Lecture 2 - MATH 327 03/29/23

TODAY - review of sets
review of functions

Before we dive into $\mathbb{R}$, let's review some concept you have learned in MATH 300

SETS: ↖ check out 1.2 in Abbott!

- a set is a collection of objects; we call them elements.
- for us, sets will mostly be sets of real numbers (i.e. subsets of $\mathbb{R}$)
- Notation: $A \subseteq B$ "A is a subset of B"
  $x \in A$ "x belongs to A"
  $x \notin A$ "x does NOT belong to A"

Remark: $A \subset B$ sometimes means properly (a.k.a. strictly) contained (that is, $A \subseteq B$ and $A \neq B$).
For us $A \subset B \equiv A \subseteq B$. Use $A \subsetneq B$ if you need.

Thinking of proofs: how do we prove that $A = B$?

• we prove $A \subseteq B$ and $B \subseteq A$.

how do we prove $A \subseteq B$?

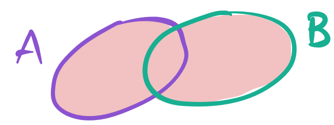
we prove that $\forall x \in A$, $x \in B$ too.

* UNION and INTERSECTION

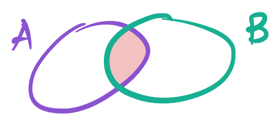
if $B \subseteq A$

$A \cup B = A$

$A \cup B$ is the set of elements that are either in A OR B



$A \cap B$ is the set of elements that are both in A and B



if $B \subseteq A$

$A \cap B = B$

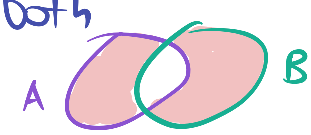
* OTHER SET OPERATIONS

$A \setminus B$ (set difference): elements that are in A but NOT in B.

usually $B \subseteq A$ in this case, but if not reads as $A \setminus (A \cap B)$

$A \Delta B$ (symmetric difference): elements that are in A or B but NOT in both

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$



* COMPLEMENT (Not compliment, although all sets are beautiful).

only makes sense if we are in some 'ambient' set.

If $B \subseteq A$, the complement of B in A is

$B^c = A \setminus B$ review interval notation in $\mathbb{R}$!

Example. In $\mathbb{R}$, $([0,1])^c = (-\infty, 0) \cup (1, +\infty)$

* De Morgan's laws: $(A \cup B)^c = A^c \cap B^c$

draw a picture!!! $(A \cap B)^c = A^c \cup B^c$

* CARTESIAN PRODUCT

$A \times B = \{ (a,b) \mid a \in A, b \in B \}$

ordered pairs

$(a,b) \neq (b,a)$

in curly brackets order does not matter

$\{1,2,3\} = \{2,3,1\}$

Example: $A = \{1,2,3\}$

$B = \{\circ, \bullet\}$

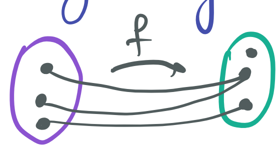
$A \times B = \{ (1,\circ), (1,\bullet), (2,\circ), (2,\bullet), (3,\circ), (3,\bullet) \}$

(this is to show that A and B need not be related)

Example $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

FUNCTIONS:

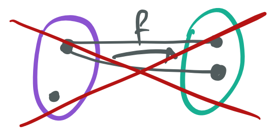
<u>Def</u>. Given sets A and B, a function from A to B is a rule that takes each element $x \in A$ and associates to it a single $y \in B$.



A is the domain of f

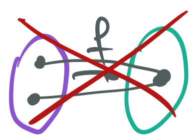
the range is

$$\{ y \in B \mid \exists x \in A \text{ s.t. } f(x) = y \}$$



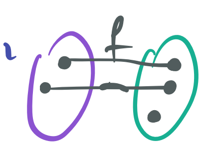
some people use "one-to-one" but that's ambiguous so let's NOT.

* INJECTIVE: $f: A \to B$ is injective if



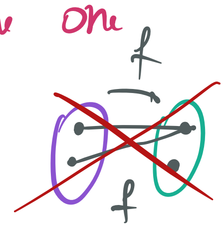
$\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

"implies"



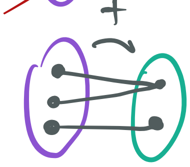
$\forall x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

(this is the contrapositive of the one above!)



* SURJECTIVE: $f: A \to B$ is surjective if

$$\forall y \in B \ \exists x \in A \text{ s.t. } f(x) = y.$$



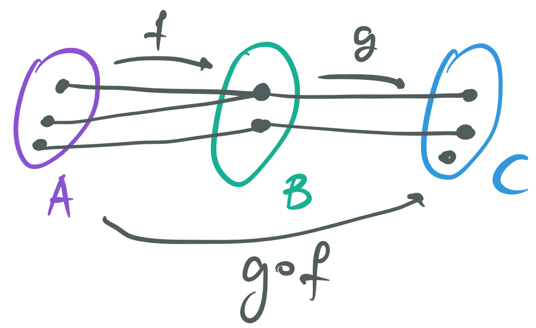
(i.e. if B = range)

* BIJECTIVE = INJECTIVE + SURJECTIVE

* compostion $f: A \to B$, $g: B \to C$

$$g \circ f : A \to C$$

$$g \circ f(x) = g(f(x))$$



LECTURE 3 - MATH 327 03/31/23

Today: logic review
equiv. relations review
$\mathbb{N}, \mathbb{Z}$

Logic and proof

Symbols: $\forall$ - for all, $\exists$ exists, $\Rightarrow$ implies
$\Leftrightarrow$ iff if and only if

* Contrapositive (sometimes it may feel easier to prove this)
"if A then B" is the same as "if not B then not A"

(here A is sufficient for B and B is necessary for A)

Iff = necessary and sufficient

* negation "if A then B" is the negation of
"A and not B [are true]"

Example: negation of "$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\delta < \varepsilon$"
is "$\exists \varepsilon > 0$ $\forall \delta > 0$ $\delta \geq \varepsilon$" (which one is true?)

• Proof by contradiction: assume the claim you want to prove is false and show that this implies something false

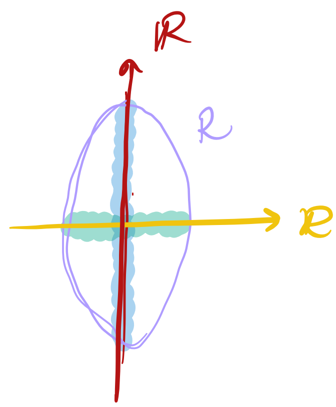
EQUIVALENCE RELATIONS:

Def A relation R from set A to set B is a subset $R \subseteq A \times B$.

Example

$$R = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid 4x^2 + y^2 = 16 \}$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$



domain $A = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} \text{ s.t. } (x,y) \in R \}$

range $B = \{ y \in \mathbb{R} \mid \exists x \in \mathbb{R} \text{ s.t. } (x,y) \in R \}$

$A = [-2, 2] \subseteq \mathbb{R}$

$B = [-4, 4] \subseteq \mathbb{R}$

Remark role of $\mathbb{R}$ and $\mathbb{R}$ is different

Example

$$D = \{ (m,n) \in \mathbb{Z} \mid m \text{ divides } n \}$$

$m \mid n$ iff $m, n \in D$.

- domain? $\mathbb{Z} \setminus \{0\}$
- range? $\mathbb{Z}$

- reflexive? YES $m|m$ ✓
- symmetric? NO $m|n \not\Leftrightarrow n|m$ (ex. $2|4$ but $4\nmid 2$)
- transitive? YES $m|n$, $n|p \Rightarrow m|p$

(Pf: $n=mk$ for some k, $p=\ell n$ for some ℓ $\Rightarrow p=\ell n=(\ell k)\cdot m$ $\rightarrow m|p$)

Example (ordered sets)

An order on a set S is a relation $<$ on S

s.t. (a) $x\in S$, $y\in S$ then one and only one of the statements holds:

$$x<y, \quad x=y, \quad y<x$$

(b) if $x<y$ and $y<z$ then $x<z$

$$\forall x,y,z\in S$$

Remark this will come up later when talking about fields and $\mathbb{Q}$ and $\mathbb{R}$

Definition (equivalence relation)

Binary relation $\sim$ on a set A that is:

(i) reflexive $\quad a \sim a \quad \forall a \in A$

(ii) symmetric $\quad a \sim b \iff b \sim a \quad \forall a, b \in A$

(iii) transitive $\quad a \sim b,\ b \sim c \Rightarrow a \sim c \quad \forall a, b, c \in A$

Example on $\mathbb{N}$, (i.e. from $\mathbb{N}$ to $\mathbb{N}$)

aRb iff $a = b$.

- $a = a$ ✓
- $a = b \iff b = a$ ✓
- $a = b,\ b, c \Rightarrow a = c$ ✓

NON-examples

- a relation which is not reflexive:

  on $\mathbb{N}$ $\quad aRb$ iff $a < b$

  $a < a$ ✗

- a relation which is not symmetric $\quad$ this one

  $2 < 3$ ✗

  $3 \not< 2$

- a relation which is not transitive

  on $\mathbb{N}$ $\quad aRb$ iff $b = a+1$ (b is the successor of a)

  if $b = a+1$ and $c = b+1 \Rightarrow c = b+1 = a+2$ ✗

  $\neq a+1$

MATH 327 - Lecture 4

02/03/2023

Equivalence classes

Def let $\sim$ be an equivalence relation on $A \neq \emptyset$. $\forall a \in A$ the equivalence class of a, denoted by $[a]$, is the subset of all elements in relation to A.:

$$[a] = \{ b \in A \mid b \sim a \}$$

Thm $A \neq \emptyset$ and $\sim$ relation on A.

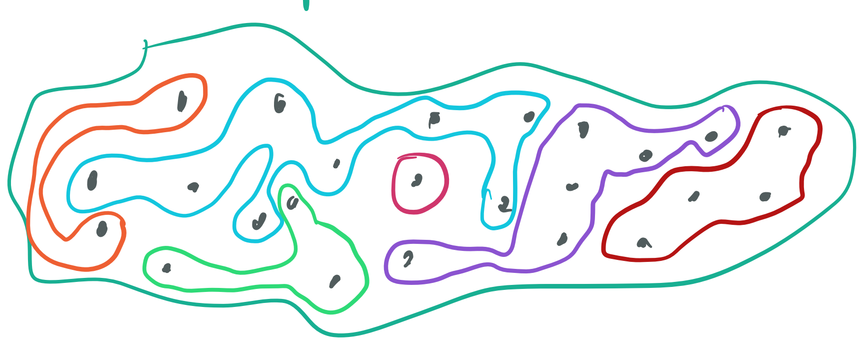
(a) $\forall a \in A \quad a \in [a]$

(b) $\forall a, b \in A, \quad a \sim b \iff [a] = [b]$

(c) $\forall a, b \in A$ either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

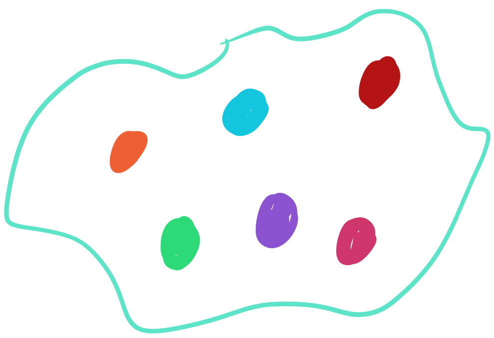
The equiv. classes $[a]$ form a partition

(b/c every $a \in [a]$ so they cover everything but they don't overlap)



example of partition

why do we care that they make a partition?
Because it gives me a natural way to define another set: the set of equivalence classes, denoted as $A/\sim$



Proof of thm

(a) Let $a \in A$. By def. of equivalence relation,
$a \in [a]$ (reflexive)

(b) $\boxed{\Rightarrow}$ if $a \sim b \Rightarrow [a] = [b]$

WTS 1: $a \sim b \Rightarrow [a] \subseteq [b]$

Let $x \in [a]$. By def. of equiv. class,
$x \sim a$. By assumption $a \sim b$.
Hence, by def. of equiv. rel (transitive),
$x \sim b$. By def of equiv. class,
$x \in [b]$.

But x was arbitrary, so $[a] \subseteq [b]$

WTS 2 : $a \sim b \Rightarrow [b] \subseteq [a]$

Let $x \in [b]$. By def. of equiv. class,
$x \sim b$. By assumption $a \sim b$. ($b \sim a$ by symmetry)
Hence, by def. of equiv. rel (transitive),
$x \sim a$ By def of equiv. class,
$x \in [a]$.
But x was arbitrary, so $[b] \subseteq [a]$

$\boxed{\Leftarrow}$ if $[a] = [b] \Rightarrow a \sim b$.

$a \in [a] = [b] \Rightarrow$ (def of equiv. class)
by (I) $\quad a \sim b$

(c) if $[a] \cap [b] = \emptyset$, there's nothing to prove

if $[a] \cap [b] \neq \emptyset$, let $x \in [a] \cap [b]$.
Then by def of equiv class, $x \sim a$
and $x \sim b$.
By transitivity (and symmetry) $a \sim b$
and so by (II) $\Rightarrow [a] = [b]$ $\square$

Def. Given a set $A \neq \emptyset$, a partition of A is a collection of subsets $\{A_i\}_{i \in I}$ s.t.

(i) $A_i \neq \emptyset \quad \forall i \in I$

index set.
usually we have $\mathbb{N}$
but could be
anything
(even uncountable)

(ii) (pairwise disjoint)

$$A_i \cap A_j = \emptyset \quad \forall i \neq j$$

(iii) $\bigcup_{i \in I} A_i = A$.

Remark: the set of equivalence classes form a partition of the set.

(ex: prove it)

MATH 327 - Lecture 5

Number systems

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} (\subseteq \mathbb{C})$

Natural numbers $\mathbb{N}$ — positive integers

Peano axioms

1. $1 \in \mathbb{N}$
2. if $n \in \mathbb{N}$ the $n+1 \in \mathbb{N}$ (successor)
3. 1 is not the successor of any $n \in \mathbb{N}$
4. if n and m have the same successor, then $n = m$ ← this allow m to cancel out $n \cancel{+1} = m \cancel{+1}$

5. if $S \subseteq \mathbb{N}$, $1 \in S$ and $\forall\, n \in S \Rightarrow n+1 \in S$ $\Rightarrow S = \mathbb{N}$ ← why induction works

Integers $\mathbb{Z}$

$(\mathbb{N}, +)$ - is missing a neutral element and inverses of its elements neutral element : 0

inverses $\quad n+(-n)=0$ and $(-n)+n=0$

$\rightsquigarrow \mathbb{Z}$

Rational numbers $\mathbb{Q}$

multiplicative inverses of elements of $\mathbb{Z}$ ✓

but it has a better structure: $\mathbb{Q}$ is a field

Field axioms

Let F be a set and $+, \cdot$, two operations on F such that:

(A1) if $x \in F$ and $y \in F \Rightarrow x+y \in F$

(A2) addition is commutative: $x+y=y+x$ $\quad \forall x,y \in F$

(A3) addition is associative $x+(y+z)=(x+y)+z$ $\quad \forall x,y,z \in F$

(A4) there's an element in F, $0 \in F$ s.t. $0+x=x$ $\forall x \in F$

(A5) for every $x \in F$ there is an element $(-x) \in F$ s.t. $x+(-x)=0$.

