EXERCISE SHEET 1

1. Example of area minimizers: calibrations

Definition 1.1. Given a d-dimensional Riemannian manifold M, we say that a k-dimensional smooth differential form ω is a calibration associated with a k-dimensional orientable surface S if the following properties hold:

(i) the form ω restricted to the tangent plane of S coincides with its volume form, that is $\langle \omega, \vec{S} \rangle = \|\vec{S}\|$, where $\vec{S}$ is a k-vector orienting S;
(ii) the form ω is closed, that is $d\omega = 0$;
(iii) for every other k-section $\tau \in TX$, the form ω does not exceed the volume form, that is $\langle \omega, \tau \rangle \le \|\tau\|$.

Exercise 1.2. *Let $S \subset \mathbb{R}^d$ be a k-dimensional, orientable, smooth surface, and assume that there exists a calibration ω associated to S. Prove that S is area minimizing in the following sense:*

$$\mathrm{vol}^k(S) \le \mathrm{vol}^k(T)$$

for every smooth orientable k-dimensional surface T such that $T \cup S = \partial U$ with $U \subset \mathbb{R}^{d+1}$.

Exercise 1.3. *Prove that in $\mathbb{C}^{d+k}$ every holomorphic subvariety Γ is area minimizing, that is:*

(1) *Recall that an holomorphic curve is the zero level set of an holomorphic map $u\colon \mathbb{C}^{d+k} \to \mathbb{C}^k$, where k is the complex codimension and d the complex dimension. It is known that the real hausdorff dimension of $\mathrm{Sing}(\Gamma)$ is less than or equal to $2d-2$ and that at each point $p \in \Gamma \setminus \mathrm{Sing}(\Gamma)$, the (real) tangent $2d$-dim. plane $T_p\Gamma$ can be identified with a complex d-dimensional plane of $\mathbb{C}^{d+k}$.*
(2) *Prove that Γ is orientable by defining an orientation on Γ.*
(3) *Consider the Kahler form $\omega := dx_1 \wedge dy_1 + \cdots + dx_{d+k} \wedge dy_{d+k}$, where $x_j := \Re(z_j)$ and $y_j := \Im(z_j)$, $j = 1, \dots, d+k$. Show that the form*

$$\omega^k := \frac{1}{k!} \underbrace{\omega \wedge \cdots \wedge \omega}_{k\, times}$$

is a calibration.
(4) *Show that $\langle \omega^k, v_1 ... v_{2k} \rangle = \|v_1 \wedge \cdots \wedge v_{2k}\|$ if and only if $v_1, \dots, v_{2k}$ is a positively oriented ($\mathbb{R}$-)base of a complex plane.*

(Compare with (1.4.2) of lecture.)

Exercise 1.4. *Let $u \in C^2(\Omega; \mathbb{R})$, $\Omega \subset \mathbb{R}^d$ a domain, and let $\mathrm{Gr}(u) := \{(x, u(x)) \subset \mathbb{R}^{d+1} : x \in \Omega\}$ be the graph of u.*

(i) *Show that* $\mathrm{Gr}(u)$ *is area minimizing in the cylinder* $\omega \times \mathbb{R}$ *if and only if* $\mathrm{Gr}(u)$ *is minimal, that is*

$$H_{\mathrm{Gr}(u)} = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

*(**Hint:** Construct a d-form whose exterior derivative is the minimal surface equation...)*

(ii) *If* Ω *is convex show that* $\mathrm{Gr}(u)$ *is area minimizing in* R^{d+1}.

(Compare with footnote 1 of lecture.)

References