

Theorem 1.3 (Compactification of the moduli of hyperelliptic K3 surfaces). *For any genus $g \geq 3$, there exists a Deligne–Mumford stack $\mathcal{M}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$, whose normalization of this stack is a compactification of the normalization of the hyperelliptic divisor $\mathcal{D}_{2,0}^g$ in the moduli stack \mathcal{F}_g of polarized K3 surfaces of genus g . If a pair appears on the boundary of $\mathcal{M}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$, it is described in Theorems 3.1, 3.2, 3.3, and 3.4.*

Outline of the paper. In Section §2, we summarize the results of [ISZ25] that will be used throughout the paper. Section §3 analyzes the double covers of the components of KSBA-stable surface pairs fibered in log Calabi–Yau curves over a nodal base, and classifies the fibers over the nodes. We then combine these ingredients to describe the KSBA moduli stack of elliptic surfaces with an unmarked bisection. Section §4 systematically studies the hyperelliptic divisor in the moduli stack of polarized K3 surfaces and the corresponding moduli stack of lattice-polarized K3 surfaces, and proves Theorem 1.3. Finally, in Section §5 we use the results of [ISZ25] to recover the descriptions of the KSBA moduli stacks of elliptic surfaces with a marked section obtained in [AB22, Inc20, Bru15].

- We work over an algebraically closed field k of characteristic 0. The reader may assume $k = \mathbb{C}$.
- We say that $Y \rightarrow (X, \frac{1}{2}D)$ is a double cover if $Y \rightarrow X$ is a double cover branched along the divisor D .
- The *coarse space* of a Deligne–Mumford (abbreviated DM) stack \mathcal{X} is its coarse moduli space $\mathcal{X} \rightarrow X$ (cf. [Sta25, Tag 04UX]). We drop the word “moduli” since many stacks we consider need not have a modular interpretation.
- A “big open” subset of an algebraic stack \mathcal{X} is a Zariski open subset containing all the codimension one points.
- Throughout the paper, we denote by A a DVR, since the letter R is reserved for the ramification divisor. For any DVR A , let η (resp. 0) be the generic (resp. closed) point of $\text{Spec } A$. If $f : X \rightarrow \text{Spec } A$ is an object over $\text{Spec } A$ (e.g. a family of pairs), we denote by X_η (resp. X_0) the generic (resp. special) fiber of f .
- Let A be a DVR with generic point η . For an object $X_\eta \rightarrow \eta$, an *extension* of X_η is an object $X \rightarrow \text{Spec } A$ such that the fiber product $X \times_A \eta$ is isomorphic to X_η . This explains the previous convention. A *limit* of X_η is the special fiber X_0 of an extension $X \rightarrow \text{Spec } A$. The extensions (resp. limits) of interest are the KSBA-stable extensions (resp. limits), which are unique.

For the reader’s convenience, we collect here the notations most frequently used throughout the paper. For the first five lines, when we write “KSBA-moduli stack” of a certain class of objects \mathfrak{C} (for example, rational Weierstrass fibrations or K3 Weierstrass fibrations), we mean the closure in the KSBA-moduli space of the locus parametrizing objects in \mathfrak{C} .

Notation	Definition/Description
$\overline{\mathcal{M}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$	KSBA-moduli stack of surface pairs fibered in log-canonical pairs of the form $(\mathbb{P}^1, \frac{1}{2}(p_1 + \cdots + p_4))$
$\overline{\mathcal{W}}_{\text{rat}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$	KSBA-moduli stack of rational Weierstrass fibrations
$\overline{\mathcal{W}}_{\text{K3}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$	KSBA-moduli stack of K3 Weierstrass fibrations
$\overline{\mathcal{W}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$	KSBA-moduli stack of Weierstrass fibrations with $\kappa = 1$
$\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$	KSBA-moduli stack of elliptic surfaces with a unmarked bi-section
$\mathcal{P}_1^{\text{GIT}}$	GIT moduli stack $[\mathcal{O}_{\mathbb{P}^1}(4) ^{ss} / \text{PGL}(2)]$ of pairs $(\mathbb{P}^1, p_1 + \cdots + p_4)$
\mathcal{P}_1	an enlargement of $\mathcal{P}_1^{\text{GIT}}$, see [ISZ25, Section 2.1]

Spec A	Spectrum of a DVR with generic point η and special point 0
\mathcal{F}_Λ	moduli stack of lattice Λ -polarized K3 surfaces

Acknowledgments. We thank Dori Bejleri, Yuchen Liu, Luca Schaffler and Roberto Svaldi for helpful conversations. GI was supported by AMS-Simons travel grant and by the NSF grant DMS-2502104. JZ was supported by AMS-Simons travel grant.

2. KSBA-MODULI STACKS FOR CERTAIN RULED SURFACES

In this section, we summarize the structural results and relevant machinery developed in [ISZ25] that will be used in this paper, for the reader's convenience. We refer the reader to [ISZ25, §1, Introduction] for a more detailed overview.

Definition 2.1. A pair (X, D) is called *KSBA-stable* if

- (1) it is semi-log canonical (abbrev. slc) and X is connected;
- (2) $K_X + D$ is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor.

The *volume* of a KSBA-stable pair (X, D) is $\text{vol}(X, D) = (K_X + D)^{\dim X}$.

Theorem 2.2 (cf. [Kol23, Theorem 8.15]). *Fix $v \in \mathbb{Q}_{>0}$ and some rational numbers $c_i \in [0, 1]_{\mathbb{Q}}$. Then there is a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{c_i, v}^{\text{KSBA}}$, whose closed points parametrize KSBA-stable surface pairs $(X, \sum c_i D_i)$ with volume $(K_X + \sum c_i D_i)^2 = v$, where each D_i is an effective \mathbb{Z} -divisor.*

Consider two-dimensional log pairs $(Y, \frac{1}{2}R)$, where Y is a proper surface and R is a Weil divisor, that are endowed with a fibration

$$\pi : (Y, \frac{1}{2}R) \longrightarrow C$$

to a curve whose fibers are one-dimensional log Calabi-Yau pairs of the form $(\mathbb{P}^1, \frac{1}{2}(p_1 + \dots + p_4))$, satisfying that

- (a) the divisor R cuts 4 distinct points on a general fiber of π ;
- (b) denoting by F the sum of the fibers of π meeting R in fewer than 4 points, the pair $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F)$ is KSBA-stable for every $0 < \epsilon_1 \ll \epsilon_2 \ll 1$.

We now recall the main result of [ISZ25] for the case $n = 1$ in *loc. cit.*, dividing it into two parts.

Theorem 2.3 (cf. [ISZ25, Theorem 1.1]). *For $0 < \epsilon_1 \ll \epsilon_2 \ll 1$, consider the closure of the locus of surface pairs $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F)$ satisfying conditions (a)-(b) as above, with all the fibers of π being log-canonical, denoted by $\overline{\mathcal{M}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$. Then $\overline{\mathcal{M}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ parametrizes KSBA-stable pairs*

$$(Y_0, (\frac{1}{2} + \epsilon_1)R_0 + \epsilon_2 F_0),$$

where

- Y_0 is a demi-normal surface which admits a morphism $g_0 : Y_0 \rightarrow C_0$ to a (possibly reducible) nodal curve with equi-dimensional fibers;
- R_0 is a divisor relatively ample over C_0 such that $g_0|_{R_0} : R_0 \rightarrow C_0$ is of degree 4, with the map $R_0 \rightarrow C_0$ possibly not finite; and

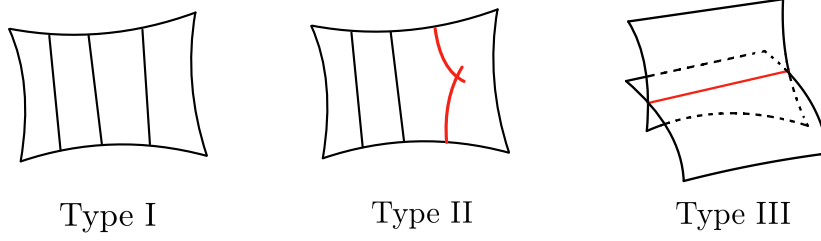


FIGURE 1. Surfaces of different types

- F_0 is a divisor whose support consists of fibers $F_{0,i}$ of $g_0 : Y_0 \rightarrow C_0$ such that $R_0|_{F_{0,i}}$ is not four distinct points.

The map $R_0 \rightarrow C_0$ is possibly not finite as it could contain some fiber components of $Y_0 \rightarrow C_0$.

Theorem 2.4 (cf. [ISZ25, Propositions 4.3, 4.4, and 4.6]). *With notation as above, let G be an irreducible component of C_0 , and let $G^n \rightarrow G$ denote its normalization. Then there exists a smooth orbifold curve \mathcal{G}^n with coarse space G^n such that the reduced structure of $Y|_{G^n}$ is the coarse moduli space of one of the following (see Figure 1):*

- (1) a projective bundle $\mathbb{P}_{\mathcal{G}^n}(\mathcal{V})$ over \mathcal{G}^n ;
- (2) a weighted blow-up of the surface $\mathbb{P}_{\mathcal{G}^n}(\mathcal{V})$;
- (3) the gluing of a surface $\mathbb{P}_{\mathcal{E}}(\mathcal{V})$ ruled over a smooth, possibly non-connected orbifold curve \mathcal{E} with at most two connected components, glued along an involution $\mathcal{E} \rightarrow \mathcal{E}$ acting transitively on the connected components of \mathcal{E} .

Remark 2.5. We refer to surfaces in the classes (1), (2), and (3) in Theorem 2.4 as being of type I, II, and III, respectively. Surfaces of type I and type II differ only over the nodal locus of C_0 . In particular, a surface of type II, when restricted to $G \cap C_0^{\text{sm}}$, is the coarse moduli space of a projective bundle as a surface of type I.

Corollary 2.6. *Let G be a component of C^{st} , and suppose that $Y|_G$ is a surface of type I or II. Then the morphism $Y|_G \rightarrow G$ is a projective bundle over $G \cap C^{\text{sm}}$.*

Proof. It follows from the results in [ISZ25] that for the surface $Y|_G$, the only stacky (i.e. nontrivial orbifold) behavior arises over the nodal locus C^{sing} . Over the smooth locus $G \cap C^{\text{sm}}$, every fiber is \mathbb{P}^1 , and hence it is a projective bundle. \square

Before stating the next result, we briefly outline the strategy used to prove the previous results. Since the moduli stack $\overline{\mathcal{M}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ is separated and every object it parametrizes is obtained as a degeneration over the spectrum of a DVR A of a general pair

$$(Y_\eta, (\tfrac{1}{2} + \epsilon_1)R_\eta + \epsilon_2 F_\eta),$$

one may instead start with such a general pair over the generic point $\eta \in \text{Spec } A$ and recover its unique KSBA-stable limit over $\text{Spec } A$. This is carried out in two steps:

- (1) Using stable quasimaps [DLI24] one constructs an extension

$$(\mathcal{Y}^{\text{tw}}, (\tfrac{1}{2} + \epsilon_1)\mathcal{R}^{\text{tw}} + \epsilon_2\mathcal{F}^{\text{tw}}) \longrightarrow \text{Spec } A$$

as a DM stack, together with a morphism to a family of twisted curves

$$\mathcal{C}^{\text{tw}} \rightarrow \text{Spec } A.$$

This fits into the diagram

$$\begin{array}{ccc} (\mathcal{Y}^{\text{tw}}, (\frac{1}{2} + \epsilon_1)\mathcal{R}^{\text{tw}} + \epsilon_2\mathcal{F}^{\text{tw}}) & \xrightarrow{\pi} & (\mathcal{C}^{\text{tw}}, \mathbf{M}^{\text{tw}}) \\ & \searrow f \quad \swarrow g & \\ & \text{Spec } A & \end{array}$$

where \mathbf{M}^{tw} is the moduli part of the canonical bundle formula for the lc-trivial fibration

$$(\mathcal{Y}^{\text{tw}}, \frac{1}{2}\mathcal{R}^{\text{tw}}) \rightarrow \mathcal{C}^{\text{tw}}.$$

Every geometric fiber of π is a log Calabi–Yau pair parametrized by the stack \mathcal{P}_1 [ISZ25, §2.1, Notation], an enlargement of the GIT moduli stack $\mathcal{P}_1^{\text{GIT}}$ of pairs $(\mathbb{P}^1, \frac{1}{2}(p_1 + \dots + p_4))$ with GIT semistable $p_1 + \dots + p_4$.

- (2) One then runs the relative MMP with scaling for the coarse space of the surface generalized pair

$$(\mathcal{C}^{\text{tw}}, \mathbf{M}^{\text{tw}}) \rightarrow \text{Spec } A,$$

and lifts this MMP to a sequence of birational contractions of the coarse space of the threefold pair using [ISZ25, Theorem 1.2]

$$(\mathcal{Y}^{\text{tw}}, (\frac{1}{2} + \epsilon_1)\mathcal{R}^{\text{tw}} + \epsilon_2\mathcal{F}^{\text{tw}}) \rightarrow \text{Spec } A.$$

Each step of the surface MMP

$$(C^{(i)}, \mathbf{M}^{(i)}) \longrightarrow (C^{(i+1)}, \mathbf{M}^{(i+1)}),$$

contracts a rational tail G of $C_0^{(i)}$, and there exists a birational contraction

$$(Y^{(i)}, (\frac{1}{2} + \epsilon_1)R^{(i)} + \epsilon_2F^{(i)}) \dashrightarrow (Y^{(i+1)}, (\frac{1}{2} + \epsilon_1)R^{(i+1)} + \epsilon_2F^{(i+1)})$$

such that $Y^{(i+1)}$ is fibered over $C^{(i+1)}$, and $\mathbf{M}^{(i+1)}$ is the moduli part of the canonical bundle formula applied to $(Y^{(i+1)}, \frac{1}{2}R^{(i+1)}) \rightarrow C^{(i+1)}$. Moreover, $Y^{(i)} \dashrightarrow Y^{(i+1)}$ is an isomorphism over $C^{(i)} \setminus G$. After finitely many steps, one obtains a relative log minimal model $(C^{\text{st}}, M^{\text{st}}) \rightarrow \text{Spec } A$, and the corresponding threefold pair $(Y^{\text{st}}, (\frac{1}{2} + \epsilon_1)R^{\text{st}} + \epsilon_2F^{\text{st}}) \rightarrow \text{Spec } A$ is KSBA-stable.

- (3) Finally one uses that the threefold $Y^{(i)}$ admits a “ruled model”: a birational model $Y^{(i)} \dashrightarrow Z^{(i)}$ such that $Z^{(i)}$ is easier to describe than $Y^{(i)}$. To obtain our main results we study the birational transformations relating $Z^{(i)}$ and $Y^{(i)}$.

Corollary 2.7. *With the notation as in Theorem 2.3, let $p \in C^{\text{st}}$ be either*

- *a node, or*
- *a smooth point of C^{st} , except for finitely many such points.*

Then, locally near p , the family $(Y^{\text{st}}, R^{\text{st}}) \rightarrow (p \in C^{\text{st}})$ is the coarse space of the pullback of the universal family $(\mathcal{Y}, \mathcal{R}) \rightarrow (\mathfrak{p} \in \mathcal{C})$ along a stable quasimap from \mathcal{C} to \mathcal{P}_1 . In particular, the fiber $(\mathcal{Y}_{\mathfrak{p}}^{\text{st}}, \mathcal{R}_{\mathfrak{p}}^{\text{st}})$ is such that $\mathcal{R}_{\mathfrak{p}}^{\text{st}}$ is supported over four distinct points.

Proof. By the construction of the birational contractions, the two families of curves C^{tw} and C^{st} are isomorphic in a neighborhood of p , and likewise the pairs $(Y^{\text{tw}}, R^{\text{tw}})$ and $(Y^{\text{st}}, R^{\text{st}})$ are isomorphic in a neighborhood of the fiber over p . The final assertion follows from the stability conditions for stable quasimaps; see [ISZ25, Definition 2.1]. \square

We conclude this section with the following technical remark, which will be used several times later in the paper. Readers who prefer to focus on the main ideas may wish to skip it on a first reading.

