Jan18

Sunday, January 16, 2022 1:39 PM

Welcome to Math 113

- Course Website
  - Videos of lectures
  - HW & Solutions
  - Up to date info.
- Discord: ✓
- Office Hours:

  W: 9:30 – 11:00
  Th: 11:00 – 12:30
  } Evans 895 or Zoom.

- GSI: M, T, Th F 10:00-12:00 (Zoom)
- Take Home: 3 of them.

  Out: Fri
  Due: Mon
  } 50%

- Homework: Every week due Friday } 50%

  @ 12:00 Noon.

Extensions: get in touch early.

Motivation & Example

⇝ A first pass @ algebra.

Discover algebra

Phase 1:

The concept of unity... of anything.

Mathematically $\rightsquigarrow \boxed{1}$

Phase 2:

The concept of distinction.

I can observe multiple things.... an count them.

$$1, \quad 2, \quad 3, \quad \ldots = \mathbb{N}$$

The natural numbers.

Bonus! Get a binary operation.

23 apples, 37 bananas

How many peices of fruit?

$$23 + 37 = 60$$

$$\longrightarrow \quad (\mathbb{N}, +)$$

Bonus! "10 boxes of 8 apples"

$$\wr$$

$$10 \underset{\uparrow}{\times} 8 = 80$$

$$\longrightarrow \quad (\mathbb{N}, +, \times)$$

Observe + & × are related.

$$a \times (b + c) = a \times b + a \times c$$

Distributive law.

Phase 3 The concept of removal.

$$8 - 6 = 2$$

Formally: $\underline{2 - 3}$ not in $\mathbb{N} = \{1, 2, 3, \dots\}$

Expand $\mathbb{N}$ to make $2 - 3$ make sense.

$$\mathbb{Z} = \{\dots, -2, \overset{\downarrow}{-1}, \underset{\uparrow}{0}, \overset{\downarrow}{1}, 2, 3, \dots\}$$

$(\mathbb{Z}, +, \times)$, + is really nice.

1) Identity: $a + 0 = a$
2) Inverses $a + (-a) = 0$

Phase 4 Notion of equal division

$$\underline{2x = 3} \; ?$$

← Need $\frac{1}{2}$ an apple.

Discovered fractions

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; b \neq 0 \right\}$$

Also has + & ×.

× is also really nice.

1) Identity $a \times 1 = a$.
2) Inverse $\underset{\uparrow}{a} \times \underset{\uparrow}{\frac{1}{a}} = \underset{\uparrow}{1}$. $(a \neq 0)$

$\{$ + geometry

$\downarrow$

$\mathbb{R}$

$\Big\downarrow$ Formally allow all $\sqrt{\ }$

$\mathbb{C}$

At each step, + & × gained nice structural properties which allow us to solve equation.

Model for kind of objects we will study

- $(\mathbb{Z}, +) \longleftarrow$ Group. very nice ✓
  set w/ one binary operation.

- $(\mathbb{Z}, +, \times) \longleftarrow$ Ring.
  - set w/ 2 binary operations. One is really nice.
  - Distributive law

Note only + is really nice.

No integer solution to $2 \cdot x = 1$.

- $(\mathbb{Q}, +, \times) \leftarrow$ Field.

· Set w/ 2 very nice binary operations.

Non-Numerical Examples

Example: Polynomials.

$f(x) = x^2+5 \qquad g(x) = 2x-7$

$$f(x)+g(x) = x^2+2x-2$$

$$f(x)\,g(x) = 2x^3 - 7x + 10x - 35$$

$0(x) = 0 \leftarrow$ additive identity $f + 0(x) = f$

$i(x) = 1 \leftarrow$ multiplicative identity $f \cdot i(x) = f$

Can subtract polynomials!!

$(\mathbb{Z}[x], +, \times)$ a ring.

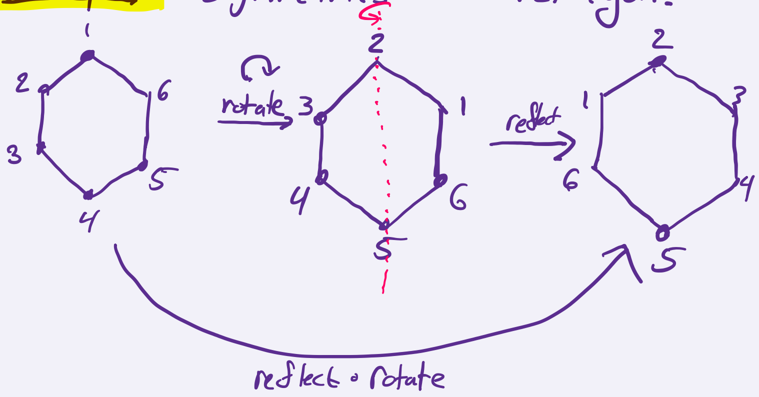
↑
set of polynomials

Sometimes divide but not always.

There is a remarkably rich world of groups, & $(\mathbb{Z}, +)$ is the most simple example.

Most groups will arise from notion of symmetry.

Example Symmetries of hexagon.



Observe $\text{rot}, \text{ref} \in \text{Sym}(⬡)$

$\text{ref} \circ \text{rot} \in \text{Sym}(⬡)$

Analogy w/ $(\mathbb{Z}, +)$

Numbers ⟷ Symmetries

hexagon.

Add $\longleftrightarrow$ Compose $(g \cdot f)$

$\uparrow$ addition law

(This addition law given by composition is "very nice")

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1) Identity $\longleftrightarrow$ Do nothing.

