EXERCISE SHEET 3

Exercise 0.1. *Let $u\colon B_2 \to \mathbb{R}^n$ be a Lipschitz function. For every $\lambda \in (0,1)$, there exists a closed set $K \subset B_1$ and a Lipschitz function $v\colon B_1 \to \mathbb{R}^n$ such that*

$$u \equiv v \ \text{ on } K \qquad \text{and} \qquad \mathrm{Lip}(v) \leq C_0\, E^{\lambda} \tag{0.1}$$

$$|B_1 \setminus K| \leq C_0\, E^{1-2\lambda} \tag{0.2}$$

where $E := \frac{1}{2^{m+1}} \int_{\mathrm{Gr}(u,B_2)} |\vec{T}_p \mathrm{Gr}(u) - \vec{\pi}_0|^2 \, \mathrm{dvol}^m(p)$.
*(**Hint:** Prove the following two facts and use them to conclude the result.*

- *Suppose $f \in L^1(B_1)$, then $|B_1 \setminus K| := |\{x \in B_1 : \mathcal{M}f(x) > t\}| \leq \frac{5^n}{t} \int_{B_1} |f|$ (**Hint:** the number 5 is meaningful!).*
- *If $f \in W^{1,2}$ then $\mathrm{Lip}(f|_K) \leq C_0\, t^{1/2}$, where K is the same as in the previous bullet with $f = |Df|^2$. (**Hint:** Recall that $f(x) := \lim_{r\to 0} (f)_{x,r}$ and use Poincaré inequality)*

Exercise 0.2 (Lemma 3.4.1. of the Lecture Notes)**.** *Consider $h\colon B_r(x) \to \mathbb{R}^n$ harmonic and let $\rho < r$. Then*

$$\int_{B_\rho(x)} |Dh - (Dh)_{x,\rho}|^2 \leq C \left(\frac{\rho}{r}\right)^{m+2} \int_{B_r(x)} |Dh|^2 \,.$$

*(**Hint:** Use Fourier series and mean value property of harmonic functions.)*

Exercise 0.3 (Theorem 4.2.1. (b) of the Lecture Notes)**.** *Prove that if $(\phi_k)_k$ is the sequence of glued interpolation of u defined in class, then*

$$\lim_{k\to\infty} \|\phi_k - u\|_{C^0([-\sigma,\sigma]^m)} = 0 \,.$$