(M1) if $x \in F$ and $y \in F \Rightarrow x \cdot y \in F$

(M2) multiplication is commutative: $x \cdot y = y \cdot x \quad \forall x, y \in F$

(M3) multiplication is associative $x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall x, y, z \in F$

(M4) there's an element in F, $1 \in F$ s.t. $1 \cdot x = x \quad \forall x \in F$

(M5) for every $x \in F \setminus \{0\}$ there is an element $\frac{1}{x} \in F$ s.t. $x \cdot \frac{1}{x} = 1$.

(D) distributive law:

$$x(y+z) = xy + xz \qquad \forall x, y, z \in F.$$

Examples
- $\mathbb{Q}$ is a field (see HW1)
- $\mathbb{R}$, $\mathbb{C}$
- $\mathbb{Z}_p$, p prime

when you solve the problem remember that, in spite of the notation, that set is $\mathbb{Q}$!

NON-examples
- $\mathbb{N}$ (many reasons: no 0, no $-x$, no $1/x$)
- $\mathbb{Z}$ (no $1/x$)
- $\mathbb{Z}_m$ m not prime (no $1/x$ for everybody)

Def an ordered field is a field which is also an ordered set, s.t.

(a) if $y < z$ then $x+y < x+z$ $\forall x, y, z \in F$

(b) if $x > 0, y > 0$ then $xy > 0$ $\forall x, y \in F$.

Next:

- upper and lower bounds
- least upper bounds, greatest lower bound
- Archimedean property
- Completeness axioms

MATH 327 - LECTURE 6 04/07/23

Axiom of Completeness

We won't construct real numbers (see Pb 6 in HW2) but let's agree of what $\mathbb{R}$ is for us.
$(\mathbb{R}, +, \cdot)$ is an ordered field, $\mathbb{Q} \subseteq \mathbb{R}$ subfield
What makes $\mathbb{R}$ "better" than $\mathbb{Q}$?
It has no gaps. What does that mean? In order to make this more mathematically precise, let's start with a few definitions

Def. We say that a set $A \subseteq \mathbb{R}$ is bounded above if there exists $b \in \mathbb{R}$ such that $a \leq b$ $\forall a \in A$.
The number b is called an upper bound.
We say that a set is bounded below if $\exists \ell \in \mathbb{R}$ s.t. $a \geq \ell$ $\forall a \in A$. The number ℓ is called a lower bound.

Def. $s \in \mathbb{R}$ is the least upper bound for $A \subseteq \mathbb{R}$ if

(i) s is an upper bound;

(ii) if b is any upper bound for A, $b \geq s$.

supremum = least upper bound.
We write $s = \sup A$.

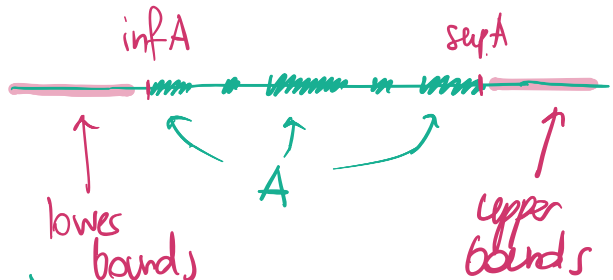
Def. $u \in \mathbb{R}$ is the greatest lower bound for $A \subseteq \mathbb{R}$ if
(i) u is a lower bound,
(ii) if l is any lower bound for A, $l \leq u$

infimum = greatest lower bound
We write $u = \inf A$.

Remark. sup and inf are unique. In fact, by (ii) if s_1 and s_2 are both least upper bounds for A then $s_1 \leq s_2$ (b/c $s_1 = \sup$) and $s_2 \leq s_1$ ($s_2 = \sup$)
$\Rightarrow s_1 = s_2$.

Examples

- $A = (-\infty, 3)$
- $A = [0,1]$
- $A = [0,1)$
- $A = [0,1) \cup [17, 32)$



The examples show that the supremum and infimum of a set may or may not belong to the set itself.

Def. We say that $M \in \mathbb{R}$ is a maximum of the set A if $M \in A$ and $M \geq a \ \forall a \in A$
We say that $m \in \mathbb{R}$ is a minimum if $m \in A$ and $m \leq a \ \forall a \in A$.

Remark: Both $(0,1)$ and $[0,1]$ are bounded above and below, $(0,1)$ doesn't have any min or max while $[0,1]$ does. The axiom of completeness states that $(0,1)$ is guaranteed to have inf and sup.

Axiom of Completeness:

Every nonempty set of real numbers that is bounded above has a least upper bound.

(and what about greatest lower bounds? see HW2.)

This is not true in $\mathbb{Q}$ (can you think of an example?)

A super important thing in mathematics is to prove characterizations for concepts we define.
By doing this we gain different perspectives, and we get to choose the most convenient one depending on the situation we're in.

Proposition

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$.
Then $s = \sup A$ if and only if $\forall \varepsilon > 0 \ \exists a \in A$ such that $s - \varepsilon < a$

(an upper bound is the least upper bound iff any number smaller than it is not an upper bound)

Pf $\boxed{\Rightarrow}$ Assume that $s = \sup A$. Let $\varepsilon > 0$.

By (iii) (or more precisely its contrapositive) if $a < s$, then a is not an upper bound.

then $s - \varepsilon < s$ is not an upper bound, which by definition means that $\exists\, a \in A$ s.t. $s - \varepsilon < a$.

$\boxed{\Leftarrow}$ Now assume that $s \in \mathbb{R}$ is an upper bound, and that $\forall \varepsilon > 0\ \exists\, a \in A$ s.t. $s - \varepsilon < a$.

We need to check that such s satisfies (ii).

Observe that if $b < s$ then b is not an upper bound. To see this, since $s - b > 0$ we can choose $\varepsilon = s - b$ and get that

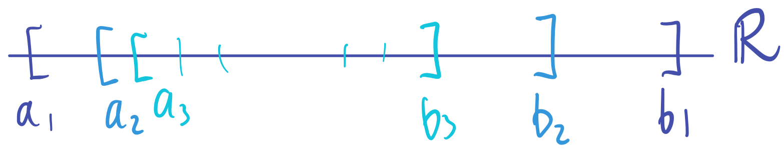
$\exists\, a \in A$ st $b < a$.

contrapositive: $\underbrace{\text{if } b \text{ is an upper bound, then } s \leq b}_{\text{this is exactly (ii)}}$. $\square$

MATH327 - Lecture 7 04/10/23

Thm (Nested intervals property)
For every $n \in \mathbb{N}$ let $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \le x \le b_n\}$ be a closed interval, and assume $I_{n+1} \subset I_n$.
Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Pf



We want to use the Axiom of Completeness to show that $\exists x \in \bigcap_{n=1}^{\infty} I_n$. (that is, $x \in I_n \ \forall n$).
Let $A = \{a_n \mid n \in \mathbb{N}\}$. (the left endpoints)
and let $x = \sup A$.
Because the intervals are nested, all b_n's are upper bounds for A. Then b/c $x = \sup A$ we have that
$\forall n \quad x \geq a_n$ and $x \leq b_n \Rightarrow x \in I_n \ \forall n$

$\Leftrightarrow x \in \bigcap_{n=1}^{\infty} I_n$ $\square$

Let's turn to the relationship between $\mathbb{N}$ and $\mathbb{R}$

Theorem (Archimedean Property)

$\mathbb{N}$ is NOT bdd above

(i) given any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $n > x$

(ii) given any $y \in \mathbb{R}$, $y > 0$ $\exists\, n$ s.t. $0 < 1/n < y$.

Remark: Also $\mathbb{Q}$ (not complete) has this property but it's a pain in the butt to prove w/o Axiom of Completeness.

Proof

$\boxed{(i)}$ for the sake of contradiction assume that $\mathbb{N}$ is bounded above then by the Axiom of Completeness $\mathbb{N}$ has a least upper bound, $\alpha = \sup \mathbb{N}$. then $\alpha - 1$ is NOT an upper bound and so $\exists\, n \in \mathbb{N}$ s.t.

$$\alpha - 1 < n$$
$$\alpha < n + 1$$

But $n \in \mathbb{N}$ and so $n+1 \in \mathbb{N}$ and we have a contradiction, hence $\mathbb{N}$ has to be unbounded.

$\boxed{(ii)}$ Let $x = 1/y$ and use (i) $\square$

Remark. You may have seen this as: "If $a, b > 0$ then $\exists\, n \in \mathbb{N}$ s.t. $na > b$". The proof is very similar (using the set $S = \mathbb{N}a = \{na \mid n \in \mathbb{N}\}$.

Thm (Archimedean Property)

If $a, b > 0$ then $\exists\, n \in \mathbb{N}$ s.t. $na > b$.

Proof

Assume not, then $\exists\, a, b$ s.t. $na \le b \;\; \forall n \in \mathbb{N}$,
that is b is an upper bound for the set
$S = \{ na \mid n \in \mathbb{N} \}$ which means S is bounded
above and so by the axiom of completeness,
$\exists\, s_0 = \sup S$. $s_0 - a < s_0$ and so it is not an
upper bound $\Rightarrow$ $\exists\, n$ st

$$s_0 - a < na$$
$$s_0 < na + a$$
$$s_0 < (n+1)a$$

But $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N} \Rightarrow (n+1)a \in \mathbb{N}$. Contradiction $\square$

Cor

(i) given $x \in \mathbb{R}$, $x > 0$, $\exists\, n \in \mathbb{N}$ s.t. $n > x$

(ii) given any $y \in \mathbb{R}$, $y > 0$ $\exists\, n$ s.t. $0 < 1/n < y$.

Pf.
$a = 1$ then $\Rightarrow$ (i)
$b = 1$ then $\Rightarrow$ (ii)

$\square$

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

For every $a, b \in \mathbb{R}$, $a < b$ $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

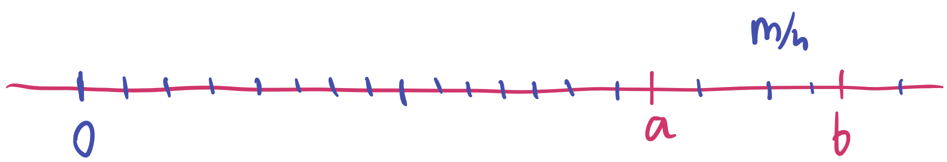
Proof

If $a < 0 < b$ then $r = 0$.

Assume now $0 < a < b$ (the other case follows from this one, why?)

We need to show that there exist $m, n \in \mathbb{Z}$, $n > 0$ s.t. $a < \frac{m}{n} < b$.

$na < m < nb$ m steps of size $\frac{1}{n}$



By the Archimedean property, $\exists n$ s.t.

$\frac{1}{n} < b - a$. (*) $\underline{a < b - 1/n}$

Now I need to choose m. I want m to be bigger than na, but not too much:

choose m s.t. $m - 1 \leq na < m$ ✓
(**)

(*) + (**): $m \overset{(**)}{\leq} na + 1$

$\overset{(*)}{<} n(b - 1/n) + 1$

$= nb \cancel{-1} \cancel{+1}$

□

MATH 327 - Lecture 8 04/12/2023

Theorem

For $c>0$ and $n\in\mathbb{N}$, $\exists!\, x\in\mathbb{R}$ s.t. $x^n=c$

Pf. (buckle up!)

Consider the set $E=\{t\in\mathbb{R} \mid t^n<c\}$.

$E\neq\emptyset$: $t=\frac{c}{1+c} \Rightarrow 0<t<1 \Rightarrow t^n<t<c \Rightarrow t\in E$.

E bdd above: if $t>1+c$ then $t^n>t>c$ so $t\notin E$

$\Rightarrow$ $1+c$ is an upper bound

$\Rightarrow$ (Axiom of completeness) $\exists\ x=\sup E$.

WTS: $x^n=c$.

We will show that both $x^n<c$ and $x^n>c$ are impossible

First observe that:

$$b^n-a^n=(b-a)(\underbrace{b^{n-1}+b^{n-2}a+\ldots+a^{n-1}}_{n \text{ terms}})$$

if $a<b$ $\Rightarrow$

$$b^{n-1-j}a^j<b^{n-1} \quad \forall j=0,\ldots,n-1$$

that's a way to write all terms

$$\Rightarrow \quad b^n-a^n<(b-a)[\underbrace{b^{n-1}+b^{n-1}+\ldots+b^{n-1}}_{n \text{ times}}]$$

$$\Rightarrow \boxed{b^n - a^n < n(b-a)\,b^{n-1}} \quad (*)$$

- assume $x^n < c$.

We want to apply $(*)$ to

$a = x$

$b = x+h$, where $h \in (0,1)$ is such that

$$h < \frac{c - x^n}{n(x+1)^{n-1}} \xleftarrow{} c - x^n > 0$$

$$\begin{aligned}
(x+h)^n - x^n &< n(x+h-x)(x+h)^{n-1} \\
&= n\,h\,(x+h)^{n-1} \\
h<1 \quad &< n\,h\,(x+1)^{n-1} \\
&< \cancel{n}\cancel{(x+1)^{n-1}} \cdot \frac{(c - x^n)}{\cancel{n}\cancel{(x+1)^{n-1}}} \\
&= c - x^n
\end{aligned}$$

$\Rightarrow (x+h)^n < c \Rightarrow x+h \in E$, which is a contradiction to $x = \sup E$.

- assume $x^n > c$

We now want to use $(*)$ with

$a = y - k$

$b = y$, where $k = \dfrac{x^n - c}{n x^{n-1}}$.

Clearly $k > 0$ and also

$$k = \frac{x^n - c}{n x^{n-1}} < \frac{x^n}{n x^{n-1}} = \frac{x}{n} < x.$$

If $t \geq x - k$, then

$$\begin{aligned} x^n - t^n &\leq x^n - (x-k)^n. \\ &< k n x^{n-1} \\ &= \frac{x^n - c}{n \cancel{x^{n-1}}} \cdot n \cancel{x^{n-1}} \\ &= x^n - c \end{aligned}$$

$\Rightarrow$ $t^n > x$, and $t \notin E$.

This means that $x-k$ is an upper bound for E, but that's a contradiction b/c $x - k < x$ and x = least upper bound $\square$

Some (OPTIONAL) stuff on countable and uncountable sets is on Canvas in Lecture 9.

