March18

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# Fundamental Theorem for Finite Abelian Groups

**Setup** $G$ a finite abelian group

$$|G| = n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$$

**Idea**

$G$ is a direct product of cyclic groups in a (sort of) unique way.

## 2 perspectives

### Invariant Factors

① $G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$

- i) $n_i \geq 2$
- ii) $n_{i+1} \mid n_i$
- iii) $\prod n_i = n$

② Unique.

### Elementary Divisors

① $G \simeq A_1 \times \cdots \times A_t$ w/ $|A_i| = p_i^{\alpha_i}$

Inv. Factors on Ai

② $A_i = \mathbb{Z}_{p_i^{\beta_1}} \times \cdots \times \mathbb{Z}_{p_i^{\beta_n}}$

- i) $\beta_j \geq 1$
- ii) $\beta_{j+1} \leq \beta_j$
- iii) $\sum \beta_j = \alpha_i$

③ Uniqueness

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These are indeed equivalent

**Main tool**

**Prop** (HW #5(c))

① $Z_n \times Z_m \cong Z_{nm} \iff \gcd(n,m)=1$

② If $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ then

$$Z_n = Z_{p_1^{\alpha_1}} \times \dots \times Z_{p_t^{\alpha_t}}$$

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Elementary divs $\Longrightarrow$ Inv. factors

Ex $|G| = 1800 = 2^3 \cdot 3^2 \cdot 5^2$

$$Z_8 \times Z_9 \times Z_{25} \simeq Z_{1800}$$

$$(\underset{\uparrow}{Z_4} \times \underset{\uparrow}{Z_{\underline{2}}}) \times \underset{\uparrow}{Z_9} \;{\uparrow} \times (\underset{\uparrow}{Z_5} \times \underset{\uparrow}{Z_5})$$

$$(\underset{\uparrow}{Z_4} \times \underset{''}{Z_9} \times \underset{\uparrow}{Z_5}) \times (Z_{\underline{2}} \times Z_{\underline{5}})$$

$$Z_{180} \times Z_{10} \qquad 10 \mid 180$$

In general

$$G = (Z_{p_1^{\alpha_{11}}} \times \dots \times Z_{p_1^{\alpha_{1n_1}}}) \times (Z_{p_2^{\alpha_{21}}} \times \dots \times Z_{p_2^{\alpha_{2n_2}}}) \times$$

$$\dots \times (Z_{p_t^{\alpha_{t1}}} \times \dots \times Z_{p_t^{\alpha_{tn_t}}})$$

$$n_1 = p_1^{\alpha_{11}} p_2^{\alpha_{21}} \cdots p_t^{\alpha_{t1}}$$

$$n_2 = p_1^{\alpha_{12}} p_2^{\alpha_{22}} \cdots p_t^{\alpha_{t2}}$$

$$\vdots$$

$$\alpha_{12} \leq \alpha_{11}$$

$$\Longrightarrow p_1^{\alpha_{12}} \mid p_1^{\alpha_{11}}$$

$$\cdots \rightarrow n_2 \mid n_1$$

Prop says $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s} \simeq G$.

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Inv Factor $\Longrightarrow$ Elementary Divs

Example

$(30, 30, 2)$ $\qquad$ $\mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_2$

$$\begin{aligned}
\mathbb{Z}_{30} &\simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
\mathbb{Z}_{30} &\simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
\mathbb{Z}_2 &\simeq \mathbb{Z}_2
\end{aligned}$$

$$\hookrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5)$$

**Rmk** Didn't require $G$ to be inv. factors, just a product of cyclic groups

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**General** $G \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_s}$

$$n_i = p_1^{\alpha_{i1}} \cdots p_t^{\alpha_{it}}$$

$G \simeq A_1 \times \dots \times A_t$ w/

$$A_j = \mathbb{Z}_{p_j^{\alpha_{1j}}} \times \dots \times \mathbb{Z}_{p_j^{\alpha_{sj}}} \Bigg\}$$ & reorder so exponents decrease

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**Upshot**

The 2 perspectives are equivalent.

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Uniqueness makes it easy to check if 2 abelian groups are isomorphic.