Remark 2.8. Recall that the nodal curves parametrized by \mathcal{P}_1 are *twisted* curves, i.e. Deligne–Mumford stacks rather than projective schemes; see [ISZ25, §2.1]. However, for the purpose of determining KSBA-stable limits, one may replace these twisted curves by their coarse moduli spaces, effectively ignoring the twisted nodes. More precisely, since all of our results concern the coarse spaces of stacky families of surfaces or curves, we will always take coarse spaces in two steps. Given a family $(\mathcal{Z}, \mathcal{D}) \rightarrow \mathcal{C}$ fibered in pairs parametrized by \mathcal{P}_1 , where \mathcal{C} is a twisted curve, we first take the relative coarse moduli space

$$\mathcal{Z} \longrightarrow \mathcal{Z}^{\text{rel}}$$

of $\mathcal{Z} \rightarrow \mathcal{C}$. This replaces each fiber containing a twisted dumbbell curve by the corresponding (untwisted) dumbbell curve, i.e. it replaces every twisted curve parametrized by \mathcal{P}_1 with its coarse space. We then take the absolute coarse moduli space of \mathcal{Z}^{rel} ; this agrees with taking the coarse space of \mathcal{Z} directly. Equivalently, one may describe this procedure using [ISZ25, Section 2.1]. There exists a moduli space $\mathcal{P}_1^{\text{CY}}$ parametrizing pairs $(C, p_1 + \cdots + p_4)$ that arise as the coarse moduli spaces of curves in \mathcal{P}_1 . Concretely, such pairs satisfy $C \simeq \mathbb{P}^1$ or C is a nodal union of two copies of \mathbb{P}^1 , the pair $(C, \frac{1}{2}(p_1 + \cdots + p_4))$ is slc, and the line bundle $\omega_C(\frac{1}{2}(p_1 + \cdots + p_4))$ is trivial. There is a natural morphism

$$\mathcal{P}_1 \longrightarrow \mathcal{P}_1^{\text{CY}}$$

sending each twisted curve to its coarse moduli space. Thus, given an open substack \mathcal{C} of a twisted curve and a morphism $\phi: \mathcal{C} \rightarrow \mathcal{P}_1$, let $\mathcal{Z} \rightarrow \mathcal{C}$ be the associated family of twisted curves. To compute the coarse moduli space of \mathcal{Z} , one may instead consider the composition

$$\mathcal{C} \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_1^{\text{CY}},$$

and replace \mathcal{C} by the relative coarse moduli space $\mathcal{C} \rightarrow \mathcal{C}'$ of the induced morphism to $\mathcal{P}_1^{\text{CY}}$. The resulting pulled-back family $\mathcal{Z}' \rightarrow \mathcal{C}'$ has the property that its coarse moduli space agrees with the coarse moduli space of \mathcal{Z} .

In summary, for our purposes, we may safely ignore twisted nodes and postcompose all morphisms $\mathcal{C} \rightarrow \mathcal{P}_1$ with $\mathcal{P}_1 \rightarrow \mathcal{P}_1^{\text{CY}}$. While working with \mathcal{P}_1 is convenient for proving (3), once this result is established one may work directly with $\mathcal{P}_1^{\text{CY}}$.

3. MODULI OF ELLIPTIC SURFACES WITH A BISECTION

Let $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ denote the irreducible closed substack of the KSBA moduli stack that generically parametrizes pairs $(X, \epsilon_1 R + \epsilon_2 F)$ with $0 < \epsilon_1 \ll \epsilon_2 \ll 1$, where:

- $f: X \rightarrow C$ is a smooth elliptic surface over a smooth curve, with no sections, but admitting a (non-unique) bisection B ;
- $R \subseteq X$ is the ramification divisor of the quotient $X \rightarrow Y$ induced by the linear system $|B|$;
- f has at worst irreducible nodal fibers, and F denotes the sum of all nodal fibers;
- the Kodaira dimension satisfies $\kappa(X) \geq 0$, ensuring that the moduli stack is non-empty.

The main goal of this section is to establish the following structure theorem for pairs parametrized by the moduli stack, together with a complete classification of the singular fibers and the corresponding surface singularities.

Theorem 3.1. *Any pair $(X, \epsilon_1 R + \epsilon_2 F)$ parametrized by $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ admits a fibration*

$$f : (X, \epsilon_1 R + \epsilon_2 F) \longrightarrow C$$

to a nodal curve C . Moreover, $\text{Supp } F$ is the union of some singular f -fibers, and for any irreducible component G of C with normalization G^ν , the surface $X|_{G^\nu}$ is the coarse space of one of the following:

- (1) a normal elliptic surface $X_G \rightarrow \mathcal{G}$; or
- (2) a non-normal fibered surface $X_G \rightarrow \mathcal{G}$ whose general fiber is a banana curve.

When we say elliptic surface, we mean a surface X with a morphism $X \rightarrow C$ whose generic fiber is a smooth curve of genus one.

The main idea to prove Theorem 3.1 is to reduce to the results in [ISZ25] as follows. Let $f: X \rightarrow C$ be a normal elliptic fibration over a smooth curve whose singular fibers are all irreducible curves with at worst nodal singularities, and let $B \subseteq X$ be a bisection. Then the linear system $|B|$ is base-point free over C , and therefore induces a double cover

$$\pi: X \longrightarrow Y := \text{Proj}_C \bigoplus_{m \geq 0} \text{Sym}^m f_* \mathcal{O}_X(B)$$

over C , where:

- (1) Every π -fiber is isomorphic to \mathbb{P}^1 , and fiberwise the map is the standard 2:1 cover from an elliptic curve to \mathbb{P}^1 determined by a g_2^1 .
- (2) The image of B , denoted S , is a section of $g: Y \rightarrow C$.
- (3) The ramification divisor $R \subseteq Y$ (resp. $R_X \subseteq X$) of π is a 4-section of g (resp. of f), and they satisfy the relation $\pi^* R = 2R_X$.
- (4) The pair $(X, 2aR_X + bF)$ is KSBA-stable if and only if the pair $(Y, (\frac{1}{2} + a)R + bF)$ is KSBA-stable, where $0 < a, b \leq 1$ and F are some marked fibers of f or g .

The other results of this section regard the singularities of the pairs appearing in $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$:

Theorem 3.2 (Singularities over the Smooth Locus I). *With the notation as in Theorem 3.1, assume that the surface X_{G^ν} is normal.*

- If a singular fiber of $X_G \rightarrow G$ over $G \cap C^{\text{sm}}$ is reduced, then both the fiber and the corresponding singularity of X_G appear in Table 2.
- If F_p is a non-reduced fiber of $X_G \rightarrow G$ over a point $p \in G \cap C^{\text{sm}}$, then F_p has multiplicity 2 and its reduced structure satisfies $(F_p)_{\text{red}} \simeq \mathbb{P}^1$. In this case, the corresponding singularity of X_G is of type A_{2k-1} (with $k \leq 4$), or of type D_m , or of type E_7 .

Theorem 3.3 (Singularities over smooth locus II). *With the notation as above, assume that the surface X_{G^ν} is non-normal.*

- If a singular fiber of $X_G \rightarrow G$ over $G \cap C^{\text{sm}}$ is reduced, then it is either a banana curve, or a k -cycle for $k = 3, 4$.
- If F_p is a non-reduced fiber of $X_G \rightarrow G$ over a point $p \in G \cap C^{\text{sm}}$, then F_p has multiplicity 2 and its reduced structure is a dumbbell curve. In this case, the corresponding singularity of X_G is of type A_{2k-1} (with $k \leq 2$), or of type D_m , or of type E_7 .

Theorem 3.4 (Singular fibers over nodes). *With the notation as above, a fiber of $X_G \rightarrow G$ over $G \cap C^{\text{sing}}$ is one of the following:*

- a reduced curve which is either an elliptic curve or a banana curve, or
- a non-reduced curve of multiplicity $m \in \{2, 3, 4, 6, 8\}$ whose reduced structure is \mathbb{P}^1 , or

- a non-reduced curve of multiplicity 2 whose reduced structure is an elliptic curve, or
- a non-reduced curve whose reduced structure is a banana curve, with multiplicities either (1, 2) or (2, 2) or (2, 4) or (4, 4) on its two components.

Remark 3.5. The preceding results should be understood as holding only in one direction. More precisely, we do not claim that all singularities listed above actually occur on surface pairs parametrized by $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$. Rather, our claim is that any singularity appearing on a surface pair in $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ must be among those described in the previous results.

We now outline the strategy for proving the theorems stated above. We begin by recalling several results concerning KSBA-stable limits of surface pairs equipped with a μ_2 -action; see Section §3.1. The proof of Theorem 3.1 follows immediately from the results of [ISZ25], reviewed in Section §2, together with some preparatory analysis of certain ramified covers of the surface pairs carried out in Section §3.1.

The classification of singular fibers requires more work. Theorem 3.4, which treats singular fibers lying over the nodal locus of C , is proved in Section §3.3. The analysis over the smooth locus of C is divided into two cases, according to whether the ramification divisor contains a fiber; these are handled in Sections §3.4 and §3.5, and together, these results complete the proof of Theorem 3.2 and Theorem 3.3.

3.1. Cyclic covers of KSBA-stable pairs. We now report two results on cyclic covers which we will use later, many of which are probably known to experts.

Lemma 3.6. *Let $(X, D) \rightarrow \text{Spec } A$ be a normal KSBA-stable family over the spectrum of a DVR A . Then any automorphism of the generic fiber (X_η, D_η) extends uniquely to an automorphism of (X, D) over $\text{Spec } A$.*

Proof. This follows as the moduli stack $\overline{\mathcal{M}}^{\text{KSBA}}$ is separated. Indeed, let $x \in \overline{\mathcal{M}}^{\text{KSBA}}(R)$ the object corresponding to the family $(X, D) \rightarrow \text{Spec } A$. As $\overline{\mathcal{M}}^{\text{KSBA}}$ is separated, the inertia stack $\text{Aut}(x) \rightarrow \text{Spec } A$ is a proper algebraic space over $\text{Spec } A$, which is a scheme finite over $\text{Spec } A$ as $\overline{\mathcal{M}}^{\text{KSBA}}$ is Deligne-Mumford. Therefore, the assertion follows from the valuative criterion of the properness of schemes. \square

Corollary 3.7. *Let $(X, D) \rightarrow \text{Spec } A$ be a KSBA-stable family with normal generic fiber over the spectrum of a DVR A and τ be an automorphism of (X, D) over $\text{Spec } A$ of order m . Let*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \searrow & \swarrow \\ & \text{Spec } A & \end{array}$$

be the quotient of X under the τ -action, D_Y be the a \mathbb{Q} -divisor supported over the image of D under π and such that for $U \subseteq Y$ a big open in Y where D_Y is Cartier, we have

$$\pi^*(D_Y)|_U = D|_{\pi^{-1}(U)}.$$

Let finally $R \subseteq Y$ be the divisorial part of the ramification locus. Then $(Y, \frac{m-1}{m}R + D_Y) \rightarrow \text{Spec } A$ is KSBA-stable. In particular, R does not contain any component of Y_0 .

If D has no component fixed by τ , then one can take as D_Y the image of D by τ .

Proof. By the assumptions, we have that Y is normal, $K_Y + \frac{m-1}{m}R + D_Y$ is \mathbb{Q} -Cartier by [AP12, Lemma 2.3], and

$$K_X + D = \pi^*(K_Y + \frac{m-1}{m}R + D_Y), \quad X_0 = \pi^*Y_0,$$

As $(X, D) \rightarrow \operatorname{Spec} A$ is KSBA-stable, then by the inversion of adjunction, $(X, D + X_0)$ is log canonical. By [KM98, Proposition 5.20], it follows that $(Y, \frac{m-1}{m}R + D_Y + Y_0)$ is a log canonical pair whose log canonical divisor is ample over $\operatorname{Spec} A$. Therefore, components of Y_0 are not contained in $\operatorname{Supp} R$, and $(Y, \frac{m-1}{m}R + D_Y) \rightarrow \operatorname{Spec} A$ is KSBA-stable. \square

We now record several results specific to our setting, which will be used later in the paper.

Let $(X_0, \epsilon_1 R_0 + \epsilon_2 F_0)$ be a KSBA-stable pair in $\overline{\mathcal{M}}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$. Take a DVR A and take a general smoothing $(X, \epsilon_1 R_X + \epsilon_2 F_X) \rightarrow \operatorname{Spec} A$ of $(X_0, \epsilon_1 R_0 + \epsilon_2 F_0)$. Let $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F) \rightarrow \operatorname{Spec} A$ be the quotient induced by a bisection of the generic fiber X_η . Then by Theorem 2.3 the threefold Y admits a morphism with purely 1-dimensional fibers to a family of at worst nodal curves $C \rightarrow \operatorname{Spec} A$, whose generic fiber C_η is smooth.

Lemma 3.8. *With the same notations as before, there exists a big open subscheme $C^\circ \subseteq C$ such that the ramification locus of $X \rightarrow Y$ over C° is divisorial, i.e. every irreducible component has codimension 1, and is contained in $(Y|_{C^\circ})^{\text{sm}}$. In particular, there exists a line bundle \mathcal{L}° on $Y|_{C^\circ}$ such that $R|_{C^\circ}$ is a section of $(\mathcal{L}^\circ)^{\otimes 2}$.*

Proof. Since $C \rightarrow \operatorname{Spec} A$ is a family of nodal curves with smooth geometric generic fiber, it suffices to work over the smooth locus C^{sm} of $C \rightarrow \operatorname{Spec} A$.

Let $p \in C^{\text{sm}}$ be a point. If the fiber of $Y \rightarrow C$ over p is \mathbb{P}^1 , then in a neighborhood U of p the morphism $Y|_U \rightarrow U$ is a \mathbb{P}^1 -fibration. By purity of the branch locus [Sta25, Tag 0BMB], the ramification locus over U is divisorial and lies in $(Y|_U)^{\text{sm}}$.

Now assume the fiber of $Y \rightarrow C$ over p is a dumbbell curve, defined in Definition 3.11. By Theorem 2.3, there exists an irreducible component $G \subseteq C_0^{\text{sm}}$ such that $Y|_G \rightarrow G$ is a fibration whose general fiber is a dumbbell curve. We claim that the node of a general fiber of $Y|_G \rightarrow G$ is not contained in the branch locus.

Indeed, a double cover of a dumbbell curve Γ that is branched at the node together with four additional points (two on each component) is never nodal; its normalization acquires worse-than-nodal singularities at the preimage of the node. Since by assumption $X|_G \rightarrow G$ has generically at worst nodal singularities, the double cover cannot branch at the node. Thus the branch locus avoids the nodes of fibers, and hence is contained in the smooth locus of $Y|_G$.

Combining the two cases gives the desired big open subset C° . \square

The following is a lemma which will be useful for the remaining part of the paper.

Lemma 3.9. *Let X and X' be S_2 varieties, and let $f: X \rightarrow C$ and $f': X' \rightarrow C$ be projective morphisms to a base scheme C with pure-dimensional fibers. Assume that there is a birational map*

$$\pi: X \dashrightarrow X'$$

over C which is an isomorphism over $C^\circ := C \setminus Z$, where Z is a closed subscheme of codimension ≥ 2 . Suppose further that L and L' are \mathbb{Q} -Cartier \mathbb{Q} -divisors on X and X' , respectively, which are relatively ample over C , and that

$$L|_{f^{-1}(C^\circ)} = L'|_{f'^{-1}(C^\circ)}$$

under the identification induced by π . Then X and X' are isomorphic over C , and under this isomorphism L corresponds to L' .

Proof. Choose an integer $r > 0$ such that rL and rL' are Cartier. For every integer $m \geq 0$ set

$$\mathcal{A}_m := f_*\mathcal{O}_X(mrL), \quad \mathcal{A}'_m := f'_*\mathcal{O}_{X'}(mrL').$$

Since mrL and mrL' are relatively ample over C , the graded \mathcal{O}_C -algebras

$$\mathcal{A} := \bigoplus_{m \geq 0} \mathcal{A}_m, \quad \mathcal{A}' := \bigoplus_{m \geq 0} \mathcal{A}'_m$$

are finitely generated, and we have isomorphisms

$$X \simeq \mathrm{Proj}_C(\mathcal{A}), \quad X' \simeq \mathrm{Proj}_C(\mathcal{A}'),$$

with $\mathcal{O}_{\mathrm{Proj}}(1)$ corresponding to rL and rL' , respectively. Over C° , the birational map π is an isomorphism, and by assumption

$$L|_{f^{-1}(C^\circ)} \simeq L'|_{f'^{-1}(C^\circ)}.$$

Hence for every $m \geq 0$ we obtain an isomorphism of \mathcal{O}_{C° -modules

$$\mathcal{A}_m|_{C^\circ} \simeq \mathcal{A}'_m|_{C^\circ}.$$

We may assume that C is affine. As $\mathrm{codim}_X Z \geq 2$ and f has pure dimensional fibers, and as

$$\Gamma(\mathcal{A}_m, C) = H^0(X, mrL) = H^0(X|_{C^\circ}, mrL|_{C^\circ}) = H^0(X'|_{C^\circ}, mrL'|_{C^\circ}) = \Gamma(\mathcal{A}'_m, C),$$

the isomorphism over C° extends uniquely to an isomorphism $\mathcal{A}_m \xrightarrow{\simeq} \mathcal{A}'_m$. These isomorphisms are compatible with the natural multiplication maps, and therefore assemble to an isomorphism of graded \mathcal{O}_C -algebras $\mathcal{A} \simeq \mathcal{A}'$. Taking relative Proj over C then gives the desired isomorphism $X \simeq X'$, and under this isomorphism, the class of rL corresponds to the class of rL' . Hence L corresponds to L' as \mathbb{Q} -Cartier \mathbb{Q} -divisors. \square

Corollary 3.10. *In Lemma 3.8, one can take $C^\circ = C^{\mathrm{sm}}$.*

Proof. This follows from Lemma 3.9. Indeed, consider the double cover of $Y|_{C^{\mathrm{sm}}}$ branched along $R|_{C^{\mathrm{sm}}}$. By construction, the inverse image of $R|_{C^{\mathrm{sm}}}$ is relatively ample over C^{sm} . Hence the resulting double cover satisfies the hypotheses of Lemma 3.9, and must therefore be isomorphic to X over C^{sm} . This shows that no further shrinking of the base is necessary, so one may take $C^\circ = C^{\mathrm{sm}}$. \square

3.2. Classification of fibers on double covers: preparation. One of the main goals of this section is to classify the singular fibers that appear by taking the double cover of a surface pair $(Y, \frac{1}{2}R)$ such that $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F)$ is a surface pair in $\overline{\mathcal{M}}^{\mathrm{KSBA}}(\epsilon_1, \epsilon_2)$. As preparation, we first classify certain ramified double covers of curves.