What you need to know is that

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$, $(0,1)$, ... are uncountable

Also, read 2.1 in Abbott.

Sequences of real numbers

Def. A sequence is a function whose domain is $\mathbb{N}$.

We usually write a_n instead of $f(n)$.

Examples For $n \in \mathbb{N}$

- $a_n = \frac{1}{n}$
- $a_n = \frac{1}{n^2}$
- $a_n = \cos(2\pi n)$
- $a_n = 1$
- $a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$
- $a_n = \sqrt[n]{n}$
- $a_n = (1 + 1/n)^n$
- $a_n = 27n$
- $a_n = 1 + \text{—} + n.$
- $a_n = \frac{5n+2}{3n-4}$

We are interested in studying convergent sequences, that is sequences that approach a certain value as n grows larger and larger.

Def. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \in \mathbb{N}$

$$|S_n - S| < \varepsilon$$

(eventually the sequence is very close to a)

Rk N depends on ε.

We write $a_n \xrightarrow[n\to\infty]{} a$ or $\lim_{n\to\infty} a_n = a$

a is called the limit of a_n

Example

$$a_n = \frac{5n+2}{3n-4} = \frac{\cancel{n}}{\cancel{n}} \frac{(5+2/n)}{(3-4/n)} = \frac{5+2/n}{3-4/n}$$

$2/n$, $4/n$ get smaller and smaller

so intuitively $a_n \to 5/3$.

We now need to learn how to use our intuition together with the definition to write formal proofs. Before we do that, a little theorem

Thm

Limits are unique. That is if $a, b \in \mathbb{R}$ are both limits of a sequence $\{a_n\}$, then $a = b$

Proof

Assume that $a, b \in \mathbb{R}$ are limits of $\{a_n\}$

By definition, given any $\varepsilon > 0$, there exist

N_1 s.t. $n > N_1$ $|a_n - a| < \varepsilon/2$

and

N_2 s.t. $n > N_2$ $|a_n - b| < \varepsilon/2$

Let $n > \max\{N_1, N_2\}$.

By triangle inequality

$$|a-b| = |(a - a_n) + (a_n - b)| = |a - a_n| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We can now conclude $a = b$ provided we prove the following lil' lemma $\square$

Lemma

Let $a, b \in \mathbb{R}$. $a = b \iff \forall \varepsilon > 0 \; |a-b| < \varepsilon$

Pf

$\boxed{\Rightarrow}$ if $a = b$ $|a-b| = 0$.

$\boxed{\Leftarrow}$ By contradiction, assume $a \neq b$.

then let $\varepsilon = |a-b| > 0$.

By assumption $|a-b| < \varepsilon$, hence we have a contradiction $\square$

MATH 327 - Lecture 9 04/14/23

Countable and uncountable sets - OPTIONAL

$\mathbb{Q} \subseteq \mathbb{R}$ dense (Lecture 7)

$\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$ dense (HW2)

It is tempting to say that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ have the "same size" but actually there's a lot more of $\mathbb{R} \setminus \mathbb{Q}$ than $\mathbb{Q}$.

Recall: cardinality = size of a set.

if the set is finite = # elements

What if I have infinitely many elements?

Cantor came up with an idea to put sets in 1-1 correspondence with each other.

Def A function $f: A \to B$ is a 1-1 correspondence if it is both injective and surjective

Def Two sets have the same cardinality, $|A| = |B|$ if $\exists f: A \to B$ 1-1 correspondence

Example

$E = \{2, 4, 6, \dots\} = \{2n \mid n \in \mathbb{N}\}$ = even numbers

One may be tempted to say that E is "smaller" than $\mathbb{N}$ b/c it is (strictly) contained in it.
But actually the two sets have the same cardinality:

the map $\quad f: \mathbb{N} \to E$

$\qquad\qquad n \mapsto 2n$

is a 1-1 correspondence.

$$\begin{array}{llllll} \mathbb{N}: & 1 & 2 & & n \\ & \updownarrow & \updownarrow & \dots & \updownarrow \\ E: & 2 & 4 & & 2n \end{array}$$

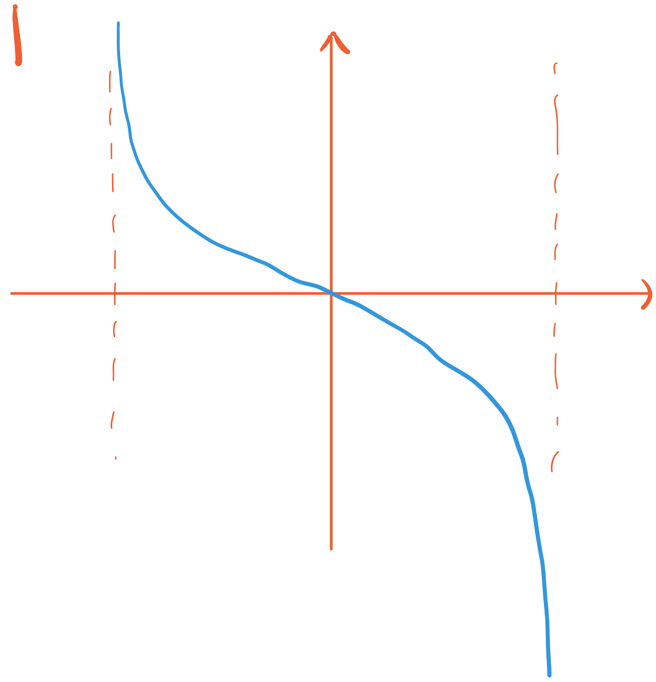
Example: $|\mathbb{Z}| = |\mathbb{N}|$.

let $f: \mathbb{N} \to \mathbb{Z}$ $\qquad f(n) = \begin{cases} (n-1)/2 & n \text{ odd} \\ -n/2 & n \text{ even} \end{cases}$

$$\begin{array}{lllllll} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z} & 0 & -1 & 1 & -2 & 2 & \dots \end{array}$$

Example $|\mathbb{R}| = |(-1,1)|$

$f : (-1,1) \to \mathbb{R}$



use calculus to show that this is a 1-1 correspondence

__Def__ A set A is countable if $|A| = |\mathbb{N}|$
An infinite set which is not countable is called uncantable

__Thm__
$\mathbb{Q}$ is cantable

__Proof__ (ish)
$\forall n \in \mathbb{N}$ let $A_n = \{ \pm p/q \mid p,q$ lowest terms and $p+q=n \}$.

$A_1 = \{ 0/1 \}$

$A_2 = \{ 1/1, -1/1 \}$

$A_3 = \{ 1/2, -1/2, 2/1, -2/1 \}$

$A_4 = \{ 1/3, -1/3, 3/1, -3/1 \}$

$A_5 = \{ 1/4, -1/4, 2/3, -2/3, 3/2, -3/2, 4/1, -4/1 \}$

Each A_n is finite and every $r \in \mathbb{Q}$ appears in one and only one A_n.

(show this)

Take for example $23/10$. $23/10 \in A_{33}$.

$A_1 \cup \_ \cup A_{32}$ is finite so if I build the correspondence as follows

$$\begin{array}{ccccccccccc}
\mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & - \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & \\
 & \underbrace{\tfrac{0}{1}}_{A_1} & \multicolumn{2}{c}{\underbrace{\tfrac{1}{1} \quad -\tfrac{1}{1}}_{A_2}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{2}{1} \quad -\tfrac{2}{1}}_{A_3}} & \cdots & -
\end{array}$$

I am bound to get to $23/10$ after finitely many numbers.

But the same reasoning applies to any $p/q \in \mathbb{Q}$ ($p/q \in A_{p+q}$ ... etc)

Every rational number appears in only one A_n so I'm done

$\square$

The standard approach:

You may have seen the following proof that the positive rationals are countable:

	1	2	3	4	5	6	7	8 — denominator
1	$1/1$	$1/2$	$1/3$	$1/4$	$1/5$	$1/6$	$1/7$	$1/8$
2	$2/1$	$\cancel{2/2}$	$2/3$	$\cancel{2/4}$	$2/5$	$\cancel{2/6}$	$2/7$	$\cancel{2/8}$
3	$3/1$	$3/2$	$\cancel{3/3}$	$3/4$	$3/5$	$\cancel{3/6}$	$3/7$	$3/8$
4	$4/1$	$\cancel{4/2}$	$4/3$					
5	$5/1$	$5/2$						
6								
7								

numerator

just follow the arrows

so that the correspondence is

$\mathbb{N}$	1	2	3	4	5	6	7	8
	|	|	|	|	|	|	|	
$\mathbb{Q}$	1	2	$1/2$	$1/3$	$3/1$	$4/1$	$3/2$	.

Thm

$\mathbb{R}$ is uncountable

Proof

By contradiction, assume $\exists\ f: \mathbb{N} \to \mathbb{R}$ bijective.

this means I can enumerate the elements of $\mathbb{R}$

Let $x_n = f(n)$, so that $\mathbb{R} = \{x_1, x_2, \ldots\}$. (*)

B/c f is a 1-1 correspondence that list contains all real numbers

Let's use the nested interval property to produce a real number not there, and obtain a contradiction.

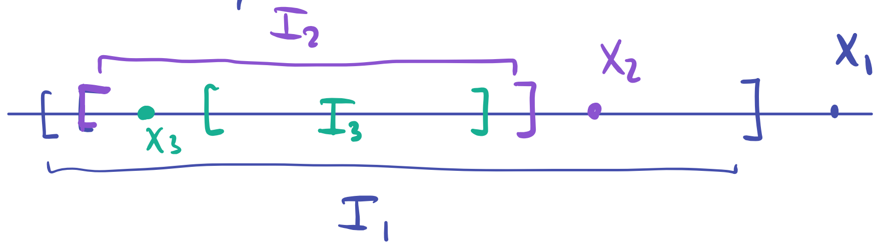
Let I_1 closed interval $x_1 \notin I_1$

Let $I_2 \subset I_1$ closed interval $x_2 \notin I_2$

(why does I_2 exist?)

$\vdots$

$I_{n+1} \subset I_n$, $x_{n+1} \notin I_{n+1}$.



If x_j is any of the list in (*), then $x_j \notin I_j$

and so

$$x_i \notin \bigcap_{n=1}^{\infty} I_n$$

But by N.I.P $\exists\ x \in \bigcap_{n=1}^{\infty} I_n$ but

$x \in \mathbb{R}$ as described above and so we're done $\square$

Cantor's diagonalization Method

Exercise: $(0,1)$ is uncountable iff $\mathbb{R}$ is

Thm

$(0,1) \subseteq \mathbb{R}$ is uncountable

Pf

By contradiction assume $\exists f: \mathbb{N} \to (0,1)$,

1-1 correspondence

$\forall m \in \mathbb{N}$, $f(m) \in (0,1)$ and we use its decimal representation (that we accept w/o formal def)

$$f(m) = .a_{m1} a_{m2} a_{m3} a_{m4} \dots$$

that is $a_{mn} \in \{0, \dots, 9\}$ is the nth digit of $f(m)$

We can look at it in the following table

$\mathbb{N}$	$(0,1)$	1st digit	2nd	$\cdots$
1	$f(1) = 0.$	a_{11}	a_{12}	$a_{13} \cdots$
2	$f(2) = 0.$	a_{21}	a_{22}	$a_{23} \cdots$
3	$f(3) = 0.$	a_{31}	a_{32}	a_{33}
4	$f(4) = 0.$	a_{41}	a_{42}	a_{43}
5	$\vdots$	$\vdots$	$\vdots$	$\vdots$
6	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Our assumption is that every number is in this list.

Now, let $x = 0.b_1 b_2 b_3, \ldots$ when

$$b_n = \begin{cases} 2 & a_{nn} \neq 2 \\ 3 & a_{nn} = 2 \end{cases}$$

(if $a_{11} = 2 \Rightarrow b_1 = 3$
$a_{11} \neq 2 \Rightarrow b_1 = 2$, etc)

Why does x not appear in the table?

$x \neq f(1)$ b/c $b_1 \neq a_{11}$

$x \neq f(2)$ b/c $b_2 \neq a_{22}$

$\vdots$ (continue the argument)

Contradiction! $\square$

MATH 327 - Lecture 10 — 07/17/23

In HW3, you will prove that limits are unique, so it makes sense to write

$$a = \lim_{n\to\infty} a_n \quad \text{or} \quad a_n \xrightarrow[n\to\infty]{} a$$

In the def of limit:

$$\forall \varepsilon > 0 \quad \exists N > 0 \quad \text{s.t.} \quad \forall n > N \quad \underbrace{|a_n - a| < \varepsilon}$$

"eventually"

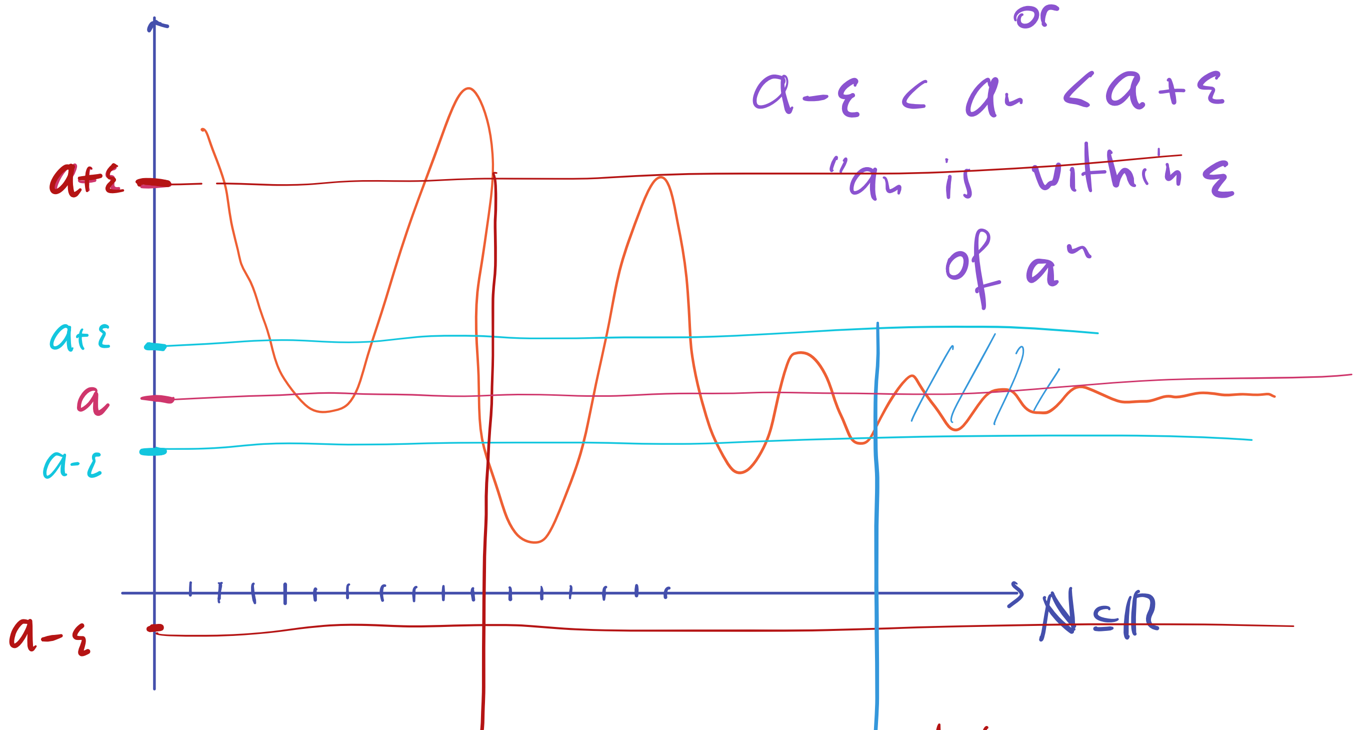
this means:

$\text{dist}(a_n, a) < \varepsilon$

or

$a - \varepsilon < a_n < a + \varepsilon$

"a_n is within ε of a"



N needs to be at least this big

N needs to be at least this big

Example $a_n = \begin{cases} 400 & n \leq 1000 \\ \frac{1}{n} & n > 1000 \end{cases}$

$a_n = \{400, 400, \_\_, 400, \frac{1}{1001}, \frac{1}{1002}, \_\_\}$

$a_n \to 0$! the first finitely many terms don't affect convergence, they just make N bigger

Remark there are three possible behavior for a sequence:
- it converges
- it doesn't — it diverges
  — it oscillates

The latter two are often bundled together but that's a questionable choice.