**Ex**

$\mathbb{Z}_6 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \leftarrow (\mathbb{Z}_3 \times \mathbb{Z}_2) \times \mathbb{Z}_5 \times \mathbb{Z}_2$

$= (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \times \mathbb{Z}_5$ ✓

$\mathbb{Z}_{30} \times \mathbb{Z}_2 \leftarrow (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5) \times \mathbb{Z}_2$

$\simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \times \mathbb{Z}_5$ ✓

$\mathbb{Z}_{60}$

$\uparrow$

$\boxed{\mathbb{Z}_4} \times \mathbb{Z}_3 \times \mathbb{Z}_5$ Different Sylow 2-sub.
Not isomorphic.

## How do you prove such a thing?

① Use Sylow's Theorems + Recognition Thm to reduce to groups of prime power order.

② Subtle induction argument to build the decomposition for abelian p-groups.

③ Uniqueness using Inv. Factor perspective & cancellation from HW8 #8.

Content is in TH2

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We will prove a special case of step 2 which should help induction go.

**Defⁿ** An elementary abelian p-group is an abelian p-group $A$ (of order $p^\alpha$) s.t. $\forall\ x \in A$, $|x| \leq p$.

**Prop** (Fund Thm: elementary case)

Let $E$ elementary abelian p-group of order $p^r$.

Then $$E \simeq \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r\text{ times.}}$$

Proof Induct on $r$

Base $r=1$ $E \cong \mathbb{Z}_p$ by TH 9. ✓

Inductive Step

$1 \neq x \in E$ so $|x| = p$.

Abelian $\Rightarrow$ $\langle x \rangle \trianglelefteq E$

Let $E' = E / \langle x \rangle$.

Notice ① $E'$ abelian.

② $|E'| = p^{r-1}$

③ $\underset{g \in E}{\bar{g} \in E'}$ $\bar{g}^p = \overline{g^p} = \bar{1} = 1$

So $E'$ elementary ab. $p$-group order $p^{r-1}$

Induction $E' \cong \underbrace{\mathbb{Z}_p \times \cdots \mathbb{Z}_p}_{r-1 \text{ times.}}$

Let $e_i$ generate $i$th factor

$e_i = (1, 1, \dots, 1, \underset{\uparrow_{i\text{'th position \& generator}}}{e_i}, 1, \dots, 1)$

"$E' = \langle e_1 \rangle \times \langle e_2 \rangle \times \cdots \times \langle e_{r-1} \rangle$"

Let $\pi : E \longrightarrow E'$ projection.

Surjective, so fix $y_i \in \pi^{-1}(e_i)$

**Define** $\phi : E' \times \langle x \rangle \longrightarrow E$

$$\phi(e_i) = y_i$$
$$\phi(x) = x$$

$\rightsquigarrow$ $\phi(e_1^{a_1}, e_2^{a_2}, \ldots, x^{a_r}) = y_1^{a_1} y_2^{a_2} \cdots y_{r-1}^{a_{r-1}} x^{a_r}$

$$E' \times \langle x \rangle \simeq \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r \text{ times}}$$

**Goal** Show $\phi$ bijective.

Since $|E' \times \langle x \rangle| = p^r = |E|$ suffice to show $\phi$ surjective.

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Fix $v \in E$.

Notice $\pi(v) = (e_1^{\alpha_1}, e_2^{\alpha_2}, \ldots, e_{r-1}^{\alpha_{r-1}}) \in E'$

Let $u = v\, y_1^{-\alpha_1} y_2^{-\alpha_2} \cdots y_{r-1}^{-\alpha_{r-1}}$

$$\pi(u) = (e_1^{\alpha_1}, \ldots, e_{r-1}^{\alpha_{r-1}})(e_1^{-\alpha_1}, \ldots, e_{r-1}^{-\alpha_{r-1}}) = 1$$

⇝ so $u \in \ker \pi \underset{\substack{\uparrow \\ \text{HW} \\ 4 \, \#8}}{=} \langle x \rangle$

So $u = x^{\alpha_r}$ some $\alpha_r$.

$$v = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{r-1}^{\alpha_{r-1}} x^{\alpha_r}$$

$$= \phi(e_1^{\alpha_1}, \dots, e_{r-1}^{\alpha_{r-1}} \cdot x^{\alpha_r}) \qquad ⊘$$

Abelian was important for $\phi$ to be a homomorphism.

