EXERCISE SHEET 4

Exercise 0.1 (Alternative construction of interpolating functions (see Proposition 5.1.2.))**.** *Let $L \in \mathcal{C}_k$ be a cube as defined in class and consider any plane π such that in the corresponding cylinder $\mathbf{C}_{16M_0 l(L)}(p_L, \pi)$ we have*

$$\bar{E} := l(L)^{-m}\mathbf{E}(\mathrm{Gr}(u), \mathbf{C}_{16M_0 l(L)}(p_L, \pi)) \leq C_0\, E\, l(L)^{2-2\delta}\,.$$

Let $\bar{f}$ be the $\bar{E}^{\gamma}$-approximation of u and let $\bar{z}$ be the function defined by

$$\begin{cases} \Delta \bar{z} = 0 & \text{in } B_{8M_0 l(L)}(p_L, \pi) \\ \bar{z} = \bar{f} & \text{in } \partial B_{8M_0 l(L)}(p_L, \pi)\,. \end{cases} \tag{0.1}$$

Show that If δ and E are sufficiently small, there is $\beta > 0$ and $C, C(j)$ geometric constants such that

$$\|\bar{z} - \bar{f}\|_{L^1(B_{4M_0 l(L)}(p_l,\pi))} \leq C\, E\, l(L)^{3+\beta} \tag{0.2}$$

$$\|\Delta D^j \bar{z}\|_{C^0(B_{4M_0 l(L)}(p_l,\pi))} \leq C(j)\, E\, l(L)^{1-j+2\beta}\,, \qquad \text{for all } j \geq 0\,. \tag{0.3}$$

(One of them is pretty easy...)

Exercise 0.2 (Interpolation inequality (Combine with previous exercise to deduce estimate in class after 5.2.2.))**.** *Suppose Suppose v is a solution of $\Delta v = 0$ in B_8 then there exists a geometric constant $C_0 > 0$ such that*

$$\|v\|_{L^\infty(B_1)} \leq C_0 \|v\|_{L^1(B_2)} \tag{0.4}$$

$$\|D^l v\|_{L^\infty(B_1)} \leq C_0 \|v\|_{L^1(B_2)}\,. \tag{0.5}$$

*(**Hint**: Use the following interpolation inequality: for $0 \leq j \leq m$ and $\frac{j}{m} \leq a \leq 1$ there exists a geometric constant $C_0 > 0$ such that*

$$\|D^j u\|_{L^p(B_1)} \leq C_0 \|D^m u\|^a_{L^s(B_1)}\, \|u\|^{1-a}_{L^q(B_1)} + C_0 \|u\|_{L^q(B_1)}$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{s} - \frac{m}{n}) + (1-a)\frac{1}{q}$.)