Def Let $\{a_n\} \subseteq \mathbb{R}$. We say that a_n diverges if $\forall M > 0\ \exists N > 0$ s.t. $\forall n > N$

$$|a_n| > M$$

We can even be a little more precise (like in Problem 5 in HW3 - see corrected version) and say

Def Let $\{a_n\} \subseteq \mathbb{R}$. We say that a_n diverges to $+\infty$ (or converges to $+\infty$) if $\forall M > 0\ \exists N > 0$ s.t. $\forall n > N$ $\quad a_n > M$

Def Let $\{a_n\} \subseteq \mathbb{R}$. We say that a_n diverges to $-\infty$ (or converges to $-\infty$) if $\forall M>0$ $\exists N>0$ s.t. $\forall n>N$ $a_n < -M$

Before we start diving into examples one last remark that we made in class, but I didn't write it down:

Rk In none of the definitions we asked that $N \in \mathbb{N}$. In fact it doesn't need to be. However by the archimedean property we know we can find a bigger natural number, and so we can assume that $N \in \mathbb{N}$, if we want.

Example:

$a_n = \frac{1}{n}$.

Scratch work:

First, we need a guess. We know $1/n$ gets smaller as n gets larger, so we will prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

given any $\varepsilon>0$ I need to find N s.t.

if $n>N$ $\quad \left|\frac{1}{n} - 0\right| < \varepsilon$.

$$\frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}.$$

Then I need to choose $N = 1/\varepsilon$ (or $\lceil \frac{1}{\varepsilon} \rceil$ if we want $N \in \mathbb{N}$)

and we get the result

<u>Proof</u> Let $\varepsilon > 0$ and let $N = \frac{1}{\varepsilon}$.

then if $n > N$ we have $\left| \frac{1}{n} - 0 \right| < \varepsilon$ D.

<u>Example</u>

$$a_n = \frac{3n+1}{7n-4}$$

<u>Scratch work.</u>

As we learned in calculus, factor n out to make a guess:

$$\frac{\cancel{n}}{\cancel{n}} \frac{(3+1/n)}{(7-4/n)} = \frac{3+1/n}{7-4/n} \to 3/7$$

$1/n \to 0$, $4/n \to 0$

We want to prove that

$$\lim_{n\to\infty} a_n = 3/7$$

Given $\varepsilon > 0$ we need to figure out how big n must be IN TERMS OF ε so that

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \varepsilon$$

$$\frac{\cancel{21n}+7-\cancel{21n}+12}{7(7n-4)} = \frac{19}{7(7n-4)}$$

$$\Leftrightarrow \quad \left|\frac{19}{7\underbrace{(7n-4)}_{>0}}\right| < \varepsilon$$

$>0 \rightarrow$ drop $|\cdot|$

$$\frac{19}{7(7n-4)} < \varepsilon$$

Now algebra

$$7n-4 > \frac{19}{7\varepsilon}$$

$$n > \left(\frac{19}{7\varepsilon}+4\right)\frac{1}{7}$$

$$n > \frac{19}{49\varepsilon}+\frac{4}{7} = N.$$

Formal proof

Let $\varepsilon > 0$ and $N = \frac{19}{49\varepsilon}+\frac{4}{7}$.

Now take $n > N$. this implies

$$n > \frac{19}{49\varepsilon} + \frac{4}{7}$$

which, by the same algebra manipulations as above, is equivalent to

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon,$$

and so we are done $\square$

<u>Thm</u> (squeeze theorem)

Show that if $x_n \le y_n \le z_n$ $\forall n \in \mathbb{N}$ (enough if it's eventually)

and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L$, then

$\lim_{n\to\infty} y_n = L$.

<u>Proof</u>.

$\forall \varepsilon > 0$ $\exists N_1 > 0$ s.t. $n > N_1$ $|x_n - L| < \varepsilon$

$\exists N_2 > 0$ s.t. $n > N_2$ $|z_n - L| < \varepsilon$

Let $N = \max\{N_1, N_2\}$. Then if $n > N$

WTS: $|y_n - L| < \varepsilon$, that is

$$-\varepsilon < y_n - L < \varepsilon$$

$$-\varepsilon \le x_n - L \le y_n - L \le z_n - L < \varepsilon \qquad \square$$

MATH 327 - Lecture 11 04/19/23

Example

$a_n = (-1)^n$

Scratch work: a_n forever oscillates b/w -1 and 1 so we guess it doesn't converge.

then it means we must prove that for every $a \in \mathbb{R}$ it can't be that

$$\lim (-1)^n = a.$$

The idea is that given any a, either 1 or -1 is at distance at least 1 from a.

Proof: By contradiction, assume $\lim (-1)^n = a$.

then choose $\varepsilon = 1$. then $\exists N$ s.t. $\forall n$

$$|(-1)^n - a| < 1.$$

n even $\Rightarrow$ $|1 - a| < 1$

n odd $\Rightarrow$ $|(-1) - a| < 1.$

$$2 = |1-(-1)| = |1-a+a-(-1)|$$
$$\leq |1-a| + |a-(-1)|$$

strictly $\longrightarrow$ $< 1 + 1$
$$= 2.$$

But $2=2$, hence we have a contradiction $\square$

Example

$$\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4$$

Scratch work: For $\varepsilon>0$ I want to understand how big n should be so that

$$\left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \varepsilon$$

$$\left| \frac{3n+24}{n^3-6} \right| < \varepsilon$$

$$\frac{3n+24}{n^3-6} < \varepsilon$$

if $n>1$
I can drop $|.|$, b/c
$n^3-6>0$

Finding the best N_ε would require solving a cubic, but we don't need that!

We can splurge with our estimates and make our life easier

if I want to bound A/B from above, I need to bound A from above and B from below (b/c $B \geq M \Rightarrow \frac{1}{B} \leq M$)

Idea: I want to end up with something like

$$\frac{\Box n}{\Box n^3}$$

so I can simplify the n

Numerator $\quad 3n+24 \leq 3n+24n = 27n$ ✓

Denominator $\quad n^3-6 \geq \underbrace{\Box}_{\text{constant in } (0,1)} n^3$

$n^3-6 \geq (1-a)n^3$ Can write it as $1-a$ Want to find a

$\cancel{n^3}-6 \geq \cancel{n^3}-an^3$

$an^3 \geq 6$

$n^3 \geq 6/a$ many choices! if I choose $a=\frac{1}{2}$ then I need to impose $n>2$

I can also have $a=3/4$ and that's good $\forall n>1$

$a = 3/4 \qquad n^3 \geq 8 \quad \checkmark \quad \forall n \geq 2$

then $\quad n^3 - 6 \geq (1 - 3/4) n^3 = \frac{1}{4} n^3.$

Finally we have

$$\frac{3n+24}{n^3-6} \leq \frac{27\cancel{n}}{\frac{1}{4}n^{\cancel{3}\,2}} = \boxed{\frac{108}{n^2} < \varepsilon}$$

$$n^2 > \frac{108}{\varepsilon}$$

$$n > \sqrt{\frac{108}{\varepsilon}}$$

But I need to recall I asked $n > 1$ so

$$N_\varepsilon = \max\left\{ \sqrt{\frac{108}{\varepsilon}}, 1 \right\}$$

Proof

Let $\varepsilon > 0$ and choose $N_\varepsilon = \max\left\{ \sqrt{\frac{108}{\varepsilon}}, 1 \right\}.$

In particular, $N_\varepsilon \geq \sqrt{108/\varepsilon}$ and so $\forall n > N_\varepsilon$

$$n > \sqrt{108/\varepsilon}$$

$$n^2 > 108/\varepsilon$$

$$\frac{108}{n^2} < \varepsilon$$

$$\frac{3n+24}{n^3-6} \leq \frac{27n}{\frac{1}{4}n^3} < \varepsilon$$

$$\left| \frac{3n+24}{n^3-6} \right| < \varepsilon$$

I can put $|\cdot|$ b/c of my choice of N_ε I know $n>1$

$$\left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \varepsilon$$

and hence convergenced is proved $\square$

MATH 327 - Lecture 12 04/21/23

Def Let $\{a_n\}$ be a sequence. We say that a_n is bounded if $\exists M>0$ s.t.

$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

Remark bounded sequence $\Rightarrow$ bounded above AND below.

Prop

Let a_n be a convergent sequence. then a_n is bounded.

Proof

By def, given any $\varepsilon>0$ $\exists N_\varepsilon>0$ s.t.

$\forall n > N_\varepsilon$ $\quad |a_n - a| < \varepsilon.$

Pick (arbitrary choice!!) $\varepsilon = 1$

$\forall n > N_1$ $\quad |a_n - a| < 1.$

see practice problems $\quad |a-b| \geq ||a|-|b||$

(Reverse) triangle inequality

$$||a_n| - |a|| \leq |a_n - a| < 1$$

$$\Rightarrow \quad |a_n| - |a| \leq ||a_n| - |a|| < 1$$

$$|a_n| < |a|+1 \qquad \forall\ n > N_\varepsilon$$

Now, let $M = \max\{|a_1|, |a_2|, \dots, |a_{N_\varepsilon}|, |a|+1\}$.

then $|a_n| \leq M$ $\square$

Example

$$\lim_{n\to\infty} \frac{n^2+3}{n+1} = +\infty$$

Scratch work

for any $M>0$ I want to find out how big n must be so that

$$\frac{n^2+3}{n+1} > M$$

$$\frac{n^2+3}{n+1} \geq$$

Need to bound from below

need to bound from above

$n^2+3 \geq n^2$

$n+1 \leq 2n$ $\Rightarrow$ $\dfrac{n^2+3}{n+1} \geq \dfrac{n^2}{2n} = \dfrac{n}{2} > M$

$n > 2M$.

Let $N_M = 2M$.

Proof Let $M>0$ and choose $N_M = 2M$

then, $\forall n > N_M$

$$n > 2M$$

$$\frac{n^2+3}{n+1} \geq \frac{n^2}{2n} > M$$

positive $\Rightarrow$ $\left|\frac{n^2+3}{n+1}\right| > M$ $\square$

Examples / exercises for today:

$\boxed{1(a)}$ $\lim_{n\to\infty} (a_n + b_n) = a+b$

provided $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

$\boxed{2(c)}$ $\lim_{n\to\infty} x^n = 0$ if $|x|<1$.

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Example

$$a_n = \frac{3n-7}{2n+3}$$

WTS $\lim_{n\to+\infty} a_n = 3/2$.

I can use limit theorems!!

Proof

Because $n>0$ $a_n = \frac{\cancel{n}(3-7/n)}{\cancel{n}(2+3/n)} = \frac{3-7/n}{2+3/n}$.

By Practice Problem 3(a)

$1/n$ converges to 0 $(p=1)$

By Practice Problem 1(b),

both $-7/n$ and $3/n$ converge to 0

By Practice Problem 1(a)

$3-7/n$ and $2+3/n$ converge to 3 and 2, respectively

and finally, By Practice Problem 1(d),

$$\lim_{n\to+\infty} a_n = 3/2 \qquad \square$$

Problem 2

Assume $\lim_{n\to\infty} a_n = a > 0$

and $\lim_{n\to\infty} b_n = +\infty$.

Prove that $\lim_{n\to+\infty} a_n b_n = +\infty$

Reasoning:

By assumption I know (*) $\forall \varepsilon > 0 \; \exists N_\varepsilon > 0$ s.t. $\forall n > N_\varepsilon$

$$|a_n - a| < \varepsilon$$

(**) $\forall M' > 0 \; \exists N_{M'} > 0$ s.t. $\forall n > N_{M'}$

$$b_n > M'$$

WTS: $\forall M > 0 \; \exists N_M > 0$ s.t. $\forall n > N_M$

$$a_n b_n > M$$

I want to prove something for all $M > 0$.

So I CANNOT CHOOSE M. But I KNOW that

(*) and (**) hold for every $\varepsilon > 0$ and $M' > 0$

So I can choose ε and M' in a convenient way for my goal.

My goal is to prove $\exists N_M > 0$ s.t.

$$a_n b_n > M.$$

If $n > N_\varepsilon$ and $n > N_{M'}$ ($\equiv n > \max\{N_\varepsilon, N_{M'}\}$)

$$|a_n - a| < \varepsilon \Rightarrow -\varepsilon < a_n - a < \varepsilon$$

$$\Rightarrow \boxed{a - \varepsilon < a_n} < a + \varepsilon$$

that's the side I need!

this is the definition of absolute value. If that's not clear to you, go review 1.1!!!!!

($|a_n - a| < \varepsilon \rightarrow -(a_n - a) < \varepsilon \Rightarrow a_n - a > -\varepsilon$
$(a_n - a) < \varepsilon$)

and $b_n > M'$.

I will not move forward if I'm afraid to write things. let's play with what we got

$$\begin{matrix} a_n > a - \varepsilon \\ b_n > M' \end{matrix} \Rightarrow a_n b_n > M'(a - \varepsilon)$$

multiply them!

I <u>want</u> $a_n b_n > M$ so if I can choose ε and M' so that $M = M'(a-\varepsilon)$ then I'm done (because choosing $N_M = \max\{N_\varepsilon, N_{M'}\}$ works)

Stare at this $M \overset{?}{=} \underbrace{M'(a-\varepsilon)} > 0$

I need this to be >0

$\Rightarrow$ I need $a - \varepsilon > 0$.