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**Upshot**
If $A$ abelian group. If I can verify $|x| = p$ $\forall$ $x \neq 1$. Then $A$ is a product of cyclic groups.

verify $x^p = 1$ $\forall$ $x$

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Leaving land of purely abelian groups

**Rmk** Next big thm is important for TH2

# Recognizing Direct Products

$G = H \times K$.

Identify $H = H \times 1 \leq G$

$K = 1 \times K \leq G.$

H & K have nice properties:

① $H, K \trianglelefteq G.$

② $H \cap K = 1$ ← Pf/ $(h,1) \in H$ $\parallel$ $(1,k) \in K$ $h = 1 = k$

③ $HK = G$ ← $(h,k) = (h,1)(1,k) \in HK$ Ø

④ $h \in H$ & $k \in K$ then $hk = kh.$

## Question

Suppose $G$ is a group & I can find $H, K \leq G$ satisfying ①-④. Is $G \simeq H \times K$? A. yes. Even drop ④ It will follow...

## Lemma A

Let $H, K \leq G$. Fix $g \in HK$.

① The number of ways to write $g = hk$ for $h \in H$ & $k \in K$ is $|H \cap K|$

② If $H \cap K = \{1\}$ $g$ is written as $hk$ for $h \in H$ & $k \in K$ in a unique way. /

Proof ① ⇒ ② ✓ so prove ①.

Fix $g = h_0 k_0$ $\quad h_0 \in H, \; k_0 \in K.$

Build a bijection

$$\left\{\begin{array}{l}\text{pairs } h,k \text{ w/} \\ ①\, h \in H \quad k \in K \\ ②\, g = hk \end{array}\right\} \longleftrightarrow H \cap K$$

$$h_0 k_0 = g = hk \quad \xmapsto{\quad \Phi \quad} \quad x$$

$$h_0^{-1} h = k_0 k^{-1} = x$$

$$\begin{array}{l} h = h_0 x \\ k = x^{-1} k_0 \end{array} \quad \xleftarrow{\quad \Psi \quad}\!\!\dashv \; x$$

Then

$$\begin{aligned} hk &= h_0 x x^{-1} k_0 \\ &= h_0 k_0 = g. \end{aligned}$$

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$\Psi(\Phi(h,k))$ $\qquad x = h_0^{-1} h = k_0 k^{-1}$

$\Psi(x)$ $\qquad h_0 x = h_0 h_0^{-1} h = h$

$= (h_0 x, x^{-1} k_0) = (h,k)$ $\qquad x^{-1} k_0 = (k_0 k^{-1})^{-1} k_0 = k k_0^{-1} k_0 = k.$

Similar to check $\Phi \Psi(x) = x.$

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**Recall** (HW 6 #4)

Given $x, y \in G$. The commutator of $x$

and g is $[x, g] = x^{-1}g^{-1}xg \in G$

**Fact**

$x$ & $g$ commute $\iff [x, g] = 1$

Pf/ $xy = yx \cdot k$ & they commute $\iff k = 1$

$$\underset{\overset{\|}{[x,y]}}{x^{-1}y^{-1}xy} = k$$

**Lemma B**

Suppose $H, K \trianglelefteq G$. & $H \cap K = 1$

Then $hk = kh$ for all $h \in H$, $k \in K$.

**Proof**

Prove $[h, k] = 1$

① $H \trianglelefteq G$. So $k^{-1}hk \in H$

So $\underset{\overset{\|}{[h,k]}}{h^{-1}k^{-1}hk} \in H$

② $K \trianglelefteq G$ so $h^{-1}k^{-1}h \in K$

so $\underset{\overset{\|}{[h,k]}}{h^{-1}k^{-1}hk} \in K$

So $\begin{array}{c}[h,k] \\ \| \\ 1\end{array} \in H \cap K = 1$ $\emptyset$

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**Theorem** (Recognition of direct products)

Let $H, K \leq G$. & suppose

① $H, K \trianglelefteq G$

② $H \cap K = 1$

Then $HK \simeq H \times K$

If also

③ $HK = G$

Then $G \simeq H \times K$.

Recall 2nd iso
$H, K \trianglelefteq G$
$\Rightarrow HK \leq G$.

Proof

Define
$$\phi : H \times K \longrightarrow HK.$$
$$(h, k) \longmapsto hk$$

① Homomorphism.

