EXERCISE SHEET 2

Exercise 0.1 (Proposition 1.3.1. of the Lecture Notes)**.** *Prove that if* Ω *is a borel set and* $u\colon \Omega \to \mathbb{R}^n$ *is a Lipschitz map, then*

$$\mathrm{vol}^m(u,\Omega) - |\Omega| = \frac{1}{2}\int_{\mathrm{Gr}(u,\Omega)} |\vec{T}_p\mathrm{Gr}(u) - \vec{\pi}_0|^2 \, d\mathrm{vol}^m(p)\,.$$

As a consequence deduce that E *in Theorem 1.2.1 is non-negative and equal to* 0 *if and only if* $\mathrm{gr}(u, B_1)$ *is horizontal.*

Exercise 0.2 (Proposition 2.2.2. of the Lecture Notes)**.** *Let* $u\colon \Omega \to \mathbb{R}^n$ *be a map with* $\mathrm{Lip}(u) \le 2$, $p = (x, u(x))$ *and* $q = (y, u(y))$*. Then there are geometric constants* C_1, C_2, C_3 *such that*

$$C_1^{-1} r^m \le \mathrm{vol}^m(\mathrm{Gr}(u) \cap B_r(p))) \le C_1\, r^m\,, \qquad \textit{if } r < \mathrm{dist}(x, \partial\Omega)\,, \tag{0.1}$$

$$|\vec{\pi}_1 - \vec{\pi}_2|^2 \le C_2(\mathbf{E}(\mathrm{Gr}(u), B_{2r}(p), \vec{\pi}_1) + \mathbf{E}(\mathrm{Gr}(u), B_\rho(p), \vec{\pi}_2))\,, \qquad \textit{if } r \le \rho \le 2r < \mathrm{dist}(x, \partial\Omega)\,, \tag{0.2}$$

$$|\vec{\pi}_1 - \vec{\pi}_2|^2 \le C_3(\mathbf{E}(\mathrm{Gr}(u), B_r(p), \vec{\pi}_1) + \mathbf{E}(\mathrm{Gr}(u), B_r(q), \vec{\pi}_2)) \tag{0.3}$$

if $r = |p - q| < \min\{\mathrm{dist}(x, \partial\Omega), \mathrm{dist}(y, \partial\Omega)\}$*.*

Exercise 0.3 (Lemma 1.1.1. and 4.4.1. of the Lecture Notes)**.** *Prove that there are constants* $c_0, C_0 > 0$ *with the following properties. Assume that*

(i) $A \in SO(m+n)$, $\|A - Id\| \le c_0$, $r \le 1$*;*
(ii) $(x_0, y_0) \in \pi_0 \times \pi_0^\perp$ *are given and* $f, g\colon B^m_{2r}(x_0) \to \mathbb{R}^n$ *are Lipschitz functions such that*

$$\mathrm{Lip}(f), \mathrm{Lip}(g) \le c_0 \qquad \textit{and} \qquad |f(x_0) - y_0| + |g(x_0) - y_0| \le c_0\, r\,.$$

Then in the system of coordinates $(x', y') = A(x, y)$*, for* $(x_1, y_1) := A(x_0, y_0)$*, the following holds:*

(a) $\mathrm{Gr}(f), \mathrm{Gr}(g)$ *are the graph of two Lipschitz functions* f', g' *in the tilted system of coordinates, whose domains of definition both contain* $B_r(x_1)$ *and*

$$\mathrm{Lip}(f'), \mathrm{Lip}(g') \le 1 + C_0\|A - Id\|\,.$$

(b) $\|f' - g'\|_{L^1(B_r(x_1))} \le C_0\, \|f - g\|_{L^1(B_{2r}(x_0))}$.
(c) *If* $f \in C^4(B_{2r}(x_0))$*, then* $f' \in C^4(B_r(x_0))$ *and*

$$\|f' - y_1\|_{C^3} \le \Phi\left(\|A - Id\|, \|f - y_0\|_{C^3}\right) \tag{0.4}$$

$$\|D^4 f'\|_{C^0} \le \Psi\left(\|A - Id\|, \|f - y_0\|_{C^3}\right)\left(1 + \|D^4 f\|_{C^0}\right)\,. \tag{0.5}$$