But I can choose ε! And $a > 0$! So I need to choose ε positive and smaller than a: one natural option is to choose

$$\varepsilon = a/2.$$

With this I have

$$a_n b_n > M'(a - a/2) = \frac{M'a}{2}.$$

Now I can choose M' so that $M = \dfrac{M'a}{2}$.

Solving for M' we get

$$M' = 2M/a.$$

Formal proof

$\boxed{a>0}$

By assumption I know (*) $\forall \varepsilon>0 \ \exists N_\varepsilon>0$ s.t. $\forall n>N_\varepsilon$

$|a_n-a|<\varepsilon$

(**) $\forall M'>0 \ \exists N_{M'}>0$ s.t. $\forall n>N_{M'}$

$b_n>M'$

$\boxed{\text{WTS: } \forall M>0 \ \exists N_M>0 \text{ s.t. } \forall n>N_M \quad a_nb_n>M}$

Let $M>0$. Choose $\varepsilon = a/2>0$

and $M' = 2M/a>0$.

By (*) and (**) there exist $N_\varepsilon>0$, $N_{M'}>0$

such that $\forall n>N_\varepsilon \quad |a_n-a|<\varepsilon$

$\forall n>N_{M'} \quad b_n>M'$.

Choose $N_M = \max\{N_\varepsilon, N_{M'}\}$.

then $\forall n>N_M \quad |a_n-a|<\varepsilon \Rightarrow a_n>a-\varepsilon$

$b_n>M'$

Multiplying, $a_n \cdot b_n > M'(a-\varepsilon) = \frac{2M}{a} \cdot (a - a/2)$

$= \frac{\cancel{2}M}{a} \cdot \frac{\cancel{a}}{\cancel{2}} = M$

and so we found $N_M > 0$ s.t.

$n > N_M \quad a_n b_n > M$.

Since M was arbitrary, this concludes the proof $\square$

MATH 327 - Lecture 14   04/26/23

We learned the def of limit of a sequence, and we talked about bounded sequences.
Another good property that sequences can have (exactly like functions) is being monotone

<u>Def</u> A sequence a_n is increasing if $a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$;
a sequence a_n is decreasing if $a_{n+1} \leq a_n$ $\forall n \in \mathbb{N}$.
A sequence is monotone if it's either increasing or decreasing

<u>Examples</u>
- $a_n = 2$ $\nearrow$ $\searrow$
- $a_n = n$ $\nearrow$
- $a_n = 1/n$ $\searrow$
- $a_n \geq 0$ $\forall n$ and $S_n = a_1 + a_2 + \dots + a_n$.

  then $S_{n+1} \geq S_n$ (b/c $a_{n+1} \geq 0$)

  and so S_n is increasing.

  S_n = partial sums of a <u>series</u>

<u>Def</u>: If a_n is a sequence an infinite series is a <u>formal expression</u> $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

b/c it could be ∞

the corresponding sequence of partial sums is

$$S_n = a_1 + \text{—} + a_n$$

and we say that $\sum_{n=1}^{+\infty} a_n$ converges to $A \in \mathbb{R}$ if

$$\lim_{n \to +\infty} S_n = A.$$

<u>Theorem</u> (MONOTONE CONVERGENCE THM)

if a sequence is monoton and bounded, then it converges

<u>Pf</u>.

Assume a_n is increasing. We need to give a candidate for the limit.

By hypothesis the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded (above, in particular) and so $\exists\ s = \sup\{a_n \mid n \in \mathbb{N}\}$.

It makes sense to guess

$$\lim_{n \to +\infty} a_n = s$$

let $\varepsilon > 0$. Because s is the supremum there exist some element a_N s.t.

$$s - \varepsilon \leq a_N.$$

But a_N is increasing, so $\forall n > N \quad a_n \geq a_N.$

then

$$S-\varepsilon \leq a_N \leq a_n \leq S \leq S+\varepsilon$$

$$-\varepsilon \leq a_n - S \leq \varepsilon$$

$$|a_n - S| \leq \varepsilon.$$

By choosing $N_\varepsilon = N$, the desired result is proven

If a_n is decreasing, repeat the same proof with infimum. $\square$

<u>Examples</u>

- $a_n = \frac{1}{n}$ decreasing $\inf\{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$

  and $\lim_{n\to\infty} 1/n = 0$

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$

  $S_n = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ is increasing (b/c $1/n^2 > 0$)

if we can find an upper bound then I know the series converge (to something)

$$S_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \dots + \frac{1}{n\cdot n}$$

$$\leq 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \dots + \frac{1}{(n-1)n}$$

$$= 1 + \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \dots + \left(\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n}\right)$$

$$= 1 + 1 - \frac{1}{n}$$

$$\leq 1 + 1$$

$$= 2.$$

then s_n is bounded and increasing, so it converges to some limit ≤ 2 (we will be back).

<u>Remark</u> (on MCT)

If a_n is monotone and unbounded, we could prove that a_n converges to $\pm\infty$

(How? HW 4)

MATH 327 - Lecture 15 05/01/2023

Subsequences

Recall (lectures and practice problems):

$a_n = (-1)^n$

$b_n = \sin(n\pi/3)$.

In both cases, to prove that they did not converge, we produced two different "special" values of n (infinitely many) such that the sequence went different ways:

$n_k = 2k$ $\quad a_{n_k} = a_{2k} = (-1)^{2k} = 1$

n - that depends on k

$n_k = 2k+1$ $\quad a_{n_k} = a_{2k+1} = (-1)^{2k+1} = -1$.

AND

$n_k = 6k+1$ $\quad b_{n_k} = \sin\left(\frac{(6k+1)}{3}\cdot\pi\right) = \sin\left(\pi/3\right) = \sqrt{3}/2$

$n_k = 6k+4$ $\quad b_{n_k} = \sin\left(\frac{(6k+4)\pi}{3}\right) = \sin\left(4\pi/3\right) = -\sqrt{3}/2$

We then concluded that the sequence

have a limit, because those two "selections" of n's uncovered infinitely many values of n s.t. a_n is very close to different values

↑ well, equal in this case

a_{n_k}

b_{n_k} → these are subsequences

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad a_{10} \quad a_{11} \quad a_{12} \ldots$

When "making" a subsequence we pick some values (a_{n_k}) or equivalently some indices (n_k) but we have 3 rules:

1. they have to be infinitely many
2. I can't repeat the same n (ok if values are the same!)
3. My choices have to be strictly increasing

$$1 \le n_2 < n_2 < \ldots < n_k < \ldots$$

Another way to look at it is w/ a selection map

$\sigma: \mathbb{N} \to \mathbb{N}$ strictly increasing

$\sigma(k) = n_k$.

Ex $(-1)^n$ $\sigma(k) = 2k$

$\sigma(k) = 2k+1$

Def A subsequence of a sequence a_n is a sequence b_k such that for every $k \in \mathbb{N}$ there exists n_k s.t.

$$1 \leq n_1 < n_2 < \_\_ < n_k < \_\_$$

and $b_k = a_{n_k}$.

Why do we care? If you look back at the first page, we've already used subsequences to prove things about convergence (or lack thereof). In fact:

Thm

A sequence a_n converges to $a \in \mathbb{R}$ if and only if every subsequence of a_n converges to the same $a \in \mathbb{R}$

Pf

$\Leftarrow$. HW4

$\Rightarrow$ let $b_k = a_{n_k}$ be a subseq. of a_n.

let $\varepsilon > 0$. WTS: $\exists N_\varepsilon > 0$ s.t. $\forall k > N$

$$|b_k - a| < \varepsilon$$

i.e. $$|a_{n_k} - a| < \varepsilon.$$

But I already know that

$$\exists \tilde{N}_\varepsilon > 0 \text{ s.t. } \forall n > N$$

$$|a_n - a| < \varepsilon.$$

But $n_k \geq k \quad \forall k \in \mathbb{N}$

induction; $n_1 \geq 1$ (b/c $n_1 \in \mathbb{N}$)
assume $n_k \geq k$
$\Rightarrow n_{k+1} > n_k \geq k$
$\Rightarrow n_{k+1} \geq k+1.$

So I can choose

$$N_\varepsilon = \tilde{N}_\varepsilon$$

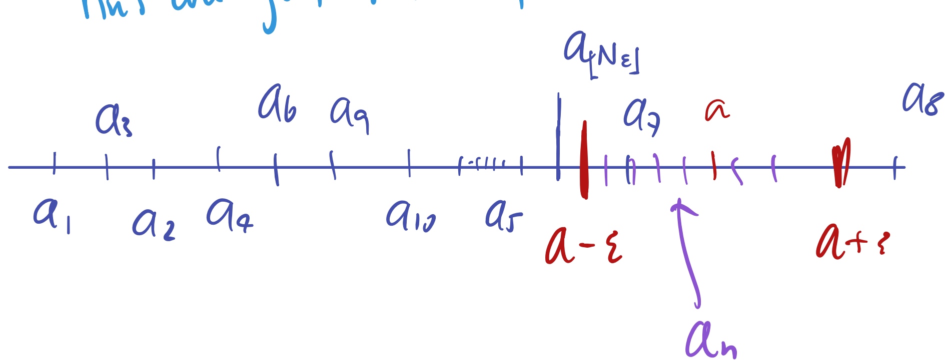
and if $k > N_\varepsilon$

$\Rightarrow n_k > N_\varepsilon$

$\Rightarrow |a_{n_k} - a| < \varepsilon$

n_k's are just some of the n's!

$\square$



__Thm__

Let a_n be a sequence

(i) If $t \in \mathbb{R}$, the set $\{n \in \mathbb{N} \mid |a_n - t| < \varepsilon\}$ is infinite if and only if $\exists\, a_{n_k} \xrightarrow[n\to\infty]{} t$.

(ii) a_n unbounded below $\Rightarrow \exists\, a_{n_k} \to -\infty$

(iii) a_n unbounded above $\Rightarrow \exists\, a_{n_k} \to +\infty$

Moreover, all these subsequences can be taken monotonic (think about it: if they aren't just throw away the n_k's you don't like, there's only many)

These are useful, for a proof see theorem 11.2 in Ross (not mandatory!).

But our favorite theorem about subsequences is

__Thm__ (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence

Bernard Bolzano
1781-1848

proven by him first

Karl Weierstrass
1815-1897

MATH 327 - Lecture 16

<u>Thm</u> (Bolzano-Weierstrass)
Every bounded sequence has a convergent subsequence.

<u>Pf 1</u>

Let a_n be bounded, that is $\exists M>0$ s.t $|a_n|<M$.
Let $A=\{a_n \mid n\in\mathbb{N}\}$. then $-M$ l.b. and M u.b., so

$$-M \leq \inf A \leq \sup A \leq M.$$

Let $a=\inf A$
$b=\sup A$

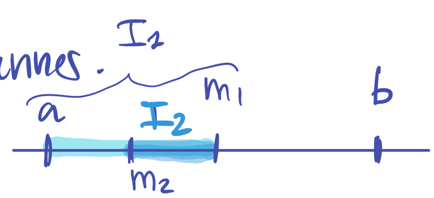
Construct a sequence of nested intervals as follows.

midpoint — m_1

$$I_1 = \begin{cases} \text{left half of } [a,b] \ \left(\left[a, \tfrac{a+b}{2}\right]\right) \text{ if the set} \\ \quad \{n\in\mathbb{N} \mid a_n\in[a,m_1]\} \text{ is infinite} \\ \text{right half of } [a,b] \ ([m_1, b]) \text{ if the set} \\ \quad \{n\in\mathbb{N} \mid a_n\in[m_1,b]\} \text{ is infinite} \end{cases}$$

(At least one of them is - kind of a pigeonhole argument)
but with a looot of pigeons

then let m_2 be the midpoint of I_1
and construct I_2 in the same manner.



By induction, we get a sequence of nonempty intervals I_n such that $I_{n+1} \subset I_n$.
By the nested interval property (NIP) there exists $x \in \bigcap_{n=1}^{\infty} I_n$.

Define a subsequence as follows:

for every k, choose $a_{n_k} \in I_k$

(I have ∞ many choices but any works)

WTS: $\lim_{k \to \infty} a_{n_k} = x$.

Let $\varepsilon > 0$ and let $\ell = |b-a| (= b-a)$ (they were sup and inf!)

$|I_k| = \ell/2^k$.

Let N be the first natural number such that $0 < \ell/2^N < \varepsilon$.

Then $\forall\ k >$, $n_k >$ and

$a_{n_k} \in I_{n_k} \subseteq I_N$

$x \in I_N$

$\Rightarrow \quad |a_{n_k} - x| \leq \ell/2^k < \varepsilon$

$\square$

The second proof uses:

Thm

Every sequence has a monotone subsequence

Pf

We say that n-th term a_n is dominant if it's greater than all the terms after it

$$a_m < a_n \quad \forall m > n.$$

Case 1 ∞ly many dominant.

a_{n_k} = subseq of dominant terms

$a_{n_{k+1}} < a_{n_k} \quad \forall k \quad \searrow$

Case 2 finitely many dominant terms

Choose n_1 s.t a_{n_1} is the last.

$\forall N \geq n_1 \quad \exists \quad m > N$ s.t $a_m \geq a_N$

$N = n_1 \longrightarrow$ select $m = n_2 \qquad a_{n_2} \geq a_{n_1}$

Suppose you selected n_{k-1}. Then choosing $N = n_{k-1}$

$\rightarrow$ select $n_k > n_{k-1}$ s.t. $a_{n_k} \geq a_{n_{k-1}}$.

Then a_{n_k} is increasing.

Pf 2.

Let a_n be bounded. Let a_{n_k} be a monotone subsequence. Then a_{n_k} is monotone and bounded so by MCT a_{n_k} converges $\square$

Def.

Let a_n be a sequence. $\forall N \in \mathbb{N}$, let

$$S_N = \sup \{a_n \mid n > N\}$$

$$s_N = \inf \{a_n \mid n > N\}.$$

Define

$$\limsup a_n = \lim_{N \to +\infty} S_N$$

$$\liminf a_n = \lim_{N \to +\infty} s_N.$$

Remark ⚠

We don't require a_n to be bounded. From now on we adopt the convention

$$\sup A = +\infty \quad \text{if } A \text{ not bdd above}$$

$$\inf A = -\infty \quad \text{if } A \text{ not bdd below}$$

We also say that "$\lim_{n \to +\infty} a_n$ is defined" if it's $a \in \mathbb{R}, \pm\infty$.