Definition 3.11. A *dumbbell curve* is the nodal union of two smooth rational curves meeting at a single point. A *twisted dumbbell curve* is a twisted curve \mathcal{C} whose coarse space is a dumbbell curve C , and which has a unique stacky point lying over the node of C with stabilizer μ_2 . The stabilizer preserves both components and acts on each component via $z \mapsto z^2$.

Equivalently, a twisted dumbbell curve is obtained by taking the nodal union of the two μ_2 -root stacks of \mathbb{P}^1 at ∞ , and gluing them along their common closed substack \mathcal{B}_{μ_2} .

Definition 3.12. A reduced projective curve C of arithmetic genus 1 is called

- a *banana curve* if $C = C_1 \cup C_2$ is the nodal union at two points of two smooth rational curves;
- a *pinched banana curve* if $C = C_1 \cup C_2$ is the union of two smooth rational curves at a singular point, which is a tacnode of C ;
- an *n-cycle*, for $n \geq 3$, if $C = C_1 \cup \dots \cup C_n$ is the nodal union of n smooth rational curves such that

$$C_i \cap C_j = \begin{cases} \text{singleton}, & |i - j| = 1 \\ \emptyset, & |i - j| \geq 2 \end{cases}$$

for any $i, j = 1, \dots, n$ where $C_{n+1} := C_2$ and $C_{n+2} := C_2$.

- A *four-leaf-clover curve* if $C = C_1 \cup_p C_2 \cup_p C_3 \cup_p C_4$ is the union of four smooth rational curves meeting at a common point p , each pair of components intersecting transversely, and such that the embedded dimension of C at p is 3. Equivalently, C is isomorphic to $\mathbb{V}(xy, z(x + y + z)) \subseteq \mathbb{P}^3$.

The next two lemmas will be used later in the classification of the singular fibers, whose proofs involve only simple computations.

Lemma 3.13. *The double cover of \mathbb{P}^1 branched along a divisor D of degree 4 is one of the following:*

- a *smooth elliptic curve*, if $\text{Supp } D$ consists of four distinct points;
- an *irreducible nodal genus one curve*, if $\text{Supp } D$ consists of three distinct points;
- an *irreducible cuspidal genus one curve*, if $\text{Supp } D$ consists of two distinct points, one of which has multiplicity 3;
- a *banana curve*, if $\text{Supp } D$ consists of two distinct points, each of which has multiplicity 2;
- a *pinched banana curve*, if $\text{Supp } D$ consists of a single point.

Lemma 3.14. *Let $C = C_1 \cup C_2$ be a dumbbell curve, and $D \subseteq C^{\text{sm}}$ be a divisor on C such that $\deg D|_{C_i} = 2$ for $i = 1, 2$. Then the double cover of \mathbb{P}^1 branched along a divisor D of degree 4 is one of the following:*

- a *banana curve*, if $\text{Supp } D$ consists of four distinct points;
- a *3-cycle*, if $\text{Supp } D$ consists of three distinct points;
- a *4-cycle*, if $\text{Supp } D$ consists of two distinct points;

Lemma 3.15. *Let $C = C_1 \cup_p C_2$ be a dumbbell curve with the node p . Let $D \subset C$ be a Cartier divisor such that for $i = 1, 2$ one has $\deg(D|_{C_i}) = 2$ and $\text{Supp}(D|_{C_i}) = \{p, p_i\}$, where $p_i \in C_i \setminus \{p\}$. Then the double cover of \mathbb{P}^1 branched along D is a pinched banana curve.*

Proof. One may verify the claim by writing a local equation near the node p . Locally at p , the dumbbell curve C is given in \mathbb{A}^2 by the equation $xy = 0$, and the divisor D is étale locally the restriction of the line $\mathbb{V}(x - y)$. Hence the double cover of C branched along D is locally given by

$$\text{Spec } k[x, y, w]/(w^2 - (x - y), xy) = \text{Spec } k[y, w]/(y(y + w^2)),$$

which is the standard local equation of a tacnode. Away from p , the cover is étale over each component C_i except at the two branch points $\{p, q_i\}$; thus its normalization restricts to a double cover of \mathbb{P}^1 branched over two distinct points, which is again isomorphic to \mathbb{P}^1 . Therefore, the resulting curve is a pinched banana curve. \square

Lemma 3.16. *Let $C = C_1 \cup_p C_2$ be a dumbbell curve. Let $D \subset C$ be a Cartier divisor such that for $i = 1, 2$ one has $\deg(D|_{C_i}) = 2$ and $\text{Supp}(D|_{C_i}) = \{p\}$. Then the double cover of C branched along D is a four-leaf-clover curve.*

Proof. The same argument as in the proof of Lemma 3.15 applies. Locally at p , the curve C is given by the node $xy = 0$, and the divisor D is the restriction of the non-reduced line $\mathbb{V}((x+y)^2)$. Therefore, the local equation of the double cover is

$$\text{Spec } k[x, y, z]/(z^2 - (x+y)^2, xy) = \text{Spec } k[x, y, z]/((z - (x+y))(z + (x+y)), xy).$$

This exhibits four smooth branches meeting transversely at the origin in embedding dimension 3, hence the double cover is a four-leaf-clover curve. \square

3.3. Classification of fibers over nodal locus of C . Let $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F) \in \overline{\mathcal{M}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ be a stable pair, and assume it is the central fiber of a one-parameter family over the spectrum of a DVR A

$$(\mathcal{Y}, (\tfrac{1}{2} + \epsilon_1)\mathcal{R} + \epsilon_2\mathcal{F}) \longrightarrow \mathcal{C} \longrightarrow \text{Spec } A$$

fibered over a twisted curve \mathcal{C} , whose generic fiber is a surface ruled over a smooth curve \mathcal{C}_η . To get the corresponding family of elliptic surfaces, one takes the double cover ramified over \mathcal{R} , which we denote by $\mathcal{X} \rightarrow \mathcal{Y}$. In this subsection, we consider a node $\mathfrak{p} \in \mathcal{C}_0$ (resp. $p \in C_0$) in the central fiber of $\mathcal{C} \rightarrow \text{Spec } A$ (resp. $C \rightarrow \text{Spec } A$, where $\mathcal{C} \rightarrow C$ is the coarse space map), and we study the fibers \mathcal{X}_p and $\mathcal{X}_{\mathfrak{p}}$. By Lemma 3.8, there exists a \mathbb{Q} -Cartier Weil divisor \mathcal{L} on \mathcal{Y} with Cartier index ≤ 2 such that \mathcal{R} is a section of $\mathcal{L}^{\otimes 2}$. Observe that \mathcal{R} is a Cartier divisor on \mathcal{Y} , so there are two possibilities: either \mathcal{L} is Cartier over p , or it is not Cartier but $\mathcal{L}^{\otimes 2}$ is Cartier. We will study these two cases separately.

Notation. Let \mathcal{G}_1 and \mathcal{G}_2 be the two irreducible components of \mathcal{C}_0 passing through p in the étale topology¹. Similarly, let \mathcal{Y}_1 and \mathcal{Y}_2 denote the surfaces over \mathcal{G}_1 and \mathcal{G}_2 , and similarly with \mathcal{X}_i . Denote by $G_1, G_2, X_1, X_2, Y_1, Y_2$ the corresponding coarse spaces. The fiber of \mathcal{Y}_i over \mathfrak{p} , denoted by $\mathcal{Y}_{\mathfrak{p}}$, is either \mathbb{P}^1 or a dumbbell curve by Corollary 2.7 and Remark 2.8.

3.3.1. \mathcal{L} is Cartier. In this case, we have line bundles \mathcal{L}_i on \mathcal{Y}_i with a section $\mathcal{R}_i := \mathcal{R}|_{\mathcal{Y}_i}$ of $\mathcal{L}_i^{\otimes 2}$. So we can take the double cover of \mathcal{Y}_i ramified over \mathcal{R}_i , this agrees with \mathcal{X}_i .

Proposition 3.17. *With the previous notations, we have the following.*

- (1) *If $\mathcal{Y}_{\mathfrak{p}} \simeq \mathbb{P}^1$, then the fiber $(X_i)_p$ has multiplicity $|\text{Stab}_{\mathcal{C}}(\mathfrak{p})|$. The reduced structure of the fiber is a quotient of an elliptic curve by a (possibly trivial) cyclic group.*
- (2) *If $\mathcal{Y}_{\mathfrak{p}}$ is a twisted dumbbell curve, then the fiber $(X_i)_p$ has either*
 - *an irreducible component of multiplicity 2 or 4 with reduced structure \mathbb{P}^1 , or*
 - *two irreducible components of multiplicities (1, 2) or (2, 2), whose reduced structure is a dumbbell curve.*

Proof. Let $\mu_n = \text{Stab}_{\mathcal{C}}(\mathfrak{p})$. We begin with the case when $\mathcal{Y}_{\mathfrak{p}} \simeq \mathbb{P}^1$. As \mathcal{L} is Cartier, the map $\mathcal{X} \rightarrow \mathcal{Y}$ is generically étale over \mathcal{Y}_p , and thus the multiplicities of the fibers $(X_i)_p$ and $(Y_i)_p$ agree. Indeed, we can write \mathcal{X} as

$$\text{Spec}_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1})$$

¹Here p could be a node of a single irreducible component in the Zariski topology of C , so étale locally the node has two branches

with the usual algebra structure of μ_2 -covers (see e.g. [KM98, Definition 2.52]), and check explicitly this is generically étale in codimension one, away from the ramification locus \mathcal{R} . We now compute the multiplicity of $(Y_i)_p$. To ease the notations, we drop the subscript i as it plays no role.

We first show that $\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n$ is generically a scheme, where the map $\mathcal{B}\mu_n \rightarrow \mathcal{G}$ is the closed embedding of the residual gerbe at \mathfrak{p} . Indeed, locally near \mathfrak{p} , \mathcal{Y} is the pull-back of the universal family via a *representable* morphism $\phi: \mathcal{G} \rightarrow \mathcal{P}_1$, hence $\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n$ can be identified with the stack-quotient of the curve corresponding to $\phi(\mathfrak{p})$ in \mathcal{P}_1 by the action of μ_n via the morphism

$$\mathrm{Stab}_{\mathcal{G}}(\mathfrak{p}) \rightarrow \mathrm{Stab}_{\mathcal{P}_1}(\phi(\mathfrak{p})).$$

The latter is injective as ϕ is representable, so $\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n$ is a stack-quotient of the form $[\mathbb{P}^1/\mu_n]$ with the action of μ_n being faithful. In particular, it is generically a scheme. Then the multiplicities of \mathcal{Y}_p and Y_p agree, and we can compute the former. We have

$$\mathcal{Y}_p = \mathcal{Y} \times_{\mathcal{G}} \{p\} = \mathcal{Y} \times_{\mathcal{G}} (\mathcal{G} \times_{\mathcal{G}} \{p\})$$

and

$$\mathcal{G} \times_{\mathcal{G}} \{p\} = [\mathrm{Spec}(k[[x]]/x^n)/\mu_n]$$

where μ_n acts by $\zeta * x = \zeta x$, as the map $\mathcal{G} \rightarrow G$ is, locally around p , a root stack at p . Then

$$\mathcal{Y}_p = \mathcal{Y} \times_{\mathcal{G}} [\mathrm{Spec}(k[[x]]/x^n)/\mu_n].$$

Now, \mathcal{Y}_p is generically a scheme, as its reduced structure (namely, $\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n \simeq [\mathbb{P}^1/\mu_n]$) is generically a scheme. Let $\mathrm{Spec} A \rightarrow \mathcal{Y}_p$ be an étale neighborhood of a schematic point. As the multiplicity of the fiber can be computed étale locally, and as the reduced structure of \mathcal{Y}_p agrees with $\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n$, the multiplicity of the fiber agrees with the multiplicity of $\mathrm{Spec} A_{\mathrm{red}}$ in $\mathrm{Spec} A$. Consider the following fiber diagrams:

$$\begin{array}{ccc} (\mathrm{Spec} A) \times \mu_n & \longrightarrow & \mathrm{Spec} A \\ \downarrow & & \downarrow \\ \mathrm{Spec} k[[x]]/(x^n) & \longrightarrow & [\mathrm{Spec}(k[[x]]/(x^n))/\mu_n] \end{array} \quad \begin{array}{ccc} (\mathrm{Spec} A_{\mathrm{red}}) \times \mu_n & \longrightarrow & (\mathrm{Spec} A) \times \mu_n \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} k[[x]]/(x^n). \end{array}$$

Here we shrink the étale neighborhood $\mathrm{Spec} A$ further so that the μ_n -torsor on the left becomes trivial. On the right, we again use the fact that $\mathcal{Y}_{\mathfrak{p}} \simeq \mathbb{P}^1$, which implies that the fiber product is reduced. Since the morphism $\mathcal{Y} \rightarrow \mathcal{G}$ is flat, then $(\mathrm{Spec} A) \times \mu_n \rightarrow \mathrm{Spec} k[[x]]/(x^n)$ is flat as well. Consequently, the fiber has multiplicity n .

Regarding the second statement, observe that $(X_i)_{p,\mathrm{red}}$ is the coarse moduli space of the double cover of $(\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n, \frac{1}{2}\mathcal{R} \times_{\mathcal{G}} \mathcal{B}\mu_n)$. In other terms, it is the coarse moduli space $E//\mu_n$ of $[E/\mu_n]$ where E is the elliptic curve that is the double cover $(\mathbb{P}^1, \mathcal{R} \times_{\mathcal{G}} \mathfrak{p})$.

We now address the second point. Proceeding as before, we have that $\mathcal{Y} \times_{\mathcal{G}} \mathcal{B}\mu_n = [C/\mu_n]$ where C is the dumbbell curve², and μ_n is a subgroup of $\mathrm{Aut}(C, R)$ where R consists of two distinct points on each branch of C . If μ_n does not fix an irreducible component of C , one can proceed exactly as before. If instead μ_n fixes an irreducible component of C and acts nontrivially on the other one, then $n = 2$. In this case, the fiber \mathcal{Y}_p will have an irreducible component of multiplicity 2 (corresponding to the component where μ_2 acts nontrivially) and an irreducible component of multiplicity 1. \square

²the dumbbell curve rather than the twisted dumbbell curve from Remark 2.8

3.3.2. \mathcal{L} is of Cartier index 2. In this case, étale locally at the generic points ξ of $\mathcal{Y}_{\mathfrak{p}}$, the morphism $\mathcal{Y} \rightarrow \mathcal{C}$ is isomorphic to

$$\mathrm{Spec} k[t, x, y, \pi]/(xy - \pi^{2n}) \longrightarrow \mathrm{Spec} k[x, y, \pi]/(xy - \pi^{2n}).$$

In particular, the local fundamental group at ξ is isomorphic to $\mathbb{Z}/2n\mathbb{Z}$, and hence, topologically, the only nontrivial double cover is

$$\mathrm{Spec} k[t, x', y', \pi]/(x'y' - \pi^n) \longrightarrow \mathrm{Spec} k[t, x, y, \pi]/(xy - \pi^{2n}),$$

where $x \mapsto (x')^2$ and $y \mapsto (y')^2$. The set-theoretic fiber of the ramified cover $\mathcal{X} \rightarrow \mathcal{Y}$ consists of a *single* point over every point of $\mathcal{Y}_{\mathfrak{p}}$. The same holds for the induced morphism $\mathcal{X}_i \rightarrow \mathcal{Y}_i$: the map $\mathcal{X}_i \rightarrow \mathcal{Y}_i$ is again a μ_2 -cover, and its codimension-one ramification locus can be identified as the set of codimension-one points of \mathcal{Y}_i over which the fiber of $\mathcal{X}_i \rightarrow \mathcal{Y}_i$ consists of a single set-theoretic point. In particular, in our situation, the morphism $\mathcal{X}_i \rightarrow \mathcal{Y}_i$ is ramified along both $\mathcal{R}|_{\mathcal{Y}_i}$ and $\mathcal{Y}_{\mathfrak{p}}$. Ramification along $\mathcal{Y}_{\mathfrak{p}}$ has the effect of preserving the reduced structure of the double locus of the central fiber of \mathcal{X} , while doubling its multiplicity.