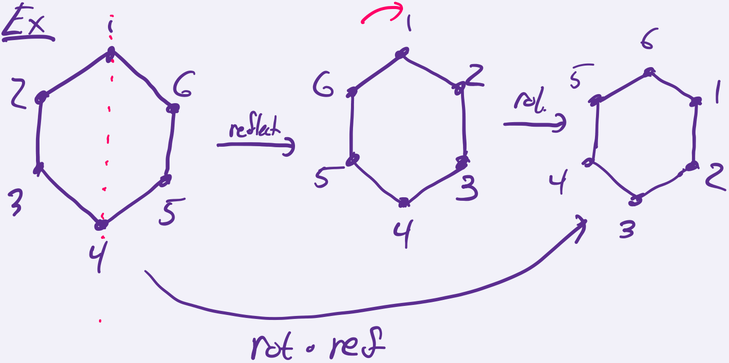
2) Inverse $\longleftrightarrow$ Undo the symmetry.

⚠ $(\mathbb{Z}, +)$ is misleadingly simple.

$(2+3 = 3+2)$

$(a+b = b+a)$

Ex



$$\text{rot} \circ \text{ref} \neq \text{ref} \circ \text{rot}$$

Remark: You've seen something like this before.

Example

$n \times n$ invertible matrices over $\mathbb{R}$. $= GL_n(\mathbb{R})$

$$M, N \in GL_n(\mathbb{R})$$

$$MN \neq NM$$

→ Symmetries of $\mathbb{R}^n$.

Remark:

- So far this has been informal.

- We will pursue this more rigorously... using the formal language of set theory.
- Group will be defined axiomatically as a set w/ extra structure satisfying axioms.

  set ↑ Collection of objects

  extra structure ↑ An "addition law" or a way to combine elements of the set.
- We will get very abstract, these properties (axioms) are natural & even obvious in examples that arise from numbers or symmetries.
- Grounding will be important!! we will look @ tons of examples.
- There is an excellent 3 blue 1 brown video "Symmetries & the monster" about groups giving a very conceptual picture.

  (put on course website).

Beginning our rigorous perspective

Our groups are sets w/ extra structure

collections of objects

ways to combine. (encode as a function)

Ex

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$+ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$+(a, b) = a + b.$$

Introducing Symbols

Logic & Logical Notation

- P, Q are logical statements.

$P \Longrightarrow Q$ means

- P implies Q
- If P is true, then so is Q.

Ex

$$(x \text{ is odd}) \Longrightarrow (x \neq 2)$$

- $P \iff Q$ (P if & only if Q)

  $\leadsto P \Rightarrow Q$ and $Q \Rightarrow P$.

Ex (x is even) $\iff$ (x is a multiple of 2)

- $\forall$ = "for all"
- $\exists$ = "there exist"
- $\exists!$ = "there exists a unque..."

Ex $\exists!\, x$ such that $2x = 8$.

Sets

Defn (Loosely Speaking)

. A set is a collection of mathematical objects. ← elements

Ex $S = \{1, 2, 7\}$ ←

Notation

$\in$ = "is an element of"

$\notin$ = "is not an element of"

Ex $2 \in S$. $3 \notin S$.

We often specify sets as follows:

$$T = \{\text{Notation for elts} \;\underbrace{|}\; \text{Property specifying which elements}\}$$

↑ "such that"

Example

$$E = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}$$

$$= \{\dots, -4, -2, 0, 2, 4, 6, \dots\}$$

More Notation $(S, T$ sets$)$

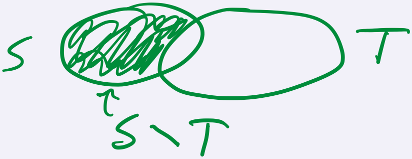
- $\emptyset = \{\ \} =$ "empty set"
- $T \subseteq S$ = "T is a subset of S"

  : $x \in T \Longrightarrow x \in S$

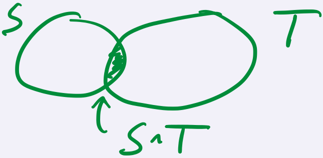
Example $E \subseteq \mathbb{Z}$

- $S \setminus T$ = "S minus T"

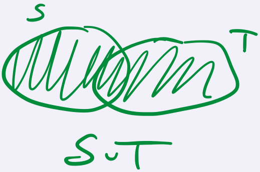
$$= \{x \in S \mid x \notin T\}$$



- $S \cap T = \{x \mid x \in S \ \underline{\text{and}} \ x \in T\}$



- $S \cup T = \{x \mid x \in S \ \underline{\text{or}} \ x \in T\}$



- Cartesian Product

$$S \times T = \{(x,y) \mid x \in S, \ y \in T\}$$

- $|S|$ = # elements in S (can be ∞)

Exercise: (will be on HW1)

Suppose two sets S, T.

Suppose $\underline{T \subseteq S}$.

Describe

a) • $T \cap S$

• $T \cup S$

• $S \cap (S \setminus T)$

• $T \cap (S \setminus T)$

Part a)

Show $\underline{\underline{T \cap S}} = \underline{\underline{T}}$

$$x \in T \cap S \iff x \in T$$

Well

($\Longrightarrow$)

$x \in T \cap S \Longrightarrow x \in T$ and $x \in S$

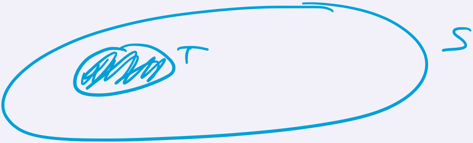
$\Longrightarrow x \in T$ ✓

($\Longleftarrow$)

$x \in T \Longrightarrow$ Since $T \subseteq S$ also have $x \in S$.

So $x \in T$ and $x \in S$ ~~$\emptyset$~~

i.e. $x \in T \cap S$.



Next time

Introduce functions, injectivity, surjectivity.