Remark

it is **NOT** true that

No

$$\limsup a_n = \sup\{a_n \mid n\in\mathbb{N}\}$$

No No

it's always true that $\limsup a_n \le \sup\{a_n \mid n\in\mathbb{N}\}$

$\limsup a_n$ is the biggest value ← why?
that a_n gets close to infinitely many times

Ex $a_n = \{1000, 1, 1, 1, 1, 1, \ldots\}$

$$\sup\{a_n \mid n\in\mathbb{N}\} = 1000$$

$$\limsup a_n = 1.$$

Thm (*)

let a_n be a sequence. Then

$\lim a_n$ is defined if and only if

$\limsup a_n = \liminf a_n$.

Moreover $\lim a_n = \limsup a_n = \liminf a_n$.

Pf

$\Rightarrow$ HW5

$\Leftarrow$ if $\limsup a_n = \liminf a_n = +\infty$, then

$\forall M>0\ \exists \bar{N}$ s.t. $\forall N > \bar{N}-1$ — just so later I don't have to make a different choice of K.

$$s_N = \inf\{a_n \mid n>N\} > M.$$

$\forall n > N$

$$a_n \geq \inf\{a_n \mid n>N\} > M.$$

and so $\lim a_n = +\infty$.

if $\limsup a_n = \liminf a_n = -\infty$ a similar proof works.

Assume $\limsup a_n = \liminf a_n = a \in \mathbb{R}$.

Let $\varepsilon>0$. then $\exists N_1$ s.t. N_1

$$\Big|\underbrace{\sup\{a_n \mid n>N_1\}}_{s_{N_1}} - a\Big| < \varepsilon$$

then

$\forall n > N_1$, $\quad a_n \leq \sup\{a_n \mid n>N_1\} < a+\varepsilon$

Also, $\exists N_2$ s.t.

$$\Big|\underbrace{\inf\{a_n \mid n>N_2\}}_{s_{N_2}} - a\Big| < \varepsilon$$

$\forall n > N_2$

$$a-\varepsilon < \inf\{a_n \mid n>N_2\} \leq a_n.$$

Putting it all together,

$\forall n > N = \max\{N_1, N_2\}$ $\quad a-\varepsilon < a_n < a+\varepsilon$.

$\Rightarrow \lim a_n = a$ $\square$

Thm

let a_n be a sequence. then $\exists$ monotonic subseq's s.t.

$$a_{n_k} \xrightarrow[k\to+\infty]{} \limsup a_n$$
$$a_{n_j} \xrightarrow[j\to\infty]{} \liminf a_n$$

MATH 327 - Lecture 17

Thm (*)

Let a_n be a sequence. Then $\exists$ monotonic subseq's s.t.

$$a_{n_k} \xrightarrow[k \to +\infty]{} \limsup a_n$$
$$a_{n_j} \xrightarrow[j \to \infty]{} \liminf a_n$$

Pf

HW5.

Def A limit point for a sequence a_n is a $\ell \in \mathbb{R}$ s.t. $\exists$ n_k s.t. $a_{n_k} \to \ell$.

We also call $+\infty$ or $-\infty$ a limit point if $\exists n_k$ s.t.

$$a_{n_k} \to \pm\infty.$$

Examples

① if a_n converges to a, then the only limit point is a.

② $a_n = (-1)^n$ limit pts: ± 1

③ $a_n = (-1)^n n^2$ limit pts $\pm\infty$.

Thm

Let S be the set of all limit points of a sequence a_n

(i) $S \neq \phi$

(ii) $\sup S = \limsup a_n$ and $\inf S = \liminf_n$

(iii) $\lim_{n \to +\infty} a_n = a$ iff $S = \{a\}$.

Pf.

(i) liminf, $\limsup \in S$. by (*)

(ii) let $t \in \mathbb{R}$ be a limit point. Then $a_{n_k} \to t$.

then $t = \liminf a_{n_k} = \limsup_{n_k}$.

But $n_k \geq k \Rightarrow \{a_{n_k} \mid k > N\} \subseteq \{a \mid n > N\}.\ \forall N.$

$\liminf_n a_n \leq \liminf_k a_{n_k} = t = \limsup_{n_k} \leq \limsup_n a_n$

True $\forall t \in S \Rightarrow \liminf_n a_n \leq \inf S$

$\sup S \leq \limsup_n a_n$

But they both belong to S so we're done

(iii) (*) (thm in lecture 16) $\square$

§12 in Ross has a lot of stuff on lim inf and limsup if that's confusing for you!

MATH 327 - Lecture 18

Cauchy sequences

Def Let a_n be a sequence. We say that a_n is Cauchy $\forall \varepsilon > 0$ $\exists N_\varepsilon > 0$ s.t. $\forall n, m > N_\varepsilon$

$$|a_n - a_m| < \varepsilon.$$

Lemma

Cauchy sequences are bounded

Pf

HW 5

Thm

A sequence converges iff it's Cauchy.

Pf

$\Rightarrow$ if $a_n \to a$ then $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n > N$

$$|a_n - a| < \varepsilon/2$$

then if $m, n > N$

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \varepsilon$$

and so a_n is Cauchy

$\Leftarrow$ Assume a_n is Cauchy. By the lemma, the sequence is bounded, so by Bolzano-Weierstrass, there exists a convergent subsequence.

Let $x = \lim_n a_{n_k}$.

(Idea: for n_k big enough it'll be close to x, and also all the terms are close to each other so triangle inequality will work again)

Let $\varepsilon > 0$. $\exists N$ s.t. $\forall m, n > N$

$$|a_n - a_m| < \varepsilon/2.$$

Also choose $\overline{n_k} > N$ such that

$$|a_{\overline{n_k}} - x| < \varepsilon/2$$

$$\begin{aligned} |a_n - x| &= |a_n - a_{\overline{n_k}} + a_{\overline{n_k}} - x| \\ &\leq |a_n - a_{\overline{n_k}}| + |a_{\overline{n_k}} - x| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

$\forall n \geq \overline{n_k}$. $\square$

Series

Recall:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to+\infty} \underbrace{\sum_{k=1}^{n} a_k}_{\substack{\shortparallel \\ S_n = a_1 + \dots + a_n}}$$

terms of the series

sequence of partial sums

$$\sum_{n=1}^{\infty} a_n = A \iff \lim_{n\to+\infty} S_n = A.$$

Thm (algebraic limit theorems for series)

Assume $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$. then

(i) $\sum_{k=1}^{\infty} c a_n = cA \quad \forall c \in \mathbb{R}$

(ii) $\sum_{k=1}^{\infty} (a_n + b_n) = A + B$

Pf

(i) We know $S_n = a_1 + \dots + a_n$ converges to A.

then $t_n = ca_1 + \dots + ca_n = cS_n$

converges to cA by Alg. limit thm for sequences.

(ii) same $\square$

Thm (Cauchy criterion for series)

$\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0$ $\exists N_\varepsilon > 0$ s.t. if

$n > m > N_\varepsilon$,

$$|a_{m+1} + \dots + a_n| < \varepsilon$$

Pf

Observe:

$$|S_n - S_m| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right|$$

$$= \left| \sum_{k=m+1}^{n} a_k \right|$$

$$= |a_{m+1} + \dots + a_n|$$

and apply Cauchy criterion for sequences □

Thm

If a series $\sum_{k=1}^{\infty} a_n$ converges then $a_n \to 0$.

Pf

Consider $n = m+1$ in thm above.

then $\forall \varepsilon > 0$ $\exists N$ s.t. $n > N$ $|a_n| < \varepsilon$

$\Rightarrow$ $a_n \to 0$. □

MATH 327. Lecture 19

Example (geometric series)

$$\sum_{k=0}^{\infty} r^k$$

Partial sums $S_n = \sum_{k=0}^{n} r^k = \dfrac{1-r^{n+1}}{1-r}$ for $r \neq 1$

Why? Because

$$(1-r)(1+r+r^2+ \_ +r^n) =$$
$$= 1+\cancel{r}+ \_ +\cancel{r^n} - (\cancel{r}+\cancel{r^2}+ \_ +r^{n+1})$$
$$= 1-r^{n+1}$$

$\Rightarrow$ if $r \neq 1$ I can divide by $1-r$ and obtain the desired result.

Now observe that if $|r| \geq 1$, then $r^k \not\to 0$ and we proved that it is a necessary condition for a series to converge.

For $|r|<1$ we know that $r^n \to 0$ when $n \to +\infty$ so we can guess the limit of S_n!

$$S_n = \frac{1-r^{n+1}}{1-r} \longrightarrow \frac{1}{1-r} \quad \text{if } |r|<1$$

Then $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ if $|r|<1$

and $\displaystyle\sum_{k=0}^{\infty} r^k$ diverges if $|r|\geq 1$.

Examples HW6

$\displaystyle\sum \frac{1}{k^p}$ $p>1$.

- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$
- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$
- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3} = L$??

Not known

we want to prove that it converges $\forall$ $p>1$ but it's hard to know to what.

we'll prove it later

For now, let's prove that $\sum \frac{1}{n}$ diverges.

harmonic series

comes from music!

We will prove that it is unbounded

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+\frac{1}{2}+\frac{1}{2} = 2$$

In general,

$$S_{2^k} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+ \text{—} \left(\frac{1}{2^{k-1}+1}+\text{—}+\frac{1}{2^k}\right)$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\text{—}\frac{1}{8}\right)+\text{—}+\left(\frac{1}{2^k}+\text{—}+\frac{1}{2^k}\right)$$

$$= 1+\frac{1}{2}+2\cdot\frac{1}{4}+4\cdot\frac{1}{8}+\text{—}+2^{k-1}\cdot\frac{1}{2^k}$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\text{—}+\frac{1}{2}$$

$$1+\frac{1}{2}k.$$

But $1+\frac{1}{2}k$ is unbounded and so is S_{2^k} (and so S_n)

One reason for which it's useful to have a bunch of known examples is that, for series of nonnegative numbers we have something similar to squeeze thm that helps us compare series

Prop (Comparison Test)

Assume $0 \le a_k \le b_k \ \forall k \in \mathbb{N}$

(i) if $\sum_1 b_n$ converges then $\sum_1 a_n$ converges

(ii) if $\sum_1 a_n$ diverges then $\sum_1 b_n$ diverges

Pf.

Observe that

$$|a_{m+1} + \dots + a_n| \le |b_{m+1} + \dots + b_n| \quad (*)$$

then by the Cauchy criterion

(i) $\sum_1 b_k$ converges $\Rightarrow$ it is Cauchy

$(*) \Rightarrow \sum_1 a_n$ Cauchy

$\Rightarrow \sum_1 a_n$ converges

(ii) $\sum_1 a_n$ diverges $\Rightarrow \sum_1 a_n$ not Cauchy

$(*) \Rightarrow \sum_1 b_n$ not Cauchy

$\Rightarrow \sum_1 b_n$ diverges $\square$

Next, we need to collect a few more tools to test whether a series converges or not.

<u>Theorem</u> (Cauchy Condensation Test)

Assume a_k is decreasing, and $a_k \geq 0$.

then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k \underbrace{a_{2^k}} \text{ converges}$$

this is a subsequence of a_k $(n_k = 2^k)$

$= a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$

Pf.

$\boxed{\Leftarrow}$ Assume $\sum 2^k a_{2^k}$ converges.

then the partial sums $t_n = a_1 + 2a_2 + \cdots + 2^n a_{2^n}$.

are bounded (b/c we know it converge)

then $\exists M > 0$ s.t. $t_n \leq M$ $\forall n \in \mathbb{N}$.

WTS: $\sum a_k$ converges because $a_k \geq 0$,

we know that

$$S_{n+1} = a_1 + \_ + a_n + a_{n+1}$$
$$= S_n + a_{n+1}$$
$$\geq S_n$$

and hence S_n is increasing, so it's enough to prove that S_n is bounded

Fix $m \in \mathbb{N}$, let n be large enough so that

$$m \leq 2^{n+1} - 1$$

$$\Rightarrow \quad S_m \leq S_{2^{n+1}-1}$$

$$S_{2^{n+1}-1} = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \_\_ + $$
$$+ (a_{2^k} + \_ + a_{2^{k+1}-1})$$
$$\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + \_ \ ) + \_\_\_$$
$$= a_1 + 2a_2 + 4a_4 + \_ + 2^n a_{2^n}$$
$$= t_n.$$

Then $S_m \leq S_{2^{n+1}-1} \leq t_n \leq M$. By MCT, S_n converges

$\boxed{\Rightarrow}$ Assume $\sum_1 2^k a_{2^k}$ diverges

Hence $t_n = a_1 + 2a_2 + \_ + 2^n a_{2^n}$ is unbounded (above, b/c we know $a_n \geq 0$).

Then $\forall M \ \exists \ n_0$ s.t. $t_{n_0} > M$

We want to show that s_n is also unbounded.

Because a_n is decreasing by hypothesis

$$0 \leq \dots \leq a_{n+1} \leq a_n \leq \dots$$

Fix $m \in \mathbb{N}$ and choose n s.t. $m > 2^n$

$$2 \cdot S_m > 2 S_{2^n} = 2\left(a_1 + a_2 + (a_3 + a_4) + (a_5 + - + a_8) + -\right)$$

$$\geq 2\left(a_1 + a_2 + (a_4 + a_4) + (a_8 + - + a_8) + -\right)$$

$$= 2\left(a_1 + a_2 + 2a_4 + 4a_8 + - + 2^{n-1} a_{2^n}\right)$$

$$= 2a_1 + 2a_2 + 4a_4 + 8a_8 + - + 2^n a_{2^n}$$

$$\geq a_1 + 2a_2 + 4a_4 + 8a_8 + - + 2^n a_{2^n}$$

$\Rightarrow \quad S_m \geq t_{2^n}/2$ and so it is also unbounded □

So far we've looked at nonnegative a_n.

We can use $\sum |a_n|$ to infer info on $\sum a_n$

Thm (Absolute convergence test)

if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Proof

Because $\sum |a_n|$ converges by the Cauchy Criterion

$\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n > m > N$

$$\big|\,|a_{m+1}| + |a_{m+2}| + \cdots + |a_n|\,\big|$$
$$= |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \varepsilon$$

By triangle inequality

$$|a_{m+1} + \cdots + a_n| \le |a_{m+1}| + \cdots + |a_n| < \varepsilon.$$

then $\sum a_n$ is Cauchy and so it converges $\square$

<u>Alternating series test</u>

(i) $a_1 \geq a_2 \geq \cdots \geq$ (a_n decreasing)

(ii) $a_n \to 0$

then $\sum (-1)^{n+1} a_n$ converges

<u>Pf</u> HW6

<u>Example</u>: alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

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Summarizing:

- if $\sum_i a_n \overbrace{<+\infty}^{\text{"converges"}} \Rightarrow a_n \to 0$
- Cauchy criterion
- absolute convergence test *
- comparison test (for ≥ 0 series)
- Cauchy Condensation test
- Alternating series test

Def We say $\sum_i a_n$ "converges absolutely" if $\sum_i |a_n|$ converges

Rk. * says that if a series converges absolutely, then it converges.