Theorem 3.18 (Fibers over nodes). *With the notation as above, a fiber of $X_G \rightarrow G$ over $G \cap C^{\mathrm{sing}}$ is one of the following:*

- a reduced curve which is either an elliptic curve or a banana curve, or
- a non-reduced curve of multiplicity $m \in \{2, 3, 4, 6, 8\}$ whose reduced structure is \mathbb{P}^1 , or
- a non-reduced curve whose reduced structure is a banana curve, with multiplicities either $(1, 2)$ or $(2, 2)$ or $(2, 4)$ or $(4, 4)$ on its two components.
- a non-reduced curve of multiplicity 2 with reduced structure an elliptic curve.

The first bullet point is the case when the twisted curve \mathcal{C} is a scheme at the nodes of the special fiber, i.e. with the previous notation the case in which $\mu_n = \{1\}$. The second case is when the fiber of the morphism $\mathcal{Y} \rightarrow \mathcal{C}$ at a node of the special fiber of \mathcal{C} is \mathbb{P}^1 , or a dumbbell curve $\mathcal{Y}_{\mathfrak{p}}$ with μ_n acting transitively on the irreducible components of $\mathcal{Y}_{\mathfrak{p}}$ (so $n = 4$). The third case is when $\mathcal{Y}_{\mathfrak{p}}$ is a dumbbell curve and μ_n does not act transitively on its irreducible components.

3.4. Singular fibers over the smooth locus: no ramified fibers. Let $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F)$ be a stable pair parametrized by $\overline{\mathcal{M}}^{\mathrm{KSBA}}(\epsilon_1, \epsilon_2)$. By Theorem 2.3, such a pair admits a fibration $Y \rightarrow C$ onto a nodal curve C . In this subsection, we classify the singularities of the components of the double cover of Y branched along R , restricted over the smooth locus C^{sm} . By Remark 2.5, it suffices to consider surfaces of either type I or type III.

Let G be an irreducible component of C , and denote by $G^\circ = G \cap C^{\mathrm{sm}}$ the smooth locus of C along G .

Proposition 3.19. *Assume that the surface $Y_G \rightarrow G$ is of type I. Let X_G be the double cover of Y_G ramified along $R|_G$. Then over G° , all possible singularities of X_G are exactly those listed in Table 2.*

Proof. Since Y_G is of type I, Corollary 2.6 implies that $Y_{G^\circ} \rightarrow G^\circ$ is a \mathbb{P}^1 -bundle. The non-slc fibers of the pair $(Y_{G^\circ}, \frac{1}{2}R|_{G^\circ})$ were classified in [ISZ25, §4.2.1]. Taking the double cover branched along $R|_{G^\circ}$, one immediately obtains the corresponding singularities on X_G .

We illustrate the argument with one example; all other cases follow by the same local analysis. Suppose that R has at a point p a singularity locally given by $y^2 - z^{k+1} = 0$,

where $z = 0$ cuts out the fiber of $Y_G \rightarrow G$ through p . Then the ramified double cover has local equation

$$x^2 + y^2 + z^{k+1} = 0,$$

which is an A_k -singularity. By running through all singularities appearing in [ISZ25, Tables 2–3], one obtains precisely the list of singularities of X_{G° recorded in Table 2. \square

Fiber	Surface	Local equation
$A_k, k = 1, 2, 3$	smooth	
A_1	$A_k, k \geq 1$	$x^2 + y^2 + z^{k+1} = 0$
$A_k, k = 2, 3$	A_1	$x^2 + yz + y^{k+1} = 0$
$A_k, k = 2, 3$	A_2	$x^2 + z^2 + y^{k+1} = 0$
A_2	$D_n, n \geq 4$	$x^2 + (y + z)(y^2 - z^{n-2}) = 0$
A_3	$D_n, n \geq 4$	$x^2 + (y^2 - z^{n-2})(z - y^2) = 0$
A_2	E_6	$x^2 + y^3 + z^4 = 0$
A_2	E_8	$x^2 + y^3 + z^5 = 0$
A_3	E_6	$x^2 + z^3 + y^4 = 0$
A_2	$E_{6k+1}, k \geq 1$	$x^2 + y^3 + yz^{2k+1} = 0$
A_2	$J_{k,0}, k \geq 2$	$x^2 + y^3 + yz^{2k} = 0$

TABLE 2. Classification of singularities of fibers and surfaces
The equation of the fiber is $z = 0$

Proposition 3.20. *Assume that the surface $Y_G \rightarrow G$ is of type III, i.e. it is generically nodal along a curve Γ , with normalization Y_G^ν . If R intersects Γ at a point q , then on each component of Y_G^ν , the restriction of R intersects Γ . Moreover, on any component of Y_G^ν , Γ and R intersect transversely. In particular, the double cover X_G of Y_G branched along R_G has a fiber which is either a pinched banana curve or a four-leaf-clover curve.*

Proof. For the first statement, since R is a \mathbb{Q} -Cartier divisor, if it meets the double locus Γ , then its pullback to the normalization of Y_G must meet the preimage of Γ on every irreducible component. This shows that the restriction of R intersects each branch of Γ . If R and Γ do not meet transversely at q , then the pair $(Y_G^\nu, (\frac{1}{2} + \epsilon)R|_{Y_G^\nu} + \Gamma)$ is not log canonical at q . The last statement follows directly from Lemma 3.15 and Lemma 3.16. \square

Proposition 3.21. *Assume that the morphism $Y_G \rightarrow G$ is of type III, i.e. it is generically nodal along a curve $\Gamma \subset Y_G$. Then away from Γ , the double cover X_G of Y_G branched along $R|_G$ has at worst A_k -singularities over G° .*

Proof. On each component of the normalization of Y_G , the divisor $R_G := R|_G$ is a bisection of the projection to G . Over G° , the morphism $Y_G \rightarrow G$ is smooth, so the only possible singularities of R_G are ordinary double points. Analytically, these are locally given by an equation of the form $x^2 - y^{k+1} = 0$. The double cover of a smooth surface branched along

such a hypersurface has an A_k -singularity. Therefore, away from Γ , the double cover X_G has at worst A_k -singularities over G° . \square

3.5. Singular fibers over the smooth locus: with ramified fibers. In this subsection we consider the case when the ramification divisor R contains a fiber F_p of the morphism $Y \rightarrow C$ over a point $p \in C$. Here, $(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F)$ is a stable pair parametrized by $\overline{\mathcal{M}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$, which is necessarily fibered over C .

Lemma 3.22. *The point p is a smooth point of C ; in particular, the fiber F_p cannot lie in the double locus of Y . Moreover, the multiplicity of R along F_p is equal to 1.*

Proof. If p were a singular point of C or if the multiplicity of R along F were greater than 1, then the pair $(Y, (\frac{1}{2} + \epsilon)R)$ would fail to be *slc*, contradicting our assumptions. \square

Now let G be the component of C such that $p \in G$, let $G^\circ := G \cap C^{\text{sm}}$, and let Y_G be the component of Y lying over G . Let X_G be the component of the double cover of Y branched along R that lies over G .

Lemma 3.23. *Assume that Y_G is of type I or II. Let $q \in F_p$ be a singular point of R , and set $\Theta := R|_G - F_p$ to be the residual curve to F_p in $R|_G$. Then:*

- Θ has at worst a double point at q ; and
- if Θ is smooth at q , then the local intersection number $(\Theta, F_p)_q \leq 4$.

Proof. If $\text{mult}_q(\Theta) \geq 3$, then the pair $(Y, (\frac{1}{2} + \epsilon)R)$ would have worse than *slc* singularities at q , which contradicts our assumptions. The second assertion follows from the fact that generically Θ is a 4-section of the fibration $Y_G \rightarrow G$. \square

Proposition 3.24. *Assume that Y_G is of type I or II. Then $X_G \rightarrow G$ has a double fiber over p , whose reduced structure is \mathbb{P}^1 . Locally over $p \in G^\circ$:*

- (1) *If Θ is smooth at q , then X_G has an A_{2k-1} -singularity with $k \leq 4$.*
- (2) *If Θ is singular at q , then X_G has either a D_k -singularity with $k \geq 4$, or an E_7 -singularity.*

Proof. For the first statement, write local equations as follows. Let $x = 0$ be a local equation of F_p . The double cover of Y branched along R has a fiber which locally is of the form

$$\text{Spec } k[x, y, w]/(w^2 - xf(x, y), x),$$

which is generically non-reduced of multiplicity 2; its reduced structure is necessarily isomorphic to $F_p \simeq \mathbb{P}^1$.

If Θ is smooth at q , then by Lemma 3.23 we have $k := (\Theta, F_p)_q \leq 4$, and the branch divisor has a local equation $x - y^k = 0$. Thus the double cover has an A_{2k-1} -singularity.

If Θ is singular at q , then by Lemma 3.23, q is a double point of Θ , so locally Θ is given by $x^2 - y^k = 0$.

- If F_p is not given by $x = 0$, then after a coordinate change we may assume F_p is given by $y = 0$. In this case the branch divisor has a D_k -singularity, hence so does X_G .
- If F_p is given by $x = 0$, then the condition that $(Y, (\frac{1}{2} + \epsilon)R)$ is *slc* forces $k \leq 3$, and the resulting singularity is of type E_7 , hence X_G has either a D_4 or an E_7 -singularity at q .

\square

Example 3.25. Let $R \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a reduced divisor of class $\mathcal{O}(4, 4)$, consisting of the union of eight distinct fibers of the two projections. Then the double cover $f : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ branched along R is a K3 surface with 16 A_1 -singularities, and the composition of f with either projection $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ has four multiple fibers of multiplicity 2.

Lemma 3.26. Assume that Y_G is of type III, i.e. it is generically nodal along a curve Γ . On any irreducible component of the normalization of Y_G , let $q \in F_p$ be a singular point of R , and write

$$\Theta := R|_G - F_p$$

for the residual curve to F_p in $R|_G$. Then:

- (1) q is disjoint from Γ ;
- (2) Θ has at worst a double point at q ; and
- (3) if Θ is smooth at q , then the local intersection number $(\Theta, F_p)_q \leq 2$.

Proof. If $q \in \Gamma$, then F_p , Θ , and Γ all pass through q . Since $R|_G = F_p + \Theta$, the divisor R contains three distinct branches through the node of Y_G at q , and hence the pair

$$(Y_G, (\tfrac{1}{2} + \epsilon)R)$$

is not slc at q . Thus $q \notin \Gamma$. The remaining assertions follow exactly as in Lemma 3.23. \square

By the same argument as in Theorem 3.18, one obtains the following analogue.

Proposition 3.27. Assume that Y_G is of type III. Then the morphism $X_G \rightarrow G$ has a double fiber over p , whose reduced structure is a dumbbell curve. Locally over $p \in G^\circ$:

- (1) If Θ is smooth at q , then X_G has an A_{2k-1} -singularity with $k \leq 2$.
- (2) If Θ is singular at q , then X_G has either a D_k -singularity or an E_7 -singularity.

4. COMPACTIFICATION OF MODULI OF HYPERELLIPTIC K3 SURFACES

An important application of Theorem 3.1 is the compactification of the moduli stack of hyperelliptic K3 surfaces.

Definition 4.1. A polarized K3 surface (S, H) consists of a projective surface S with at worst ADE singularities such that $\omega_S \sim \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$, and an ample line bundle H .

Recall that one has the following trichotomy for polarized K3 surfaces.

Theorem 4.2 ([May72, SD74]). Let (S, L) be a primitively polarized K3 surface with $(L^2) = 2g - 2$. Then exactly one of the following holds.

- (1) (Generic case) The linear system $|L|$ is very ample, and the morphism $\phi_{|L|} : S \hookrightarrow |L|^\vee \simeq \mathbb{P}^g$ embeds S as a surface of degree $2g - 2$ in \mathbb{P}^g . In this case, a general member of $|L|$ is a smooth non-hyperelliptic curve.
- (2) (Hyperelliptic case) The linear system $|L|$ is base-point free, and the induced morphism $\phi_{|L|}$ realizes S as a double cover of a normal surface of degree $g - 1$ in \mathbb{P}^g . In this case, a general member of $|L|$ is a smooth hyperelliptic curve, and $|2L|$ is very ample.
- (3) (Unigonal case) The linear system $|L|$ has a fixed component E , which is a smooth rational curve. The linear system $|L - E|$ defines a morphism $S \rightarrow \mathbb{P}^g$ whose image is a rational normal curve. In this case, a general member of $|L - E|$ is a union of smooth elliptic curves, and $|2L|$ is base-point free.

Remark 4.3. The integer g is called the *genus* of the polarized K3 surface (S, L) , and it satisfies $g \geq 2$. When $g = 2$, a general polarized K3 surface (S, L) is realized as a double cover of \mathbb{P}^2 branched along a sextic curve; this case is exceptional among all genera. Therefore, for the remainder of this section, we will assume $g \geq 3$.

Let (S, L) be a hyperelliptic K3 surface of genus g . Then the Picard group $\text{Pic}(S)$ contains a primitive rank-two sublattice

$$\Lambda^g := \Lambda_{2,0}^g := \begin{pmatrix} 2g-2 & 2 \\ 2 & 0 \end{pmatrix},$$

generated by two effective divisor classes L and F . The class L defines a double cover

$$S \longrightarrow \mathbb{F}_n, \quad 0 \leq n \leq 4,$$

see e.g. [Rei76, p. 1]. The divisor F is the pullback of the fiber class of \mathbb{F}_n , and hence it satisfies $F^2 = 0$ and is nef; it induces an elliptic fibration

$$f: S \longrightarrow \mathbb{P}^1.$$

The *branch divisor* $R_{\mathbb{F}_n}$ on \mathbb{F}_n is smooth and of the class

$$4\mathfrak{e} + 2(n+2)\mathfrak{f},$$

where \mathfrak{e} and \mathfrak{f} denote respectively the classes of the negative section and of a fiber of the ruling on \mathbb{F}_n . If moreover (S, L) is a general hyperelliptic K3 surface, then $n = 0$ when g is even and $n = 1$ when g is odd, reflecting the parity constraint imposed by the discriminant form of $\Lambda_{2,0}^g$. By the Riemann-Roch theorem, the class

$$B := L - \left\lfloor \frac{g}{2} \right\rfloor F$$

is effective and satisfies

$$B \cdot F = 2,$$

and hence defines a bisection (not necessarily integral) of the elliptic fibration f .

Lemma 4.4. *With the notations as above. Let $R \subseteq S$ be the ramification locus of $S \rightarrow \mathbb{F}_n$, and F_{sing} be the sum of the singular locus of f , counted via the canonical bundle formula. Then $(S, \epsilon_1 R + \epsilon_2 F_{\text{sing}})$ is KSBA-stable for any $0 < \epsilon_1 \ll \epsilon_2 \ll 1$.*

Proof. It suffices to show that

$$K_S + \epsilon_1 R + \epsilon_2 F_{\text{sing}} \sim_{\mathbb{Q}} \epsilon_1 R + \epsilon_2 F_{\text{sing}}$$

is ample. As $(R \cdot F_{\text{sing}}) > 0$ and $\epsilon_1 \ll \epsilon_2$, then one has $(\epsilon_1 R + \epsilon_2 F_{\text{sing}})^2 > 0$. For any integral curve $C \subseteq S$, if C is not contained in a fiber of f , then $(\epsilon_1 R + \epsilon_2 F_{\text{sing}} \cdot C) > 0$, because $(F \cdot C) > 0$ and $\epsilon_1 \ll \epsilon_2$. If C is a component of an f -fiber, then

$$(\epsilon_1 R + \epsilon_2 F_{\text{sing}} \cdot C) = \epsilon_1 (R \cdot C) = \frac{\epsilon_1}{2} (\pi^* R_{\mathbb{F}_n} \cdot C) = \frac{\epsilon_1}{2} (R_{\mathbb{F}_n} \cdot \pi_* C) = \frac{\epsilon_1}{2} (4\mathfrak{e} + 2(n+2)\mathfrak{f} \cdot m\mathfrak{f})$$

for some $f > 0$, which is positive. Therefore, the divisor $\epsilon_1 R + \epsilon_2 F_{\text{sing}}$ is ample as desired. \square

Definition 4.5. For any number $0 < c < 1$, we set $H_c := (1-c)L + cF \in \Lambda^g \otimes \mathbb{R}$ and we say that c is *very irrational* if $H_c \notin \Lambda' \otimes \mathbb{R}$ for any proper sublattice $\Lambda' \subsetneq \Lambda^g$.