$h_1, h_2 \in H$, $k_1, k_2 \in K$.

Then $\phi((h_1, k_1)(h_2, k_2)) = \phi(h_1 h_2, k_1 k_2)$

$$= h_1 h_2 k_1 k_2$$

$$= h_1 k_1 h_2 k_2 \qquad \text{Lemma B}$$

$$= \phi(h_1,k_1)\,\phi(h_2,k_2) \qquad \checkmark$$

② Injectivity is Lemma A.

$$\begin{array}{ccc} \phi(h_1,k_1) & = & \phi(h_2,k_2) \\ \| & & \| \\ h_1k_1 & = & h_2k_2 \end{array} \quad \Longrightarrow \quad \begin{array}{l} h_1 = h_2 \\ k_1 = k_2 \end{array}$$

③ Surjectivity is immediate

$$hk = \phi(h,k).$$

**Rmk** Getting around Lemma A. in finite case.

$$|HK| = \frac{|H|\cdot|K|}{|H\cap K|} = |H|\cdot|K| = |H\times K|$$

surj ⟹ inj.

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**Defn/Notation** (after [DF])

If $H,K \trianglelefteq G$ as in thm

$hk \in HK$ ← internal direct product.

$(h,k) \in H\times K$ ← external direct product.

By thm these are interchangeable

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**Rmk** (HW 8 #8)

$M \times N \simeq G \simeq M' \times N'$

& $M \simeq M'$ finite then $N \simeq N'$

**By thm**

$M, N, M', N' \trianglelefteq G$

& $M \cap N = 1$ $MN = G$ $G/N = M$ $G/M = N$

$M' \cap N' = 1$ $M'N' = G$ $G/N' = M'$ $G/M' = N'$

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**Examples** (HW4 #5c)

$\mathbb{Z}_{mn}$ w/ $(n, m) = 1$

$|x| = m$ $|y| = n$.

$H = \langle x \rangle \simeq \mathbb{Z}_m$ ($H \cap K = 1$ by Lagrange).

$K = \langle y \rangle \simeq \mathbb{Z}_n$.

$$\mathbb{Z}_{nm} = HK \simeq H \times K = \mathbb{Z}_m \times \mathbb{Z}_n$$

$$\langle x \rangle \times \langle y \rangle$$

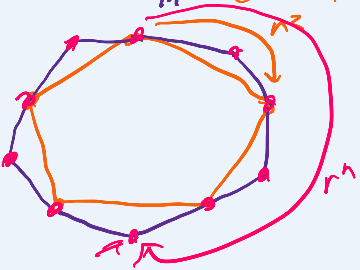
$$x^i y^j \longleftarrow\!\mapstochar\ (x^i, y^j)$$

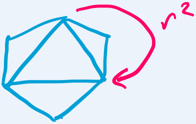
**Example**

Let $n$ be odd.

Then $D_{4n} \simeq D_{2n} \times \mathbb{Z}_2$

1) Embed $D_{2n}$ into $D_{4n}$ (cf HW7 #7).





$$D_{2n} \simeq \underset{\underset{H}{\shortparallel}}{\langle r^2, s\rangle} \le D_{4n} = \left\langle r, s \;\middle|\; \begin{array}{l} r^{2n} = s^2 = 1 \\ rs = sr^{-1} \end{array} \right\rangle$$

$$\mathbb{Z}_2 \simeq \langle r^n \rangle = K \le D_{2n}$$

① $H \trianglelefteq D_{4n}$ b/c $[D_{4n} : H] = \frac{4n}{2n} = 2$

② $K \underset{\text{HW4 \#4}}{=} Z(D_{4n}) \underset{\text{HW 6? 7?}}{\trianglelefteq} D_{4n}$

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$r^n \notin \underset{\{s^i r^{2j}\}}{\langle r^2, s\rangle}$ b/c $n$ odd

So $H \cap K = 1$

$$\begin{tikzcd}
H \arrow[r, phantom, "\lneq"] & HK \arrow[r, phantom, "\leq"] & D_{4n} \\
2n \arrow[u] & 4n \text{ by LG.} \arrow[u] & 4n \arrow[u]
\end{tikzcd}$$

So $D_{4n} = HK \simeq H \times K$

$\simeq D_{2n} \times \mathbb{Z}_2$

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Good luck with TH2 &

have a good break!!!!