Prop. (Root test)

Let $\sum_i a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$

The series $\sum_i a_n$:

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) if $\alpha = 1$ I get no information

Pf

(i). Assume $\alpha < 1$. Let $\varepsilon > 0$ s.t $\alpha + \varepsilon < 1$.

By def of limsup

$\exists\, N$ s.t.

$$\alpha - \varepsilon < \sup\{ |a_n|^{1/n} \mid n > N \} < \alpha + \varepsilon$$

In particular, $\forall n > N$ $\quad |a_n|^{1/n} < \alpha + \varepsilon$

$$|a_n| < (\alpha+\varepsilon)^n$$

But $\alpha + \varepsilon < 1 \Rightarrow \displaystyle\sum_{n=N+1}^{\infty} (\alpha+\varepsilon)^n$ converges

(geometric series)

and By comparison test

$$\sum_{n=N+1}^{\infty} |a_n| \text{ converges.}$$

(ii) Assume $\alpha > 1$. Then exists a subsequence of $|a_n|^{1/n}$ converging to α.

then $|a_n|^{1/n} > 1$ ∞ly many times

$|a_n| > 1$ ∞ly many times.

then $|a_n| \not\to 0$

(iii) $\sum \frac{1}{n}$ $\qquad \alpha = 1$

$\sum \frac{1}{n^2}$

<u>Prop</u> (Ratio test)

$\sum a_n$, $a_n \neq 0 \ \forall n$

(i) converges if $\limsup |a_{n+1}/a_n| < 1$

(ii) diverges if $\liminf |a_{n+1}/a_n| > 1$

(iii) if $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$

No information

Pf practice problems from this week

Double summation and rearrangement

If $\{a_{ij} \mid i,j \in \mathbb{N}\}$ is a double indexed set of real numbers, we want to understand what

$$\sum_{i,j} a_{ij} \quad \text{means.}$$

let's look at an example

$$a_{ij} = \begin{cases} \frac{1}{2^{j-i}} & i < j \\ -1 & i = j \\ 0 & i > j \end{cases}$$

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & \cdots & & \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

How do we add them all up.

I could add each row first:

Fix i and add up j

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} 0 = 0$$

$$\sum_{j=1}^{\infty} a_{1j} = -1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

$$= -1 + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

$$= -1 + 1 = 0$$

$$\sum_{j=1}^{\infty} a_{2j} = 0 + -1 + \frac{1}{2} + \frac{1}{4} \dots$$
$$= 0$$
and it's the same for all i

Fix j and add up i

$$\sum_{j=1}^{\infty} \Big(\underbrace{\sum_{i=1}^{\infty} a_{ij}} \Big) = \sum_{j=1}^{\infty} -\frac{1}{2^{j-1}}$$
$$= -\sum_{k=0}^{\infty} \frac{1}{2^{j-1}} = -2.$$

$$\sum_{i=1}^{\infty} a_{i1} = -1$$
$$\sum_{i=1}^{\infty} a_{i2} = -1 + \frac{1}{2} = -\frac{1}{2}$$
$$\sum_{i=1}^{\infty} a_{i3} = -1 + \frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$
$$\sum_{i=1}^{\infty} a_{ij} = -\frac{1}{2^{j-1}}$$

So two different interpretations of $\sum_{i,j} a_{ij}$ give two different answers. What's the "right" one

Is there even a definition for "the right one"?

One could argue that neither of the approaches above is the right one because they both send i and j at ∞ at different times, but that's not enough of a justification (i and j are independent variables)

Now observe that for every $n, m \in \mathbb{N}$

$$S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$ is a finite sum

and hence $+$ is commutative so I can rearrange as I please.

A more fair way to sum the double indexed sequence in our example above is by looking at

$$S_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$$

Going back up to the "matrix" and adding increasing "squares":

$$S_{11} = -1$$

$$S_{22} = -2 + \frac{1}{2}$$

$$S_{33} = -3 + 2 \cdot \frac{1}{2} + \frac{1}{4} = -2 + \frac{1}{4}$$

$$S_{44} = -2 + \frac{1}{8}$$

$$\vdots$$

$$S_{nn} = -2 + \frac{1}{2^{n-1}}$$

$$\lim_{n \to +\infty} S_{nn} = \lim_{n \to +\infty} \left(-2 + \frac{1}{2^{n-1}}\right) = -2.$$

Is this the right answer?

MATH 327 - Lecture 21 05/17/23

We were discussing double indexed sums
How is that related to products?

$$\sum a_i \cdot \sum b_j = \sum_{i,j} a_i \cdot b_j$$

we would be tempted to say

but we learned this is not a well defined sum

these are all matters of rearrangements.
let's start with that.

<u>Rearrangements</u>

<u>Example</u>. $\sum \frac{(-1)^{n+1}}{n}$ — <u>can't rearrange</u> / see practice problems

$\sum (-\frac{1}{2})^n$ — <u>can</u> rearrange

(PRACTICE PROBLEMS) <u>what's the difference?</u>

<u>Def</u> Let $\sum a_n$ be a series. A series $\sum b_n$ is called a rearrangement of $\sum a_n$ if $\exists f: \mathbb{N} \to \mathbb{N}$ bijective function s.t. $b_{f(k)} = a_k \; \forall k \in \mathbb{N}$.

Def. We say that a series converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ does not

(example $\sum \frac{(-1)^{n+1}}{n}$)

this was the issue!

Thm

If $\sum a_n$ converges absolutely then any rearrangement converges to the same limit

Pf.

Assume $\sum a_n = A$. Let $\sum b_n$ be a rearrangement.

s_n - partial sums of $\sum a_n$

t_m - partial sums of $\sum b_n$

WTS: $t_m \xrightarrow[m\to+\infty]{} A$

Let $\varepsilon>0$. Because $s_n \to A$ $\exists N_1$ s.t.

$$|s_n - A| < \varepsilon/2 \quad \forall n > N_1$$

Bc the convergence is absolute, $\sum |a_n|$ converges and so it's Cauchy $\Rightarrow \exists N_2$ s.t. $\forall n > m > N_2$

$$\sum_{k=m+1}^{n} |a_k| < \varepsilon/2 \qquad (*)$$

let $N = \max\{N_1, N_2\}$. the finite set $\{a_1, \dots, a_N\}$ must appear in $\sum b_n$ eventually. I want to go far ahead enough

Let $M = \max\{f(k) \mid 1 \leq k \leq N\}$

the last one that appears (among $a_1, \dots, a_N$)

Now, let $m \geq M$.

$t_m - S_N$ = finitely many terms and I can use ⊛ (all after a_N)

$$\Rightarrow |t_m - A| = |t_m - S_N + S_N - A|$$
$$\leq |t_m - S_N| + |S_N - A|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

if $n > M$. this is the N_2 that says $t_m \to A$.

$\square$

We observed that in general

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \neq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

Theorem

If $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges

(that is, for every $i \in \mathbb{N}$ $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ and $\sum_{i=1}^{\infty} b_i$ converges too)

then

$$\lim_{n\to\infty} S_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

Pf

See guided exercise proof in Abbott, or Rudin.

Remark

Another reasonable way to sum is to sum along diagonals when $i+j$ is constant.

$$d_2 = a_{11} \quad d_3 = a_{12} + a_{21} \quad d_4 = a_{13} + a_{22} + a_{31}$$

It can be shown (similarly as thm above) that

$$\sum_{k=2}^{\infty} d_k = \lim_{n\to\infty} S_{nn} \quad \text{too.}$$

We'll see in a moment where this came from.

Product of series

We mentioned before that when wanting to multiply series we can again into this rearrangement issue.

$$\left(\sum_i a_i\right) \cdot \left(\sum_j b_j\right) = \sum_{i,j} a_i \cdot b_j \quad ???$$

Not defined

Cauchy product of series:

$$\sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j := \sum_{k=1}^{\infty} c_n$$

where $\quad c_k = \sum_{i+j=k} a_i b_j$.

Motivation:

something super important that you have actually encountered in Calc classes:

power series ← Taylor series!

Come to 424 for more!!!

$$\sum_{i=0}^{\infty} a_i x^i$$

$$\sum_{j=0}^{\infty} b_j x^j$$

$(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$

$= (a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + b_0 a_1 x + a_1 b_1 x^2 + \dots)$

it makes sense to group them by power of x

and this is exactly

$$\sum_{k=0}^{\infty} \sum_{i+j=k} a_i \cdot b_j = \sum_{k=0}^{\infty} c_k$$

<u>Thm</u>

Assume
- $\sum a_n$ converges absolutely
- $\sum a_n = A$
- $\sum b_n = B$

and let $c_k = \sum_{i=0}^{k} a_i b_{k-i}$.

then $\sum c_k = A \cdot B$

<u>Proof</u> (see Rudin thm 3.50).

In fact if I know a priori that $\sum c_n$ converges then it must converge to the right thing! But to prove that we will need power series

<u>then</u>

if $c_k = \sum_i a_i b_{k-i}$ and

$$\sum_i a_n = A, \quad \sum_i b_n = B \quad \text{and} \quad \sum_i c_n = C$$

$$\Rightarrow \quad A \cdot B = C$$

<u>Rk</u> No need for absolute convergence here!

MATH 327 - Lecture 22

e (Rudin pages 63-65)

Def

$$e := \sum_{k=0}^{\infty} \frac{1}{k!} \qquad (0! = 1)$$

$$S_n = 1 + 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2\cdot 1} + \frac{1}{4\cdot 2\cdot 3\cdot 1} + \cdots$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \leq 3$$

b/c $2 < 3, 4, \dots$

then the series converges (by MCT)

thm

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

Pf

let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ $\quad t_n = \left(1+\frac{1}{n}\right)^n$

Binomial theorem:

$$t_n = 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{n}\right)}_{\leq 1} + \frac{1}{3!}\left(\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\right) + \cdots + \frac{1}{n!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}_{\leq 1}$$

$\Rightarrow \quad t_n \leq S_n$

$\Rightarrow \quad \limsup t_n \leq e$

if $n \geq m$

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$$

I stop before
and each term is ≥ 0.

Now, let $n \to +\infty$

$$\liminf t_n \geq \lim_{n\to+\infty}\left(\qquad\qquad\right)$$

$$= 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = S_m$$

Now let $m \to +\infty$

$e \leq \liminf t_n$. $\square$

Remark

$\forall n \in \mathbb{N}, \quad 0 < e - S_n < \frac{1}{n! \cdot n}$

Pf $\quad e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!}$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)$$

$$= \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{n+1}\right)^k$$

$$= \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}}$$

$$= \frac{1}{(n+1)!} \cdot \frac{n+1}{n}$$

$$= \frac{1}{n! n}$$

Ex: S_6 approx e with an error $< 10^{-7}$.

Thm e is irrational

Assume e is rational. then $e = p/q$, $p, q \in \mathbb{N}$.

$$0 < q!(e - S_q) < \frac{1}{q}$$

$$q! \cdot e = (q-1)! \cdot \cancel{q} \cdot p/\cancel{q} = (q-1)!\, p \in \mathbb{N}.$$

$$q! \cdot s_q = q! \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!} \right)$$

$$= q! + q! + q \cdot (q-1) \cdots 3 + \dots + 1 \in \mathbb{N}$$

$\Rightarrow \quad q!\,(e - s_q) \in \mathbb{N}.$

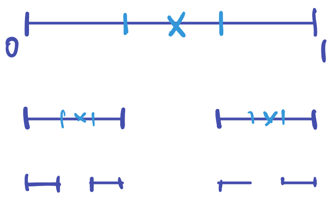
But $q \geq 1 \quad \Rightarrow \quad 0 < q!\,(e - s_q) < \frac{1}{q} \leq 1$

$q!\,(e - s_q)$ is an integer between 0 and 1,

Contradiction $\square$

MATH 327 - Lecture 23-24

Cantor set



$C_0 = [0,1]$

$C_1 = [0, 1/3] \cup [2/3, 1]$

$C_2 = [0, 1/9] \cup \ldots$

C_n = union of 2^n disjoint intervals of length $1/3^n$

$C_{n+1} \subset C_n$ and $C := \bigcap_{n=0}^{\infty} C_n$

Facts: • $C \neq \emptyset$. Note all endpoints of the intervals are in C (b/c they are in every C_n - numbers like $m/3^k \in \mathbb{Q}$)

• length$(C) = 0$

length$(C) = \ell([0,1]) - \sum$ lengths of removed middle intervals

$\sum$ lengths of removed middle intervals $= \frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \ldots + \frac{2^{n-1}}{3^n}$

$$= \frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{9} + \cdots\right)$$

$$= \frac{1}{3} \sum_{k=0}^{\infty} (2/3)^k$$

$$= \frac{1}{3} \cdot \frac{1}{1 - 2/3} = \frac{1}{3} \cdot 3 = 1.$$

$\Rightarrow$ length$(C) = 1 - 1 = 0$

- C is uncountable

For every $x \in C$, $x \in C_n$ $\forall n$.

Define $a_1 = 0$ if $x \in [0, 1/3] \subseteq C_1$

$a_1 = 1$ if $x \in [2/3, 1] \subseteq C_1$

Now define $a_2 = 0$ or 1 according to whether x falls on the left and right component

Then this is a 1-1 correspondence with ∞ sequences w/ values in $\{0,1\}$ which is uncountable.

That's weird! It has zero length (small)

but uncountable (large)

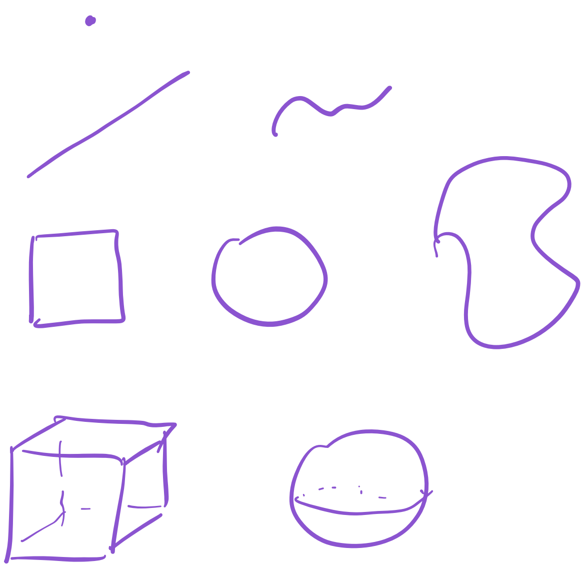
What is happening?

Dimension

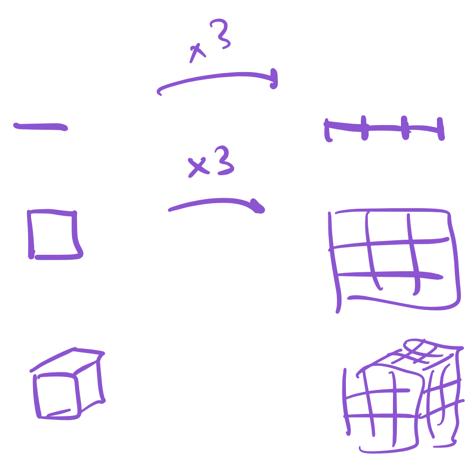
What is dimension?