The sublocus of hyperelliptic K3 surfaces inside the moduli stack \mathcal{F}_g of polarized K3 surfaces of genus g need not be normal. The correct object to consider is the moduli stack

of lattice polarized K3 surfaces, which is the normalization of the hyperelliptic locus; see Proposition 4.13.

Definition 4.6 (cf. [Dol96, AE25]). For any very irrational $0 < c < 1$, let $\mathcal{F}_{\Lambda^g, c}$ be the moduli stack which sends a base scheme T to

$$\left\{ (f : \mathcal{S} \rightarrow T; \varphi) \left| \begin{array}{l} \mathcal{S} \rightarrow T \text{ is a proper flat morphism, each geometric fiber} \\ \mathcal{S}_{\bar{t}} \text{ is an ADE K3 surface, and } \varphi : \Lambda^g \rightarrow \text{Pic}_{\mathcal{S}/T}(T) \text{ is} \\ \text{a group homomorphism such that the induced map} \\ \varphi_{\bar{t}} : \Lambda^g \rightarrow \text{Pic}(\mathcal{S}_{\bar{t}}) \text{ is an isometric primitive embedding of} \\ \text{lattices and that } \varphi_{\bar{t}}(H_c) \in \text{Pic}(\mathcal{S}_{\bar{t}})_{\mathbb{R}} \text{ is an ample class.} \end{array} \right. \right\}.$$

We call $\mathcal{F}_{\Lambda^g, c}$ a *moduli stack of lattice polarized K3 surfaces*, where the lattice is Λ^g and the polarization is given by $H_c = (1 - c)L + cF$.

Remark 4.7. In the definition of the moduli stack $\mathcal{F}_{\Lambda^g, c}$, the class L is not required to be ample; only H_c is assumed to be ample. Nevertheless, in the situations we shall consider, the class L will in fact turn out to be ample.

Theorem 4.8 (cf. [AE25, Proposition 5.4, Theorem 5.5]). *The stack $\mathcal{F}_{\Lambda^g, c}$ is a separated smooth DM stack, and is independent of the choice of c .*

Remark 4.9. More precisely, for any two parameters $0 < c \neq c' < 1$, the stacks $\mathcal{F}_{\Lambda^g, c}$ and $\mathcal{F}_{\Lambda^g, c'}$ are canonically isomorphic, although their universal families may differ. There is a natural wall-crossing structure: there exist finitely many critical values

$$0 = c_0 < c_1 < \cdots < c_n = 1$$

such that for each $i = 0, \dots, n-1$, the universal family over $c \in (c_i, c_{i+1})$ is independent of the specific choice of c . The intervals (c_i, c_{i+1}) are called *small cones*. Equivalently, the small cones are the connected components of the complement of the set of parameters c for which there exists a vector

$$v \in \Lambda_{K3} \setminus \Lambda^\perp \quad \text{with} \quad v^2 = -2,$$

such that $(H_c \cdot v) = 0$ and the lattice $\langle \Lambda, v \rangle \subseteq \Lambda_{K3}$ is hyperbolic. See [AE25, Definition 4.9, Proposition 4.14, Definition 5.2].

Lemma 4.10. *Let $0 < \epsilon \ll 1$ be a very irrational number, and let*

$$(S, (1 - \epsilon)L + \epsilon F) \in \mathcal{F}_{\Lambda^g, \epsilon}$$

be a Λ^g -polarized K3 surface of genus $g \geq 3$. Then the class F is nef. In particular, H_c is ample for any $0 < c < 1$, i.e. all the $c \in (0, 1)$ are in the same small cone.

Proof. By assumption $H_\epsilon = (1 - \epsilon)L + \epsilon F$ is ample; in particular, L is nef and big. Let $\pi : \tilde{S} \rightarrow S$ denote the minimal resolution, so \tilde{S} is a smooth K3 surface. Denote by \tilde{L} and \tilde{F} the pullbacks of L and F , respectively.

Suppose that F was not nef. Then \tilde{F} is also not nef, so there exists a connected smooth rational curve $\tilde{C} \subset \tilde{S}$ such that $C := \pi(\tilde{C})$ satisfies

$$a := (\tilde{C} \cdot \tilde{L}) = (C \cdot L) \geq 0, \quad b := (\tilde{C} \cdot \tilde{F}) = (C \cdot F) < 0.$$

Since H_ϵ is ample, one has

$$(1 - \epsilon)a + \epsilon b > 0.$$

Consider the sublattice of $\text{Pic}(\tilde{S})$ generated by \tilde{L} , \tilde{F} , and \tilde{C} ; its intersection matrix is

$$\begin{pmatrix} 2g-2 & 2 & a \\ & 2 & 0 & b \\ & a & b & -2 \end{pmatrix}.$$

By the Hodge index theorem, its determinant must be positive. A direct computation shows that this determinant is

$$(2-2g)b^2 + 8 + 4ab > 0.$$

Since $g \geq 3$, this determinant is at most $4(b(a-b)+2)$. Hence the only possible solution is $a = 0$ and $b = -1$, contradicting the inequality $(1-\epsilon)a + \epsilon b > 0$. Thus F must be nef. \square

Corollary 4.11. *For every fixed $g \geq 3$ and every $0 < c < 1$, the moduli stacks $\mathcal{F}_{\Lambda^g, c}$ are canonically isomorphic, and their universal families are isomorphic. Moreover, for each $g \geq 3$ there is a canonical isomorphism*

$$\phi_g : \mathcal{F}_{\Lambda^g, c} \xrightarrow{\sim} \mathcal{F}_{\Lambda^{g+2}, c},$$

whose universal families of K3 surfaces are isomorphic and whose effect on the polarizing lattices is given by $L \mapsto L + F$ and $F \mapsto F$.

Proof. Fix $g \geq 3$ and consider the moduli stack $\mathcal{F}_{\Lambda, \epsilon}$ with $0 < \epsilon \ll 1$. By Lemma 4.10, the divisor H_ϵ is ample for every $0 < c < 1$ and every $(S, H_\epsilon) \in \mathcal{F}_{\Lambda, \epsilon}$. Therefore, all $0 < c < 1$ lie in the same generalized small chamber, and the corresponding stacks $\mathfrak{F}_{\Lambda, c}$ have isomorphic universal families.

Now let $(S, (1-\epsilon)L + \epsilon F) \in \mathcal{F}_{\Lambda^g, \epsilon}$ with $g \geq 3$. Lemma 4.10 implies that F is nef, hence $L + F$ and $(1-\epsilon)(L + F) + \epsilon F$ are ample. This defines a natural morphism

$$\phi_g : \mathcal{F}_{\Lambda^g, \epsilon} \longrightarrow \mathcal{F}_{\Lambda^{g+2}, \epsilon}, \quad (S, (1-\epsilon)L + \epsilon F) \longmapsto (S, (1-\epsilon)(L + F) + \epsilon F).$$

Conversely, given $(S, (1-\epsilon)L + \epsilon F) \in \mathcal{F}_{\Lambda^{g+2}, \epsilon}$, we claim that $L - F$ is nef, and hence the inverse morphism of ϕ_g is constructed. Since $(L - F)^2 > 0$, nefness implies that $(1-\epsilon)(L - F) + \epsilon F$ is ample, giving the inverse morphism. If $L - F$ were not nef, then for some curve C we would have

$$a := (L - F) \cdot C < 0, \quad b := F \cdot C \geq 0.$$

Repeating the intersection-matrix argument in the proof of Lemma 4.10 shows that the only numerical possibility is $a = -1$ and $b = 0$, contradicting the fact that L is nef. Thus $L - F$ must be nef, completing the proof. \square

Therefore, it is reasonable to denote $\mathcal{F}_{\Lambda^g, c}$ simply by \mathcal{F}_{Λ^g} for any $0 < \epsilon < 1$. Moreover, there are essentially only two stacks, depending on the parity of g . Since the results in the remainder of this section are insensitive to g , we henceforth write \mathcal{F}_Λ for \mathcal{F}_{Λ^g} , regardless of parity.

By the proof of Corollary 4.11, for any $g \geq 5$ and any $(S, (1-c)L + cF) \in \mathcal{F}_\Lambda$, the divisor L is ample. We observe that that this fails for $g = 3, 4$. Indeed, for $g = 3$, let $\pi : S \rightarrow \mathbb{F}_2$ be the double cover branched along a smooth divisor of class $4(\mathbf{e} + 2\mathbf{f})$. Then S is a K3 surface, and $L := \pi^*(\mathbf{e} + 2\mathbf{f})$ is not ample, although $(S, (1-c)L + c\pi^*\mathbf{f})$ is a Λ^3 -polarized K3 surface for $0 < c < 1$. Similarly, for $g = 4$, let $\pi : S \rightarrow \mathbb{F}_3$ be the double cover branched along a smooth divisor of class $4(\mathbf{e} + 2\mathbf{f})$. Then S is a K3 surface,

and $L := \pi^*(\mathfrak{e} + 3\mathfrak{f})$ is not ample, but $(S, (1 - c)L + c\pi^*\mathfrak{f})$ is a Λ^4 -polarized K3 surface for $0 < c < 1$.

For every $g \geq 3$, the hyperelliptic polarized K3 surfaces (S, L) form a divisor in the moduli stack \mathcal{F}_g of primitively polarized K3 surfaces of genus g . When $g = 4k + 3 \geq 7$, the hyperelliptic locus has two components; one of these generically consists of double covers of \mathbb{F}_4 branched along a divisor of the class $4\mathfrak{e} + 12\mathfrak{f}$, which is a disjoint union of the negative section and a trisection. The K3 surfaces parametrized by this component are always elliptic with a section, and we will not consider them here. For all other genera $g \geq 3$, the hyperelliptic locus is irreducible. We refer to this irreducible component as the *hyperelliptic divisor* and denote it by $\mathcal{D}_{2,0}^g$.

Remark 4.12. As stated in [Rei76, Page 2], the hyperelliptic locus in \mathcal{F}_g has two components for all odd genera $g \geq 5$. However, when $g = 4k + 1$, the polarization is not primitive, so this case does not lie in the moduli stack \mathcal{F}_g of *primitively* polarized K3 surfaces.

Proposition 4.13. *For $g \geq 4$, the moduli stack \mathcal{F}_Λ is isomorphic to the normalization $\mathcal{D}_{2,0}^{g,\nu}$ of the hyperelliptic divisor $\mathcal{D}_{2,0}^g$.*

Proof. We prove the statement for g even; the odd case is identical. Let $0 < \epsilon \ll 1$ be very irrational, and let

$$(\mathcal{S}, (1 - \epsilon)\mathcal{L} + \epsilon\mathcal{F}) \longrightarrow \mathcal{F}_{\Lambda, \epsilon} = \mathcal{F}_\Lambda$$

be the universal family. Then \mathcal{L} is big and nef over \mathcal{F}_Λ ; taking the ample model of \mathcal{L} (which is an isomorphism for $g \geq 5$) yields a natural morphism

$$\mathcal{F}_\Lambda \longrightarrow \mathcal{F}_g$$

by the universality of \mathcal{F}_g , whose image is contained in $\mathcal{D}_{2,0}^g$. Since \mathcal{F}_Λ is smooth, there exists a morphism

$$\alpha : \mathcal{F}_\Lambda \longrightarrow \mathcal{D}_{2,0}^{g,\nu}.$$

We claim that α is representable, birational, and finite; the result then follows from Zariski's main theorem for DM stacks.

Quasi-finiteness. Let $[(\bar{S}, \bar{L})] \in \mathcal{D}_{2,0}^g$ be a hyperelliptic K3 surface. Let $\pi : \tilde{S} \rightarrow \bar{S}$ be the minimal resolution. If

$$[(S, (1 - \epsilon)L + \epsilon F)] \in \mathcal{F}_{\Lambda, \epsilon}$$

lies above it, then \tilde{S} factors as

$$\tilde{S} \rightarrow S \rightarrow \bar{S},$$

contracting finitely many (-2) -curves; thus there are finitely many choices of S . The class L is the pullback of \bar{L} . By [Huy16, Proposition 11.1.3], a K3 surface has only finitely many elliptic fibrations up to isomorphism, so there are finitely many choices of F . Hence α is quasi-finite.

Birationality. A general $(S, L) \in \mathcal{D}_{2,0}^g$ is a double cover of \mathbb{F}_1 , and $L = \pi^*(\mathfrak{e} + (\frac{g}{2} - 1)\mathfrak{f})$. If $[(S, (1 - c)L + cF)]$ lies in the fiber of α over $[(S, L)]$, then F must be the pullback of \mathfrak{f} . Hence α is generically injective. Since both stacks are irreducible of dimension 18, then α is birational.

Representability. It suffices to check that stabilizers inject. As ϵ is very irrational, any stabilizer of $(S, (1 - \epsilon)L + \epsilon F)$ preserves L and hence acts on the ample model of L . If the induced automorphism is trivial, then the stabilizer is trivial, proving the representability.

Properness. Finally, we prove that α is proper, and therefore finite. Since both stacks are separated, it suffices to verify the existence part of the valuative criterion. Let $h : (\overline{\mathcal{S}}, \overline{\mathcal{L}}) \rightarrow (0 \in B)$ be a one-parameter family of hyperelliptic K3 surfaces of genus g such that there exists a commutative diagram

$$\begin{array}{ccc} (\mathcal{S}^\circ, (1-c)\mathcal{L}^\circ + c\mathcal{F}^\circ) & \xrightarrow{g^\circ} & (\overline{\mathcal{S}}^\circ, \overline{\mathcal{L}}^\circ) \\ & \searrow \tilde{h}^\circ & \swarrow h^\circ \\ & B^\circ & \end{array}$$

where $B^\circ = B \setminus \{0\}$, and g° is the ample model morphism over B° with respect to \mathcal{L} . We may assume that g° is an isomorphism. Furthermore, possibly after a finite base change, there exists a section of \mathcal{F}° over B° ; we continue to denote it by \mathcal{F}° . Let $\overline{\mathcal{F}}$ be the closure of \mathcal{F}° inside $\overline{\mathcal{S}}$ as a subscheme. Take a \mathbb{Q} -factorialization of $\overline{\mathcal{S}}$ and run a relative $\overline{\mathcal{F}}$ -MMP over $\overline{\mathcal{S}}$. This yields a family of K3 surfaces

$$\mathcal{S} \rightarrow B,$$

a birational morphism $g : \mathcal{S} \rightarrow \overline{\mathcal{S}}$ which is isomorphic in codimension one, and a \mathbb{Q} -Cartier divisor \mathcal{F} such that the divisor $(1-\epsilon)g^*\mathcal{L} + \epsilon\mathcal{F}$ is relatively ample over B . We claim that \mathcal{F} is a Cartier divisor. Then every fiber of the pair $(\mathcal{S}, (1-\epsilon)g^*\mathcal{L} + \epsilon\mathcal{F})$ over $b \in B$ is a Λ -polarized K3 surface: this follows because the general fiber has this property and the intersection numbers remain constant throughout the family. The line bundle $\overline{\mathcal{L}}$ is ample and base-point free over B , and it induces a double cover

$$\overline{\mathcal{S}} \rightarrow \overline{\mathcal{T}},$$

where the general fiber of $\overline{\mathcal{T}} \rightarrow B$ is a Hirzebruch surface \mathbb{F}_n , and the special fiber over $0 \in B$ is either \mathbb{F}_n or $\mathbb{P}(1, 1, m)$ for some $m \leq 4$.

In the former case, there exists a prime divisor $\overline{\mathcal{G}}$ on $\overline{\mathcal{T}}$, Cartier over B , such that for a general $b \in B$ its restriction to the fiber is the fiber class of the fibration $\mathbb{F}_n \rightarrow \mathbb{P}^1$. The divisor \mathcal{F} is then the pullback of $\overline{\mathcal{G}}$ along

$$\mathcal{S} \rightarrow \overline{\mathcal{S}} \rightarrow \overline{\mathcal{T}},$$

and therefore is Cartier. In this situation, $\overline{\mathcal{F}}$ is already Cartier, and no \mathbb{Q} -factorialization or MMP is required.