We all have the intuition:

dim	shape	measure
0		# pts
1		length
2		area
3		volume



Let's think about rescaling by a factor of 3.



3 copies

$9 = 3^2$ copies

$27 = 3^3$ copies

What about the Cantor set?

If I want to rescale it by a factor of 3,

then I obtain

So I basically obtain two copies.

Then, intuitively, dimension should be

$$3^x = 2$$

ln.

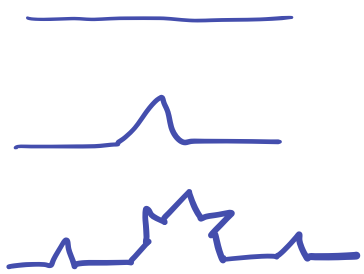
$$x = \log_3 2 = \frac{\log 2}{\log 3} \in (0,1)$$

That is correct!

Another way, b/c C has a nice self-similar structure is

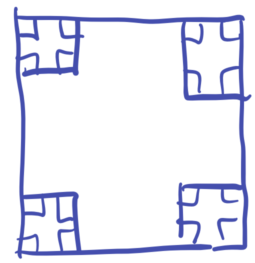
$$\frac{\log(\#\text{ copies})}{\log(1/\text{scale})}.$$

other examples



Von Koch snowflake

$$\dim = \frac{\log(4)}{\log(3)}$$



4 corner Cantor set

$$\dim = \frac{\log(4)}{\log(4)} = 1$$

like a line,
yet so different

Open and closed set

Def if $a \in \mathbb{R}$ we call the ε-neighborhood of a
(ε-nbhd)

the set $$V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x-a| < \varepsilon\}.$$
$$= B(a, \varepsilon)$$
$$= (a-\varepsilon, a+\varepsilon)$$

Def A set $O \subseteq \mathbb{R}$ is open if for all $a \in O$
$\exists\ \varepsilon > 0$ s.t. $V_\varepsilon(a) \subseteq O$

Example

- $\mathbb{R}$ is open - $\forall x \in \mathbb{R}$ and $\forall \varepsilon > 0$

$$V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R}$$

- ϕ gotta be empty too
  (b/c of the logical structure of the def.)

- (a, b), $a, b \in \mathbb{R}$ is open.
  let $x \in (a, b)$ and let $\varepsilon = \min\{x-a, b-x\} > 0$
then $V_\varepsilon(x) \subseteq (a, b)$.

- $[0,1]$ NOT open. Take $x=0$ there's NO $\varepsilon>0$ s.t $(-\varepsilon,\varepsilon)\subset[0,1]$.

Thm

- if A_λ, $\lambda\in\Lambda$ is an arbitrary collection of open sets, then $A=\bigcup_{\lambda\in\Lambda} A_\lambda$ is open
- if $A_1,\dots,A_N$ are finitely many open sets, then $\bigcap_{i=1}^{N} A_i$ is also open

Pf

- Let $\{A_\lambda \mid \lambda\in\Lambda\}$. Let $A=\bigcup_\lambda A_\lambda$.

  Take $x\in A$ $\exists\bar{\lambda}$ s.t. $x\in A_{\bar{\lambda}}$. $A_{\bar{\lambda}}$ open $\Rightarrow$ $\exists\,\varepsilon>0$ s.t. $V_\varepsilon(x)\subseteq A_{\bar{\lambda}}\subseteq A$.

- Let $A=A_1\cap\dots\cap A_N$. If $x\in A$ $\Rightarrow$ $x\in A_i$ $\forall i=1,\dots,N$

  Then $\exists\,\varepsilon_1,\dots,\varepsilon_k$ s.t

$$V_{\varepsilon_i}(x)\subseteq A_i \qquad i=1,\dots,N.$$

Let $\varepsilon=\min_{1\le i\le k}\varepsilon_i$. Then $V_\varepsilon(x)\subseteq V_{\varepsilon_i}(x)\subseteq A_i\ \forall i$

$$\Rightarrow V_\varepsilon(x)\subseteq A \qquad \square$$

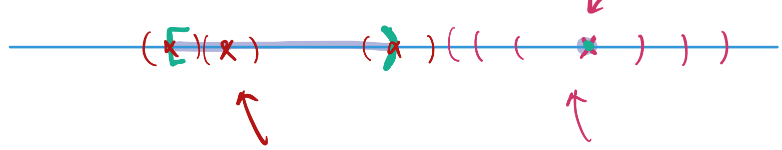
MATH 327 A&B - Lecture 25-26

Closed sets

Def A point x is a limit point for A if $\forall$ ε-nbhd $V_\varepsilon(x)$ intersects A in some other pt other than x

this one doesn't!

$E = [0,1) \cup \{2\}$



0, 1 and all in between have nbhds that always $\cap$

Proof that 0 limit pt of $(0,1)$: $\forall \varepsilon > 0$ $\exists$ $0 \neq x \in (-\varepsilon, \varepsilon) \cap (0,1)$. In fact, choose $0 < x < \varepsilon$.

Def A point $x \in E$ is isolated if it's not a limit point

$x \in E$ → limit point (e.g. $0 \in [0,1]$)
$x \in E$ → isolated point (e.g. $2 \in [0,1] \cup \{2\}$)
$x \notin E$ → limit point (e.g. $0 \notin (0,1]$)
$x \notin E$ → no relation (e.g. $2 \notin (0,1]$)

Rk A limit point for a set doesn't necessarily belong to the set itself.

We want to find an easier way than using the definition to find limit points. Luckily, limit pts from seq's weren't that far off.

Thm

A point x is a limit point for a set E if and only $\exists$ sequence $\{a_n | n \in \mathbb{N}\} \subseteq E$, s.t.

$$a_n \to x \quad \text{and} \quad a_n \neq x \quad \forall n$$

Pf

$\boxed{\Rightarrow}$ Assume x l.p. for E then $\forall \varepsilon \; \exists \; y_\varepsilon \in E \cap V_\varepsilon(x)$, $y_\varepsilon \neq x$. let $\varepsilon = 1 \Rightarrow$ choose $a_1 \in E \cap V_1(x)$

$\varepsilon = \frac{1}{2} \Rightarrow$ choose $a_2 \in E \cap V_{1/2}(x)$

any sequence that goes to 0 works

$\varepsilon = \frac{1}{n} \Rightarrow$ choose $a_n \in E \cap V_{1/n}(x)$.

$$\Rightarrow \quad a_n \in \left(x - 1/n, \; x + 1/n\right)$$

$$\Rightarrow \quad 0 \leq |a_n - x| < 1/n$$

$$\downarrow$$
$$0$$

$\Rightarrow$ squeeze thm $|a_n - x| \to 0 \Rightarrow a_n \to x$

$\boxed{\Leftarrow}$ Assume $\exists \{a_n\} \subseteq E$, $a_n \neq x$, $a_n \to x$.

let $\varepsilon > 0$. By def $\exists N$ s.t $\forall n \geq N$

$$|a_n - x| < \varepsilon$$

i.e. $a_n \in V_\varepsilon(x)$, and $a_n \neq x$.

and because $\{a_n\} \subset E$ $a_n \in E \cap V_\varepsilon(x)$.

which proves x is a l.p for E □

Limit pts are important, b/c they are all a set can "reach". Sets that contain all their limit points are special

Def A set is closed if it contains all its limit points.

Examples

① $E = (0,1)$. Let's prove it's open

$\forall x \in (0,1)$ let $\varepsilon = \min\{dist(0,x), dist(1,x)\}$
$= \min\{x, 1-x\}$.

Then $(x-\varepsilon, x+\varepsilon) \subseteq (0,1)$ and so $(0,1)$ is open

② $E = [0,1)$ let's prove it's not open.

For $x=0$ no matter how small I choose $\varepsilon>0$

$(-\varepsilon,\varepsilon) \not\subset [0,1)$, because $(-\varepsilon,\varepsilon)$ contains

some $-\varepsilon < -\delta < 0$ and $-\delta \notin [0,1)$

Let's prove it's <u>not closed</u>

For $x=1$, I can use the theorem.

$a_n = 1 - 1/n$. $\forall n \in \mathbb{N}$ $0 \le a_n < 1$

(b/c $a_1 = 0$ and $a_n \nearrow$)

i.e. $a_n \in [0,1)$, $a_n \neq 1$

and $a_n \to 1$.

then 1 is a limit pt but $1 \notin [0,1)$.

③ $E = [0,1]$ is closed.

Every pt in $[0,1]$ is a limit point because

I can construct a sequence that converges to it

($x \in [0,1]$. if $x \le 1/2$ $a_n = x + 1/3n$

$x > 1/2$ $a_n = x - \frac{1}{2n}$.)

No other point can be a limit point because

for every $\{a_n\} \subseteq [0,1]$

i.e $0 \leq a_n \leq 1$

By the order limit theorem if a_n converges to $x \in \mathbb{R}$ then $0 \leq x \leq 1$.

Then $F =$ its limit pts, in particular it's closed.

<u>Remark</u> In examples ①–③ all points of E were limit points! But this is not always the case:

④ $E = [0,1] \cup \{2\}$.

E is closed (its limit pts are $[0,1]$) and 2 is isolated.

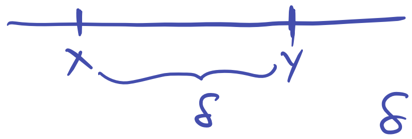
⑤ $E = \{x\}$.

What are the limit points of E?

x can't be a limit pt b/c $(x-\varepsilon, x+\varepsilon) \cap \{x\} = \{x\}$ and no $y \neq x$ can be a limit pt b/c there's no sequence I can build that converges to $y \neq x$

then E has <u>No</u> limit pts and in particular it is closed.

__Rk__ Same is true for $E = \{x, y\}$.



$\delta = |x-y| = dist(x,y)$

then x, y can't be limit pts b/c if $0 < \varepsilon \le \delta$

$$V_\varepsilon(x) \cap E = \{x\}, \quad V_\varepsilon(y) \cap E = \{y\}$$

and the same reasoning about not being able to build a sequence applies.

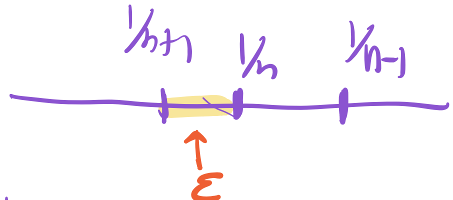
Similarly one can prove

__Prop__

If $|E| < +\infty$ (i.e. E has finitely many elements) then E is closed and all its elements isolated points.

One could ask. What about countable?

__Ex__ 5 $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.



Every $x = \frac{1}{n} \in E$ is isolated: choose $\varepsilon = \frac{1}{n} - \frac{1}{n+1} > 0$

then $\left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right) \cap E = \{1/n\}$

So all pts are isolated, but 0 is a limit pt

b/c $a_n = \frac{1}{n}$ $\{a_n\} \subseteq E$ (in fact $\{a_n\} = E$)

$a_n \neq 0$ and $a_n \to 0$.

But $0 \notin E$ and so E is <u>not</u> closed.

<u>Thm</u>

$F \subseteq \mathbb{R}$ closed $\iff$ $\forall$ Cauchy sequence $\{a_n\} \subset F$,

$\lim a_n \in F$.

<u>Pf</u>

HW7

<u>Def</u>. Given a set E, let $L_E = \{x \in \mathbb{R} \mid x$ l.p. for $E\}$.

set of limit pts

Then the closure of E is defined as

$$\bar{E} = E \cup L_E$$

(I am "forcing" my set to be closed)

<u>Ex</u>

- $E = (0,1) \Rightarrow \bar{E} = [0,1]$
- $E = (0,1] \Rightarrow \bar{E} = [0,1]$
- $E = [0,1] \Rightarrow \bar{E} = [0,1]$

(in general, if F closed) $\overline{F} = F$

needs to be proven

(b/c the limit points of $F \cup L_F$ could be more than L_F - but they're not, one just need to prove it)

Exercise the limit points of $E \cup L_E$ are the same as E

- $E = [0,1) \cup \{2\} \Rightarrow \overline{E} = [0,1] \cup \{2\}$.

Recall that given $A \subseteq \mathbb{R}$ $A^c := \mathbb{R} \setminus A$ the complement. Also recall $(A^c)^c = A$.

e

Thm

(i) A is open iff A^c is closed

(ii) F is closed iff F^c is open

Pf.

First observe that (ii) follows from (i) by letting $F = A^c \Rightarrow F^c = (A^c)^c = A$. Now to prove (i):

$\boxed{\Rightarrow}$ Assume A open. WTS A^c closed, that is, that it contains all its limit pts..

If x l.p. of A^c, then <u>all</u> its ε-nbhds intersect A^c. But $\forall y \in A$ there is at least one nbhd fully contained in A, and hence it does not

intersect A^c. then $x \notin A \Rightarrow x \in A^c \Rightarrow A^c$ closed.

$\boxed{\Leftarrow}$ Now assume A^c closed. WTS: A open.

let $x \in A$. then, b/c A^c closed, x is <u>NOT</u> a l.p. for A^c.

then $\exists \varepsilon > 0$ s.t. $V_\varepsilon(x) \cap A^c$ doesn't contain anything other than x, but $x \notin A^c \Rightarrow V_\varepsilon(x) \cap A^c = \emptyset$

$$\Rightarrow V_\varepsilon(x) \subseteq A.$$

and so A is open $\square$

<u>Remark</u> closed sets are usually defined as the complements of the open sets (and declaring which subsets are open means to give a <u>topology</u>).

But because we are in $\mathbb{R}$ we have a beautiful

metric structure (that is, a distance) - actually more...
so we have all these properties.

thanks to De Morgan's laws:

$$\Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c \quad , \quad \Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$$

the next thm follows immediately from the previous one

<u>thm</u>

(i) if F_λ closed, $\lambda\in\Lambda$ then

$$F = \bigcap_{\lambda\in\Lambda} F_\lambda \text{ is closed}$$

(ii) if $F_1, \dots, F_N$, $N<+\infty$ closed, then

$$F = F_1 \cup \dots \cup F_N \text{ closed}$$

<u>Pf</u>

(i) $F^c = (\bigcap F_\lambda)^c = \bigcup \overset{\text{open}}{F_\lambda^c}$ open $\Rightarrow$ F closed

(ii) $F^c = (F_1 \cup \dots \cup F_N)^c = \underbrace{F_1^c}_{\text{open}} \cap \dots \cap \underbrace{F_N^c}_{\text{open}}$ open $\Rightarrow$ F closed

□