In the latter case, we take a \mathbb{Q} -factorialization of $\overline{\mathcal{T}}$, denoted by \mathcal{T} , such that the special fiber over $0 \in B$ becomes isomorphic to \mathbb{F}_n , and the strict transform of $\overline{\mathcal{G}}$ on \mathcal{T} , denoted by \mathcal{G} , is ample over \mathcal{T} . Composing with the morphism $\mathcal{S} \rightarrow \overline{\mathcal{S}}$ yields a generically 2:1 rational map

$$\mathcal{S} \dashrightarrow \mathcal{T}.$$

Let \mathcal{R} denote the closure of the ramification locus of this map. Let $\widetilde{\mathcal{S}}$ be the normalization of the double cover of \mathcal{T} branched along \mathcal{R} . Then $\widetilde{\mathcal{S}}$ is isomorphic to \mathcal{S} in codimension one, and both are ample models of the \mathbb{Q} -Cartier divisor $(1-\epsilon)\mathcal{L} + \epsilon\mathcal{F}$. Hence they are in fact isomorphic. Consequently, \mathcal{F} is the pullback of the Cartier divisor \mathcal{G} on \mathcal{S} , and is therefore Cartier. \square

Remark 4.14. The above proposition does not hold for $g = 3$: the morphism $\mathcal{F}_{\Lambda^3} \rightarrow \mathcal{D}_{2,0}^3$ is even not birational. Indeed, it is generically 2 : 1: a general $(S, L) \in \mathcal{D}_{2,0}^3$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$, with L the pullback of $\mathcal{O}(1, 1)$. However, there are two choices for F , namely the pullbacks of $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$.

Let

$$\pi : (\mathcal{S}, (1-c)\mathcal{L} + c\mathcal{F}) \longrightarrow \mathcal{F}_\Lambda$$

be the universal family. Then \mathcal{L} induces a double cover

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad} & \mathcal{Y} \\ & \searrow & \swarrow \\ & \mathcal{F}_\Lambda & \end{array},$$

which fiberwise is the aforementioned ramified cover $S \rightarrow \mathbb{F}_n$. Denote by \mathcal{R} the ramification locus. As the divisor \mathcal{F} is relatively nef over \mathcal{F}_Λ by Lemma 4.10, it induces an elliptic fibration

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad} & \mathcal{C} := \mathbb{P}_{\mathcal{F}_\Lambda}(\pi_*\mathcal{F}) \\ & \searrow & \swarrow \\ & \mathcal{F}_\Lambda & \end{array}.$$

Denote by $\mathcal{F}_{\text{sing}} \subseteq \mathcal{S}$ the closure of the sum of the singular fibers. Then by Lemma 4.4, the family $(\mathcal{S}, \epsilon_1\mathcal{R} + \epsilon_2\mathcal{F}_{\text{sing}}) \rightarrow \mathcal{F}_\Lambda$ is KSBA-stable, and hence there exists a natural morphism $\beta : \mathcal{F}_\Lambda \rightarrow \mathcal{M}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$, which is dominant. Let $\mathcal{M}_{\text{ell},2}^{\text{KSBA},\nu}(\epsilon_1, \epsilon_2)$ be the normalization of $\mathcal{M}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$, and

$$\beta^\nu : \mathcal{F}_\Lambda \rightarrow \mathcal{M}_{\text{ell},2}^{\text{KSBA},\nu}(\epsilon_1, \epsilon_2)$$

be the morphism induced by β .

Proposition 4.15. *The morphism β^ν is an open immersion.*

Proof. Since both \mathcal{F}_Λ and $\mathcal{M}_{\text{ell},2}^{\text{KSBA},\nu}(\epsilon_1, \epsilon_2)$ are separated DM stacks of finite type over \mathbb{C} , Zariski's main theorem reduces the claim to proving that β^ν is representable, quasi-finite, and birational.

Let $(S, \epsilon_1 R + \epsilon_2 F_{\text{sing}})$ be a general object of $\mathcal{M}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ and consider any $(S, (1-c)L + cF)$ mapping to it. For such S , the Picard rank is $\rho(S) = 2$, and there is a unique elliptic fibration $f : S \rightarrow \mathbb{P}^1$ for which F_{sing} is a multiple of the fiber class. Hence the class F is uniquely determined. Moreover, f admits a smooth bisection, which induces a double cover $S \rightarrow \mathbb{F}_n$. Since $\rho(S) = 2$, every bisection of f is the pullback of a section of \mathbb{F}_n , and they all induce the same double cover $S \rightarrow \mathbb{F}_n$. Therefore the polarization class L is uniquely determined as well, and β^ν is birational.

Next we verify representability. For any $(S, (1-c)L + cF) \in \mathcal{F}_\Lambda$, an automorphism σ fixes L , and in particular it fixes the ramification locus R of the double cover induced by L . Similarly, since σ fixes F , it also fixes F_{sing} . This yields a natural homomorphism

$$\text{Aut}(S, (1-c)L + cF) \longrightarrow \text{Aut}(S, \epsilon_1 R + \epsilon_2 F_{\text{sing}}),$$

which is injective because both groups are subgroups of $\text{Aut}(S)$. Hence β is representable, and therefore so is β^ν .

Finally, to check that β is quasi-finite, let $(S, \epsilon_1 R + \epsilon_2 F_{\text{sing}})$ be in the image of β . A preimage under β consists of S together with the class F , which is uniquely determined by F_{sing} . Thus it remains to show that there are only finitely many possibilities for the choice of L . Let $\tilde{S} \rightarrow S$ be the minimal resolution and let \tilde{L} be the pullback of L . Since \tilde{L} realizes \tilde{S} as a double cover of one of the Hirzebruch surfaces $\mathbb{F}_0, \dots, \mathbb{F}_4$, and since \tilde{L} is the pullback of a unique line bundle on \mathbb{F}_n , it follows that there are only finitely many possibilities for \tilde{L} , and hence for L . Therefore β is quasi-finite.

Having established representability, quasi-finiteness, and birationality, Zariski's main theorem implies that β^ν is an open immersion. \square

Corollary 4.16 (Compactification of the moduli of hyperelliptic K3 surfaces). *For any genus $g \geq 3$, there exists a Deligne–Mumford stack $\mathcal{M}_{\text{ell},2}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ whose objects are described in Theorems 3.1, 3.2, 3.3, and 3.4. The normalization of this stack is a compactification of the normalization of the hyperelliptic divisor $\mathcal{D}_{2,0}^g$ in the moduli stack \mathcal{F}_g of polarized K3 surfaces of genus g .*

5. MODULI OF WEIERSTRASS FIBRATIONS

In this section, we use the results in [ISZ25] to classify the singular objects parametrized by the KSBA moduli stack for Weierstrass fibrations.

5.1. Geometry of Weierstrass fibrations. Let $(X, S) \rightarrow C$ be a general Weierstrass fibration. We begin by observing that the linear series $|2S|$ is base-point free. Therefore, the doubled section $2S$ induces a double cover

$$\pi : X \longrightarrow Y := \text{Proj}_C \oplus_{m \geq 0} \text{Sym}^m f_* \mathcal{O}_X(2S)$$

over C , where

- (1) every π -fiber is a \mathbb{P}^1 , and fiberwise it is a standard hyperelliptic involution quotient from an elliptic curve to \mathbb{P}^1 induced by a g_2^1 ;
- (2) the image of S is a section of $g : Y \rightarrow C$, which we denote by Σ ;
- (3) the ramification divisor $R \subseteq Y$ (resp. $R_X \subseteq X$) of π is a 4-section of g (resp. f) which satisfies that $\pi^*R = 2R_X$;
- (4) Σ is an irreducible component of R ;

The surface Y is a ruled surface over C , and hence one can identify Y with $\mathbb{P}\mathcal{E}$ for some vector bundle \mathcal{E} of rank 2 on C . Moreover, one can assume that $H^0(C, \mathcal{E}) \neq 0$ and $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any $\mathcal{L} \in \text{Pic}(C)$ with $\deg \mathcal{L} < 0$. Let $\sigma \in H^0(Y, \mathcal{O}_Y(1)) \simeq H^0(C, \mathcal{E})$ be a section, where $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ under the identification

$$\text{Pic}(Y) = \text{Pic}(\mathbb{P}\mathcal{E}) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)] \oplus g^* \text{Pic}(C).$$

If X is smooth, then R is a smooth curve, and hence $R = \Sigma \sqcup T$ for some smooth divisor $T \sim 3\sigma + g^*D$, where D is a divisor on C . The letter T stands for “trisection” since $T \rightarrow C$ is of degree 3.

5.2. KSBA-moduli stack of Weierstrass fibrations. Let $0 < \epsilon \ll 1$ be a sufficiently small rational number, and let

$$\vec{a} = (a_1, \dots, a_n), \quad 0 < a_i \leq 1.$$

Consider a Weierstrass fibration

$$(X_\eta, \epsilon S_\eta + \vec{a} F_\eta) \rightarrow C_\eta$$

over the generic point η of the spectrum of a DVR A , whose fibers are at worst irreducible nodal curves, and such that the pair $(X_\eta, \epsilon S_\eta + \vec{a} F_\eta)$ is KSBA-stable. Here:

- S_η is a section of $X_\eta \rightarrow C_\eta$;
- $p_\eta = p_{\eta,1} + \dots + p_{\eta,n}$ is a sum of marked points on C_η , and

$$\vec{a} p_\eta := \sum_{i=1}^n a_i p_{\eta,i};$$

- $F_\eta = F_{\eta,1} + \cdots + F_{\eta,n}$ is the corresponding sum of fibers, and

$$\vec{a}F_\eta := \sum_{i=1}^n a_i F_{\eta,i}.$$

Denote by $\overline{\mathcal{W}}^{\text{KSBA}}(\epsilon, \vec{a})$ the KSBA moduli stack which generically parametrizes pairs of the form $(X_\eta, \epsilon S_\eta + \vec{a}F_\eta)$.

Theorem 5.1. *Let \mathbf{M}_η be the moduli part of the canonical bundle formula for $X_\eta \rightarrow C_\eta$. Then the following statements hold:*

- (1) *After a finite base change of $\text{Spec } A$, the KSBA-stable extension $(X, \epsilon S + \vec{a}F)$ of $(X_\eta, \epsilon S_\eta + \vec{a}F_\eta)$ over $\text{Spec } A$ admits a fibration over a generalized pair*

$$(C, \vec{a}p + \mathbf{M}) \rightarrow \text{Spec } A,$$

which is a family of nodal pointed curves with $K_C + \vec{a}p + \mathbf{M}$ ample, and whose generic fiber is $(C_\eta, \vec{a}p_\eta + \mathbf{M}_\eta)$.

- (2) *The family $C \rightarrow \text{Spec } A$ is the coarse space of a family of twisted curves $\mathcal{C} \rightarrow \text{Spec } A$ equipped with a fibration $(\mathcal{X}, \mathcal{S}) \rightarrow \mathcal{C}$, whose geometric fibers $(\mathcal{X}_p, \mathcal{S}_p)$ have equation*

$$(y^2x = x^3 + axz^2 + bz^3, [0, 1, 0]),$$

and such that the induced morphism on coarse spaces is $(\mathcal{X}, \mathcal{S}) \rightarrow \mathcal{C}$ giving $(X, S) \rightarrow C$.

- (3) *The fibers of $(X, S) \rightarrow C$ over codimension-one points of C have at worst nodal singularities.*

The following diagram summarizes the situation:

$$\begin{array}{ccccc} (X_\eta, \epsilon S_\eta + \vec{a}F_\eta) & \longrightarrow & (X, \epsilon S + \vec{a}F) & \xleftarrow{\text{cms}} & (\mathcal{X}, \epsilon \mathcal{S} + \vec{a}\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \text{Weierstrass} \\ (C_\eta, \vec{a}p_\eta + \mathbf{M}) & \longrightarrow & (C, \vec{a}p + \mathbf{M}) & \xleftarrow{\text{cms}} & \mathcal{C} \\ \downarrow & & \downarrow & & \swarrow \\ \eta & \hookrightarrow & \text{Spec } A & & \end{array}$$

Theorem 5.1 describes the KSBA-stable limits of Weierstrass fibrations.

5.3. Proof of Theorem 5.1. Take the quotient of X_η by the involution induced by the linear system $|2S_\eta|$, and denote the result by $(Y_\eta, \frac{1}{2}R_\eta)$. Then for any $0 < \epsilon \ll 1$, the pair $(X_\eta, 2\epsilon S_\eta + \vec{a}F_\eta)$ is KSBA-stable if and only if the pair

$$(Y_\eta, \frac{1}{2}R_\eta + \epsilon \Sigma_\eta + \vec{a}F_\eta) = (Y_\eta, \frac{1}{2}T_\eta + (\frac{1}{2} + \epsilon)\Sigma_\eta + \vec{a}F_\eta)$$

is KSBA-stable, where we abuse notation by denoting the marked fibers of $X_\eta \rightarrow C_\eta$ and those of $Y_\eta \rightarrow C_\eta$ uniformly by F_η . Moreover, we denote by R_η the ramification locus, which can be written as $R_\eta = \Sigma_\eta + T_\eta$, where Σ_η is a section and T_η a 3-section.

To find the KSBA-stable limit over $\text{Spec } A$, we apply Corollary 3.7. Namely, we obtain the KSBA-stable extension of $(Y_\eta, (\frac{1}{2} + \epsilon)\Sigma_\eta + \frac{1}{2}T_\eta + \vec{a}F_\eta)$, denoted by $(Y, (\frac{1}{2} + \epsilon)\Sigma + \frac{1}{2}T + \vec{a}F)$, by first taking the KSBA-stable limit of

$$(Y_\eta, (\frac{1}{2} + \epsilon_1)(\Sigma_\eta + T_\eta) + \vec{a}F_\eta + \epsilon_2 F'_\eta)$$

where F'_η is the sum of all fibers for which the support of the restriction of R to them has fewer than four points; and then decreasing the coefficient of T to $\frac{1}{2}$ and decreasing ϵ_2 to 0. The procedure has two steps.

- (1) The first step is preparatory: we study the pair (Y, Σ) in more detail; see §5.3.1.
- (2) In the second step, we construct the desired stable limits by first taking the canonical model of

$$(Y, (\tfrac{1}{2} + \epsilon_1)\Sigma + \tfrac{1}{2}T + \vec{a}F + \epsilon_2F'),$$

and then letting ϵ_2 decrease to 0; see §5.3.2.

5.3.1. *Step 1: the family $(Y, \Sigma) \rightarrow \operatorname{Spec} A$.* By Theorem 2.3, the pair $(Y_\eta, (\tfrac{1}{2} + \epsilon_1)D_\eta + \vec{a}F_\eta + \epsilon_2F'_\eta)$ extends to a KSBA-stable family

$$\begin{array}{ccc} (Y, (\tfrac{1}{2} + \epsilon_1)D + \vec{a}F + \epsilon_2F') & \xrightarrow{\quad\quad\quad} & C \\ & \searrow & \swarrow \\ & \operatorname{Spec} A & \end{array}.$$

Here, $0 < \epsilon_1 \ll \epsilon_2 \ll 1$ are two rational numbers. Let Σ be the closure of Σ_η and T the closure of $D_\eta \setminus \Sigma_\eta$. We have the following key observation.

Lemma 5.2. *The divisor Σ is \mathbb{Q} -Cartier.*

Proof. By Corollary 2.7, away from finitely many smooth points $x_1, \dots, x_r \in C$, the map $\pi: (Y, \tfrac{1}{2}D) \rightarrow C$ is the induced morphism between the coarse moduli spaces of

$$(\mathcal{Y}, \tfrac{1}{2}\mathcal{D}) \longrightarrow \mathcal{C},$$

where \mathcal{C} is a family of twisted curves over $\operatorname{Spec} A$, and the universal family comes from a morphism $\mathcal{C} \rightarrow \mathcal{P}_1$. The fibers of π are either \mathbb{P}^1 or the nodal union Γ of two copies of \mathbb{P}^1 . Moreover, \mathcal{D} intersects Γ at the smooth locus of Γ . In particular, since \mathcal{C} is \mathbb{Q} -factorial, the space \mathcal{Y} is also \mathbb{Q} -factorial in a neighborhood of \mathcal{D} . Thus Y is \mathbb{Q} -factorial near D , and away from the points x_1, \dots, x_r .

It remains to show that Y is \mathbb{Q} -factorial over a neighborhood of each x_i . Let G_i denote the irreducible component of the special fiber C_0 that contains x_i . In cases (1) and (2) of Theorem 2.4, the surface $G_i \times_C Y$ is smooth over x_i . In case (3), the surface $G_i \times_C Y$ has a unique singularity of the form

$$\operatorname{Spec}(k[[x, y, s, t]]/(xy - s^n))$$

lying over x_i , for some integer n . Here t is a uniformizer of the DVR A , while s is the pullback of a local equation of $G_i \subset C$ at x_i . But this singularity is \mathbb{Q} -factorial; hence Y is \mathbb{Q} -factorial in a neighborhood of x_i . \square

Lemma 5.3. *The divisors Σ and T are disjoint.*

Proof. For the generic Weierstrass fibration $(X_\eta, S_\eta) \rightarrow C_\eta$, every fiber is either a smooth elliptic curve or an integral nodal curve, and S is away from singularities of fibers. Thus the double cover $Y_\eta \rightarrow X_\eta$ is branched along Σ_η of ramification index 1, and Σ_η is disjoint from T_η on X_η . Since both Σ and T are \mathbb{Q} -Cartier by Lemma 5.2 and the fact that ϵ_1 is small enough, then

$$(\Sigma_\eta \cdot T_\eta) = (\Sigma_0 \cdot T_0) = 0,$$

which implies that either T and Σ do not intersect, or they intersect along a curve Γ . By inversion of adjunction, the pair $(Y, (\tfrac{1}{2} + \epsilon_1)(T + \Sigma) + \vec{a}F + \epsilon_2F' + Y_0)$ is log canonical, and thus the latter case cannot occur: otherwise, it would not be log canonical along Γ . Therefore, Σ and T are disjoint. \square

Remark 5.4. In what follows, the fibers $\vec{a}F$ do not play any role. To simplify notation, we will therefore omit them. More precisely, we will focus on the case $\vec{a}F = 0$; the case where $\vec{a}F \neq 0$ is entirely analogous.

5.3.2. *Step 2: KSBA-stable extension* $(Y, (\frac{1}{2} + \epsilon)\Sigma + \frac{1}{2}T) \rightarrow \text{Spec } A$. We now decrease the coefficients in

$$(Y, (\frac{1}{2} + \epsilon_1)R + \epsilon_2 F) = (Y, (\frac{1}{2} + \epsilon_1)\Sigma + (\frac{1}{2} + \epsilon_1)T + \epsilon_2 F)$$

to obtain

$$(Y, (\frac{1}{2} + \epsilon_1)\Sigma + \frac{1}{2}T),$$

and take its log canonical model over $\text{Spec } A$. Throughout this process, we keep track of the birational transformations in order to describe the geometry of the log canonical model of $(Y, (\frac{1}{2} + \epsilon_1)\Sigma + \frac{1}{2}T)$.

We proceed in two steps: first, we decrease the coefficient of T to $\frac{1}{2}$ and take the canonical model over C ; then, we decrease the coefficient ϵ_2 of F to 0. We begin with the following, which is a consequence of Lemma 5.2.

Corollary 5.5. *The threefold pair $(Y, \Sigma + \epsilon_2 F')$ is log canonical.*

Proof. We first show that the pair $(Y, \Sigma + \epsilon_2 F')$ is slc over the smooth points of C_0 , the central fiber of $C \rightarrow \text{Spec } A$. Since the fibers F intersect Σ transversely, it suffices to prove that (Y, Σ) is log canonical.

Let G be an irreducible component of C_0 , and let $x \in G$ be a point which is not a node of C_0 . Denote by F_x the fiber over x . Then

$$(F_x \cdot \Sigma) = 1,$$

since F_x is numerically equivalent to a fiber of the generic fiber. In particular, $\Sigma \rightarrow C$ remains a section and is therefore smooth over such x . As Σ does not intersect T by Lemma 5.3, and $Y_G := Y|_G \rightarrow G$ is smooth over x in cases (1) and (2) of Theorem 2.4, the desired local log canonicity follows. Hence, to show that $(Y, \Sigma + \vec{a}F + \epsilon_2 F')$ is slc over the smooth points of C_0 , it remains only to analyze Σ in case (3) of Theorem 2.4.

For this remaining case, observe that Σ does not meet the horizontal double locus of Y_G . Indeed,

- (1) Σ is \mathbb{Q} -Cartier, so the pair $(Y, t\Sigma)$ is locally stable for every $0 < t \leq \frac{1}{2}$. In particular, the reduced pair

$$(Y_G, t\Sigma|_{Y_G})$$

is slc for all $t \in [0, \frac{1}{2}]$;

- (2) as $\Sigma \rightarrow C$ is a section, the intersection $\Sigma \cap F_x$ consists of a single reduced point.

If Σ met the double locus of Y_G , then étale locally near such a point, Y_G would have two irreducible components, with Σ lying on exactly one of them. Consider $Y_G^n = (\Gamma_1, \Delta_1) \sqcup (\Gamma_2, \Delta_2)$ where Γ_i are the irreducible components of the normalization of Y_G , and Δ_i the preimages of the double locus. The proper transform Σ_1 of Σ is contained in Γ_1 but not Γ_2 , and if Σ intersected the double locus, then there would be a point $p \in \Delta_1$ such that the different $\text{Diff}_{\Delta_1}((\Gamma_1, \Delta_1 + t\Sigma_1))$ depends on t . On the other hand, as Σ_1 is not contained in Γ_2 , there is no point in $\text{Diff}_{\Delta_2}((\Gamma_2, \Delta_2 + t\Sigma_1))$ which depends on t . This contradicts Kollár's gluing theorem.

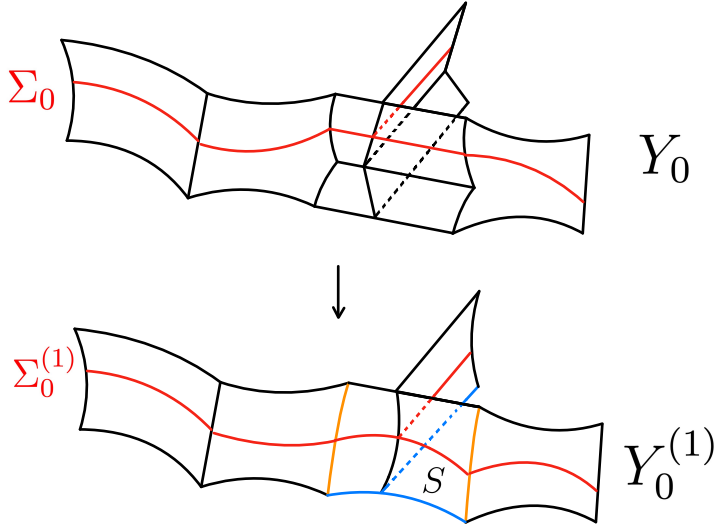


FIGURE 2. Log canonical model morphism $Y \rightarrow Y^{(1)}$. In blue, the curves to which some irreducible components of Y_0 are contracted to.

We now analyze the pair over the nodes of C . By the construction of the ruled model, after possibly restricting C to a neighborhood of its nodal locus, the family

$$(Y, \tfrac{1}{2}\Sigma + \tfrac{1}{2}D) \rightarrow C$$

is obtained by pulling back the universal family from a twisted curve $\mathcal{C} \rightarrow \mathcal{P}_1$ and then passing to coarse spaces. Since Σ is a section and is disjoint from T , every fiber has the form

$$(\mathcal{R}, \tfrac{1}{2}p_\Sigma + \tfrac{1}{2}(p_1 + p_2 + p_3)),$$

with $p_\Sigma \neq p_i$ for all i . Such fibers satisfy that $(\mathcal{R}, p_\Sigma + \tfrac{1}{2}(p_1 + p_2 + p_3))$ is slc, and therefore the original pair is slc over the nodal locus as well. \square

Due to the presence of the section Σ , if $G \subseteq C_0$ is an irreducible component such that Y_G falls into case (3) of Theorem 2.4, then Y_G necessarily has two irreducible components: one meeting Σ and one disjoint from Σ . Moreover, we have the following.

Corollary 5.6. *The pair*

$$(Y, (\tfrac{1}{2} + \epsilon_1)\Sigma + \tfrac{1}{2}T + \epsilon_2F)$$

is relatively minimal over C .

Proof. This is because $0 < \epsilon_1 \ll \epsilon_2$ and $(Y, \tfrac{1}{2}(\Sigma + T))$ is relatively minimal over C . \square

Thus, taking the log canonical model over C

$$(Y, (\tfrac{1}{2} + \epsilon_1)\Sigma + \tfrac{1}{2}T + \epsilon_2F) \longrightarrow (Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)} + \epsilon_2F^{(1)})$$

contracts precisely those irreducible components of the surface pairs appearing in case (3) of Theorem 2.4 that do not meet Σ ; see Figure 2. For a detailed description of the geometry of these components, see [ISZ25, Proposition 4.6 and Figure 10]. Furthermore, the fibers of $\pi^{(1)} : Y^{(1)} \rightarrow C$ over the codimension-one points of C are isomorphic to \mathbb{P}^1 .

Corollary 5.7. *Each irreducible component of $Y_0^{(1)} \rightarrow C_0$ is the coarse space of a \mathbb{P}^1 -bundle over an orbifold curve.*

Proof. Let $\mathcal{C} \rightarrow C$ be the smooth canonical covering stack. Since C is a family of nodal curves, its singularities are analytically of the form

$$\mathrm{Spec}(k\llbracket x, y, t \rrbracket / (xy - t^n)).$$

The canonical smooth covering stack replaces such a singularity with

$$[\mathrm{Spec}(k\llbracket u, v \rrbracket) / \mu_n],$$

where μ_n acts via $\zeta \cdot u = \zeta u$ and $\zeta \cdot v = \zeta^{-1}v$, and the coarse space map sends

$$x \mapsto u^n, \quad y \mapsto v^n, \quad t \mapsto uv.$$

Away from finitely many points of C_0 , the morphism $Y^{(1)} \rightarrow C$ is a \mathbb{P}^1 -fibration. Since \mathcal{C} and C agree away from the nodes of C_0 , there exists a big open subscheme $U \subseteq \mathcal{C}$ together with a \mathbb{P}^1 -fibration over U . This produces a morphism

$$U \longrightarrow \mathcal{B}\mathrm{PGL}_2,$$

and by [DLI24, Lemma 2.1] this extends uniquely to a morphism

$$\mathcal{C} \longrightarrow \mathcal{B}\mathrm{PGL}_2.$$

Consequently, we obtain a \mathbb{P}^1 -fibration $\mathcal{Y} \rightarrow \mathcal{C}$, and denote its coarse space by $Y' \rightarrow C$. Now, $Y^{(1)}$ carries a divisor—namely $\Sigma^{(1)}$ —which is relatively ample over C . Its proper transform in Y' gives a divisor Σ' that is also relatively ample over C . As both Y' and $Y^{(1)}$ are S_2 , then by Lemma 3.9, Y' is isomorphic to $Y^{(1)}$. In particular, each component of $Y_0^{(1)}$ is the coarse space of a \mathbb{P}^1 -bundle over an orbifold curve. \square

We now run an MMP to decrease ϵ_2 to 0 in the pair

$$(Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)} + \epsilon_2F).$$

The canonical bundle formula for the lc-trivial fibration

$$(Y^{(1)}, \tfrac{1}{2}(\Sigma^{(1)} + T^{(1)})) \longrightarrow C$$

yields

$$K_{Y^{(1)}} + \tfrac{1}{2}(\Sigma^{(1)} + T^{(1)}) \sim_{\mathbb{Q}} \pi^{(1)*}(K_C + \mathbf{M}),$$

where the discriminant divisor is trivial and \mathbf{M} is the moduli divisor. Since

$$(Y, \tfrac{1}{2}(\Sigma + T)) \rightarrow (Y^{(1)}, \tfrac{1}{2}(\Sigma^{(1)} + T^{(1)}))$$

is crepant, then these two pairs have the same moduli part; hence $K_C + \mathbf{M}$ is nef. To take the log canonical model of

$$(Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)}),$$

one considers the log canonical model

$$(C, \mathbf{M}) \longrightarrow (C^{\mathrm{st}}, \mathbf{M}^{\mathrm{st}}),$$

which contracts precisely:

- (1) rational bridges $B \subseteq C_0$ along which $\deg_B(\mathbf{M}) = 0$, and
- (2) rational tails G of C_0 along which $\deg_G(\mathbf{M}) = 1$.

Rational bridges with $\deg_B(\mathbf{M}) = 0$. Let B be such a rational bridge. By Corollary 5.7, the restriction (see the component S in Figure 2)

$$(Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)})|_B$$

is birational to (see Figure 3)

$$(\mathbb{P}_B, (\tfrac{1}{2} + \epsilon_1)\Sigma_{\mathbb{P}_B} + \tfrac{1}{2}\Sigma_2 + \Delta + F),$$

where:

- $\mathcal{B} \rightarrow B \simeq \mathbb{P}^1$ is a (possibly trivial) root stack;
- \mathbb{P}_B is the coarse space of a \mathbb{P}^1 -bundle $\mathbb{P}_B(\mathcal{E}) \rightarrow \mathcal{B}$;
- Δ is the image in $Y^{(1)}$ of the horizontal double locus of $Y_0|_B$, i.e. where the two components of $Y|_B$ intersect;
- $\Sigma_{\mathbb{P}_B}$ is a section of $\mathbb{P}_B \rightarrow B$, and Σ_2 is determined by the relation

$$\tfrac{1}{2}T|_{Y_0^{(1)}} - \Delta - \tfrac{1}{2}\Sigma_{\mathbb{P}_B} = \tfrac{1}{2}\Sigma_2;$$

- $F = F_1 + F_2$ is the coarse space of the two fibers of $\mathbb{P}_B \rightarrow \mathcal{B}$.

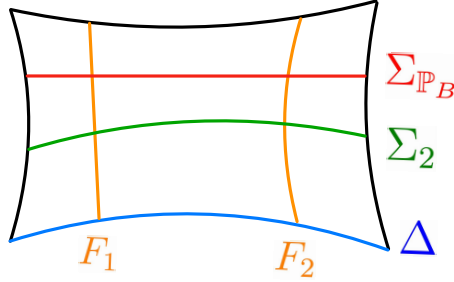


FIGURE 3. The pair $(\mathbb{P}_B, \Sigma_{\mathbb{P}_B} + \Sigma_2 + \Delta + F)$

Observe that $\Sigma_{\mathbb{P}_B}$, Σ_2 , and Δ must be pairwise disjoint in order for $\deg \mathbf{M}|_B = 0$. Indeed, $\Sigma_{\mathbb{P}_B}$ and Σ_2 are disjoint from Lemma 5.3, and Σ and Δ are disjoint as before the contraction $Y \rightarrow Y^{(1)}$ the section does not intersect the horizontal double locus of the central fiber. We just need to argue that Δ and Σ_2 are disjoint. But if they met over a point x , the boundary part over that point would not be 0, as Δ has coefficient 1 and Σ_2 has coefficient $\frac{1}{2}$. The degree of the moduli part $\deg \mathbf{M}|_B$ agrees with the sum of the degree of the moduli part and the boundary part for $(Y^{(1)}|_B, \frac{1}{2}\Sigma_{\mathbb{P}_B} + \frac{1}{2}\Sigma_2 + \Delta)$, so $\deg \mathbf{M}|_B > 0$ which is a contradiction.

Consequently, when taking the canonical model of

$$(Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)}),$$

the surface $Y^{(1)}|_B$ is contracted to the two fibers to which it is attached in neighboring components of $Y_0^{(1)}$

Rational tails where \mathbf{M} has degree 1. We now consider components of $Y_0^{(1)}$ lying over a rational tail G of C_0 such that $\deg \mathbf{M}|_G = 1$.

Lemma 5.8. *The component $Y_G^{(1)} := Y_0^{(1)}|_G$ is contracted by taking the log canonical model of*

$$(Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)}).$$

Proof. By the classification of the components of $Y_0^{(1)}$, the pair $(Y^{(1)}, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)} + \tfrac{1}{2}T^{(1)})|_G$ is of the form

$$(\mathbb{P}_G, (\tfrac{1}{2} + \epsilon_1)\Sigma^{(1)}|_{\mathbb{P}_G} + \tfrac{1}{2}T^{(1)}|_{\mathbb{P}_G} + F),$$

where \mathbb{P}_G is the coarse space of a \mathbb{P}^1 -bundle \mathbb{P}_G over a stacky curve \mathcal{G} , and F is the coarse space of one fiber over the unique point of \mathcal{G} lying above the node of C_0 . By assumption,

$$(5.1) \quad K_{\mathbb{P}_G} + \tfrac{1}{2}\Sigma^{(1)}|_{\mathbb{P}_G} + \tfrac{1}{2}T^{(1)}|_{\mathbb{P}_G} + F \sim_{\mathbb{Q}} 0.$$

Thus the desired log canonical model of \mathbb{P}_G is also the ample model of $\Sigma^{(1)}|_{\mathbb{P}_G}$. Since $\Sigma^{(1)}|_{\mathbb{P}_G}$ is irreducible, it suffices to show that $(\Sigma^{(1)}|_{\mathbb{P}_G})^2 \leq 0$.

By Corollary 5.5, and since Σ and T don't intersect from Lemma 5.3, the pair

$$(\mathbb{P}_G, \Sigma|_{\mathbb{P}_G} + \frac{1}{2}T|_{\mathbb{P}_G} + F)$$

is lc, and hence the pair

$$(\mathbb{P}_G, \Sigma_{\mathbb{P}_G} + \frac{1}{2}\mathcal{T}_{\mathbb{P}_G} + \mathcal{F})$$

is also lc, where we use curly letters to denote the reduced preimages under the coarse space map $\phi: \mathbb{P}_G \rightarrow \mathbb{P}_G$. The morphism ϕ is étale in codimension 1 away from the unique point over which the tail attaches, and otherwise ramified exactly along F , which appears with coefficient 1 in the boundary. Thus one has

$$\phi^*(K_{\mathbb{P}_G} + F) \sim_{\mathbb{Q}} K_{\mathbb{P}_G} + \mathcal{F}, \quad \phi^*(\Sigma|_{\mathbb{P}_G}) \sim_{\mathbb{Q}} \Sigma_{\mathbb{P}_G}, \quad \phi^*(T|_{\mathbb{P}_G}) \sim_{\mathbb{Q}} \mathcal{T}_{\mathbb{P}_G},$$

and consequently

$$(5.2) \quad (K_{\mathbb{P}_G} + \Sigma|_{\mathbb{P}_G} + F \cdot \Sigma|_{\mathbb{P}_G}) = (\phi^*(K_{\mathbb{P}_G} + \Sigma|_{\mathbb{P}_G} + F) \cdot \phi^*\Sigma|_{\mathbb{P}_G})$$

$$(5.3) \quad = (K_{\mathbb{P}_G} + \Sigma_{\mathbb{P}_G} + \mathcal{F} \cdot \Sigma_{\mathbb{P}_G})$$

$$(5.4) \quad = \deg \omega_{\Sigma_{\mathbb{P}_G}}(p),$$

where $p = \mathcal{F} \cap \Sigma_{\mathbb{P}_G}$ is a single (stacky) point with reduced structure. If $\Sigma_{\mathbb{P}_G}$ is an orbifold \mathbb{P}^1 with a single stacky point, and $\psi: \Sigma_{\mathbb{P}_G} \rightarrow \mathbb{P}^1$ is the coarse map, then

$$\omega_{\Sigma_{\mathbb{P}_G}}(p) = \psi^*(\omega_{\mathbb{P}^1}(p)).$$

On the other hand, from (5.1),

$$K_{\mathbb{P}_G} + \frac{1}{2}\Sigma|_{\mathbb{P}_G} + \frac{1}{2}T|_{\mathbb{P}_G} + F \sim_{\mathbb{Q}} \pi^*(\omega_{\mathbb{P}^1}(p) + \mathbf{M}|_G) \sim_{\mathbb{Q}} 0,$$

where $p \in G$ is the attaching point. Since $\deg \mathbf{M}|_G \geq 0$, we obtain

$$(K_{\mathbb{P}_G} + \frac{1}{2}\Sigma|_{\mathbb{P}_G} + \frac{1}{2}T|_{\mathbb{P}_G} + F) \cdot \Sigma|_{\mathbb{P}_G} = 0,$$

$$(K_{\mathbb{P}_G} + \Sigma|_{\mathbb{P}_G} + \frac{1}{2}T|_{\mathbb{P}_G} + F) \cdot \Sigma|_{\mathbb{P}_G} = -\deg(\mathbf{M}|_G) \leq 0.$$

Thus $(\Sigma|_{\mathbb{P}_G})^2 \leq 0$, completing the proof. \square

As a consequence, denoting by

$$(Y^{(1)}, (\frac{1}{2} + \epsilon_1)\Sigma^{(1)} + \frac{1}{2}T^{(1)}) \longrightarrow (Y^{\text{st}}, (\frac{1}{2} + \epsilon_1)\Sigma^{\text{st}} + \frac{1}{2}T^{\text{st}})$$

the log canonical model morphism, we see that $\pi^{\text{st}}: Y^{\text{st}} \rightarrow C^{\text{st}}$ has 1-dimensional fibers and is a \mathbb{P}^1 -fibration in codimension 1. Proceeding as in Corollary 5.7, and using [DLI24, Lemma 2.1], we obtain that C^{st} is the coarse moduli space of a family of twisted curves C^{st} admitting a \mathbb{P}^1 -fibration $\mathcal{Y}^{\text{st}} \rightarrow C^{\text{st}}$ whose coarse space is $Y^{\text{st}} \rightarrow C^{\text{st}}$:

$$\begin{array}{ccc} \mathcal{Y}^{\text{st}} & \longrightarrow & Y^{\text{st}} \\ \downarrow & & \downarrow \\ C^{\text{st}} & \longrightarrow & C^{\text{st}}. \end{array}$$

Taking the double cover of Y^{st} as in Corollary 3.7, branched along $D^{\text{st}} = \Sigma^{\text{st}} + T^{\text{st}}$, yields a family of elliptic surfaces

$$(X^{\text{st}}, 2\epsilon_1 S^{\text{st}}) \longrightarrow C^{\text{st}},$$

where S^{st} is the reduced preimage of Σ^{st} . Finally, the fibers of $Y \rightarrow C^{\text{st}}$ over the smooth locus of $C^{\text{st}} \rightarrow \text{Spec } A$ are of the Weierstrass form $y^2z = x^3 + Axz^2 + Bz^3$. It follows that

all geometric fibers are of this form, again using [DLI24, Lemma 2.1] and the fact that the good moduli space of $[\mathbb{A}^2/\mathbb{G}_m]$, the moduli of planar Weierstrass models, is a point; see [BFH⁺24, Remark 7.10] for a similar argument.

This completes the proof of Theorem 5.1.

5.4. Applications of Theorem 5.1. The first application of Theorem 5.1 is that it allows us to prove the following statement *without running any explicit MMP*. Indeed, the result is a direct consequence of Theorem 5.1.

Let $\epsilon > 0$ be a rational number, and $\overline{\mathcal{W}}^{\text{KSBA}}(\epsilon)$ denote the irreducible component of the KSBA moduli stack whose general point parametrizes a pair $(X, \epsilon S)$ admitting a Weierstrass fibration $(X, S) \rightarrow C$ over a smooth curve C .

Theorem 5.9 (cf. [Inc20, Theorem 1.2]). *Let $0 < \epsilon \ll 1$ be a (rational) number. Then any pair $(X, \epsilon S)$ parametrized by $\overline{\mathcal{W}}^{\text{KSBA}}(\epsilon)$ admits a (not necessarily flat) fibration*

$$f : (X, \epsilon S) \rightarrow C$$

to a nodal curve C with purely 1-dimensional fibers such that

$$X|_{C^{\text{sm}}} \rightarrow C^{\text{sm}}$$

is flat with integral fibers. Moreover, for any irreducible component G of C with normalization G^ν , the surface $X|_{G^\nu}$ is one of the following:

- (1) *a normal elliptic surface $X|_{G^\nu} \rightarrow G^\nu$, or*
- (2) *a non-normal fibered surface $X|_{G^\nu} \rightarrow G^\nu$, whose general fiber is an integral nodal curve or arithmetic genus 1.*

In particular, the surface pairs appearing on the boundary of the KSBA moduli space parametrizing pairs

$$(Y_\eta, (\tfrac{1}{2} + \epsilon_1) \Sigma_\eta + \tfrac{1}{2} T_\eta + (\tfrac{1}{12} + \epsilon_2) F_\eta),$$

which arise as degenerations of the 2:1 quotients of elliptic surfaces with j -map of degree 12, have KSBA-stable limits fibered over a family of nodal curves. The base family is a stable generalized pair $(C, \mathbf{M} + \tfrac{1}{12} \mathbf{p})$, where \mathbf{p} is a divisor of degree 12 and $\deg \mathbf{M} = 1$.

The special fiber consists of a nodal curve

$$C_0 = C_{0,1} \cup C_{0,2}$$

with marked points p_1, \dots, p_{12} and a non-negative \mathbb{Q} -divisor \mathbf{M}_0 such that

$$K_{C_0} + \tfrac{1}{12}(p_1 + \dots + p_{12}) + \mathbf{M}_0$$

is ample. A direct combinatorial check shows that C_0 has at most two irreducible components; if it has two, then each component carries a j -map of degree 6 and exactly six of the marked points.

Now increase the coefficient of the section from $\tfrac{1}{2} + \epsilon_1$ to $\tfrac{1}{2} + 12\epsilon_2$. For a general fiber, the pair

$$(Y_\eta, (\tfrac{1}{2} + \epsilon_1) \Sigma_\eta + \tfrac{1}{2} T_\eta + (\tfrac{1}{12} + \epsilon_2) F_\eta)$$

is no longer KSBA-stable. Nevertheless, for any KSBA-stable limit

$$(Y, (\tfrac{1}{2} + \epsilon_1) \Sigma + \tfrac{1}{2} T + (\tfrac{1}{12} + \epsilon_2) F)$$

the log canonical divisor is nef. Indeed, it suffices to check that for every irreducible component Γ of the special fiber of the section Σ ,

$$(K_Y + (\tfrac{1}{2} + \epsilon_1) \Sigma + \tfrac{1}{2} T + (\tfrac{1}{12} + \epsilon_2) F) \cdot \Gamma = 0.$$

This follows from the equalities

$$(K_Y + \tfrac{1}{2}\Sigma + \tfrac{1}{2}T + \tfrac{1}{12}F) \cdot \Gamma = 0, \quad T \cdot \Gamma = 0, \quad (K_Y + \Sigma) \cdot \Gamma = -1,$$

where the first equality comes from the canonical bundle formula, and the latter two follow from the fact that (Y, Σ) is slc and smooth along Σ away from the nodes of the central fiber of $Y \rightarrow \text{Spec } A$.

Thus, taking the canonical model under the specialization $\epsilon_1 = 12\epsilon_2$ contracts the section, and yields the following. In particular, the wall-crossing morphism of [ABIP23, MZ23] give a map $\overline{\mathcal{W}}_{\text{rat}}^{\text{KSBA}}(\epsilon_1, \frac{1}{12} + \epsilon_2) \rightarrow \overline{\mathcal{M}}^{\text{KSBA}}$ to the KSBA moduli space with coefficient $\frac{1}{12} + \epsilon_2$ and given volume, which contracts the divisor with coefficient ϵ_1 .

Theorem 5.10 (rational elliptic surfaces, cf. [AB22, Theorem 1.1 (a), (b)]). *Let $0 < \epsilon_1 \ll \epsilon_2 \ll 1$ be two (rational) numbers. Then any pair $(X, \epsilon_1 S + (\frac{1}{12} + \epsilon_2)F)$ parametrized by $\overline{\mathcal{W}}_{\text{rat}}^{\text{KSBA}}(\epsilon_1, \frac{1}{12} + \epsilon_2)$ admits a (not necessarily flat) fibration*

$$f : (X, \epsilon_1 S + (\tfrac{1}{12} + \epsilon_2)F) \longrightarrow C$$

to a nodal curve C with purely 1-dimensional fibers satisfying that

- S is a section of $X \rightarrow C$;
- F consists of some f -fibers over C^{sm} ; and
- $X|_{C^{\text{sm}}} \rightarrow C^{\text{sm}}$ is flat with integral fibers.

Any irreducible component G of C is isomorphic to \mathbb{P}^1 . If C is irreducible, then $X|_G$ is

- (1) either a normal elliptic surface $X|_G \rightarrow G$ with ADE singularities; or
- (2) a non-normal fibered surface $X|_G \rightarrow G$, whose general fiber is an integral nodal curve or arithmetic genus 1.

If C is reducible, then C has two components G_1, G_2 , and $X|_{G_i} \rightarrow G_i$ is a normal elliptic surface for $i = 1, 2$. Taking the canonical model for $\epsilon_1 = 12\epsilon_2$ contracts the section, and gives the KSBA-stable limits of smooth degree 9 Del Pezzo surfaces with marking being the images of the 12 singular fibers.

Similarly, the following follows from Theorem 5.1.

Theorem 5.11 (elliptic K3 surfaces, cf. [Bru15, Theorem 9.1.4]). *Let $0 < \epsilon_1 \ll \epsilon_2 \ll 1$ be two (rational) numbers. Then any pair $(X, \epsilon_1 S + \epsilon_2 F)$ parametrized by $\overline{\mathcal{W}}_{\text{K3}}^{\text{KSBA}}(\epsilon_1, \epsilon_2)$ admits a (not necessarily flat) fibration*

$$f : (X, \epsilon_1 S + \epsilon_2 F) \longrightarrow C$$

to a nodal curve C with purely 1-dimensional fibers satisfying that

- S is a section of $X \rightarrow C$;
- F consists of some f -fibers over C^{sm} ; and
- $X|_{C^{\text{sm}}} \rightarrow C^{\text{sm}}$ is flat with integral fibers.

Any irreducible component C is isomorphic to \mathbb{P}^1 , and for any irreducible component G of C , the surface $X|_G$ is one of the following:

- (1) a normal elliptic surface $X|_G \rightarrow G \simeq \mathbb{P}^1$, or
- (2) a non-normal fibered surface $X|_G \rightarrow G \simeq \mathbb{P}^1$, whose general fiber is an integral nodal curve or arithmetic genus 1.

REFERENCES

- [AB19] Kenneth Ascher and Dori Bejleri. Moduli of fibered surface pairs from twisted stable maps. *Math. Ann.*, 374(1-2):1007–1032, 2019.

- [AB21] Kenneth Ascher and Dori Bejleri. Moduli of weighted stable elliptic surfaces and invariance of log plurigenera. *Proc. Lond. Math. Soc. (3)*, 122(5):617–677, 2021. With an appendix by Giovanni Inchiostro.
- [AB22] Kenneth Ascher and Dori Bejleri. Compact moduli of degree one del Pezzo surfaces. *Math. Ann.*, 384(1-2):881–911, 2022.
- [AB23] Kenneth Ascher and Dori Bejleri. Compact moduli of elliptic K3 surfaces. *Geom. Topol.*, 27(5):1891–1946, 2023.
- [ABE22] Valery Alexeev, Adrian Brunyate, and Philip Engel. Compactifications of moduli of elliptic K3 surfaces: stable pair and toroidal. *Geom. Topol.*, 26(8):3525–3588, 2022.
- [ABIP23] Kenneth Ascher, Dori Bejleri, Giovanni Inchiostro, and Zsolt Patakfalvi. Wall crossing for moduli of stable log pairs. *Ann. of Math. (2)*, 198(2):825–866, 2023.
- [AE25] Valery Alexeev and Philip Engel. On lattice-polarized k3 surfaces. *arXiv preprint arXiv:2505.22557*, 2025.
- [AP12] Valery Alexeev and Rita Pardini. Non-normal abelian covers. *Compositio Mathematica*, 148(4):1051–1084, 2012.
- [AV02] Dan Abramovich and Angelo Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.*, 15(1):27–75, 2002.
- [BFH⁺24] Dori Bejleri, Josiah Foster, Andres Fernandez Herrero, Giovanni Inchiostro, Svetlana Makarova, and Junyan Zhao. Moduli of elliptic surfaces of Kodaira dimension one fibered over rational curves. *arXiv preprint arXiv:2407.05539*, 2024.
- [Bru15] Adrian Brunyate. A modular compactification of the space of elliptic K3 surfaces. *PhD Thesis*, 2015.
- [DLI24] Andrea Di Lorenzo and Giovanni Inchiostro. Stable maps to quotient stacks with a properly stable point. *arXiv preprint arXiv:2411.16141*, 2024.
- [Dol96] I. V. Dolgachev. Mirror symmetry for lattice polarized K3 surfaces. *J. Math. Sci.*, 81(3):2599–2630, 1996. Algebraic geometry, 4.
- [Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158. Cambridge University Press, 2016.
- [Inc20] Giovanni Inchiostro. Moduli of Weierstrass fibrations with marked section. *Adv. Math.*, 375:107374, 57, 2020.
- [ISZ25] Giovanni Inchiostro, Roberto Svaldi, and Junyan Zhao. Moduli of surfaces fibered in log Calabi-Yau pairs. *arXiv preprint arXiv:2509.14145*, 2025.
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134. Cambridge university press, 1998.
- [Kol23] János Kollár. *Families of varieties of general type*, volume 231 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2023. With the collaboration of Klaus Altmann and Sándor J. Kovács.
- [LN02] Gabriele La Nave. Explicit stable models of elliptic surfaces with sections. *arXiv preprint math/0205035*, 2002.
- [May72] Alan L Mayer. Families of K3 surfaces. *Nagoya Mathematical Journal*, 48:1–17, 1972.
- [Mir81] Rick Miranda. The moduli of Weierstrass fibrations over \mathbb{P}^1 . *Mathematische Annalen*, 255(3):379–394, 1981.
- [MZ23] Fanjun Meng and Ziquan Zhuang. MMP for locally stable families and wall crossing for moduli of stable pairs. *arXiv preprint arXiv:2311.01319*, 2023.
- [Rei76] Miles Reid. Hyperelliptic linear systems on a K3 surface. *J. London Math. Soc. (2)*, 13(3):427–437, 1976.
- [SD74] B. Saint-Donat. Projective models of $K - 3$ surfaces. *Amer. J. Math.*, 96:602–639, 1974.
- [Sta25] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>, 2025.

C-138 PADEFORD, BOX 354350, SEATTLE, WA 98195, USA

Email address: ginchios@uw.edu

WILLIAM E. KIRWAN HALL, 4176 CAMPUS DR, COLLEGE PARK, MD 20742, USA

Email address: jzhao81@umd.edu