March16

Tuesday, March 16, 2021 1:53 PM

Outlook

This week & first week after break:

$\rightsquigarrow$ Groups. (Direct / Semidirect products Classifications)

Then....

$\rightsquigarrow$ Rings

TH2 Out Friday

Due Tuesday

Direct Products

Setup G, H 2 group.

$$G \times H = \{(g,h) : g \in G,\ h \in H\}$$

Inclusion

$$\begin{tikzcd}
\iota_1 : G \arrow[r, hook] & G \times H \\
g \arrow[r, mapsto] & (g,1) \\
i_2 : H \arrow[r, hook] & G \times H \\
h \arrow[r, mapsto] & (1,h)
\end{tikzcd}$$

Exhibit G & H as normal subgroups of $G \times H$.

Rmk $G \trianglelefteq G \times H$

$H \trianglelefteq G \times H$ $\Big\}$ Identifying $G = G \times 1$

$g \in G \times H$

$(g,1)$

cf/ HW #8 $\quad G \simeq M \times N \simeq M' \times N'$

$\rightsquigarrow M, N, N', M' \trianglelefteq G$

Projection maps

$$\begin{aligned} \pi_1 : G \times H &\longrightarrow G \\ (g,h) &\longmapsto g \\ \pi_2 : G \times H &\longrightarrow H \\ (g,h) &\longmapsto h. \end{aligned}$$

surjective &

$\ker \pi_1 = \{(g,h) \mid g = 1\} = 1 \times H = H$

$\ker \pi_2 = G$

$\rightsquigarrow$

A cancellation law

$$\frac{G \times H}{G \times 1} \simeq H \qquad\qquad \frac{G \times H}{1 \times H} \simeq G$$

Rmk IS $\quad K \leq G$

$\rightsquigarrow K \leq G \times H \quad$ as $\quad K \times 1$

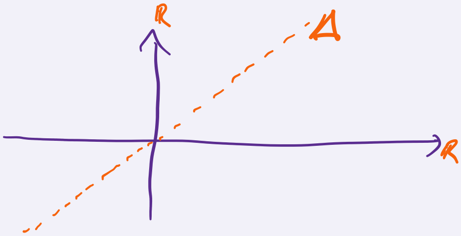
⚠ There could be other subgroups of $G\times H$ isomorphic to G. (Don't call them G).

Defⁿ The diagonal subgroup is

$$\begin{array}{ccc} \Delta & \le & G\times G \\ \shortparallel & & \\ \{(g,g) \mid g\in G\} & & \end{array}$$

$$\begin{tikzcd} G \arrow[r, "\cong"] & \Delta \\ g \arrow[r, mapsto] & (g,g) \end{tikzcd}$$

Ex $\Delta \le \mathbb{R}\times\mathbb{R}$



Defⁿ (Graph of a homomorphism)

Let $\phi : G \longrightarrow H$ a homomorphism.

Then the graph of ϕ is

$$\Gamma_\phi := \{(g,\phi(g)) : g\in G\} \le G\times H$$

Prop Γ_ϕ is a subgroup isomorphic to G.

Pf/
$$\begin{tikzcd} G \arrow[r, "\Phi"] & G \times H \\ g \arrow[r, mapsto] & (g, \phi(g)) \end{tikzcd}$$

This is an injective homomorphism.

$$\Phi(g_1 g_2) = (g_1 g_2, \phi(g_1 g_2))$$
$$= (g_1 g_2, \phi(g_1)\phi(g_2))$$
$$= (g_1, \phi(g_1))(g_2, \phi(g_2))$$
$$= \Phi(g_1)\Phi(g_2)$$

$$\begin{array}{c} \operatorname{im}\Phi \\ \wr\!\mid \\ G \end{array} = \Gamma_\phi \leq G \times H \qquad \varnothing$$

Ex $\Delta = \Gamma_{id: G \to G}$

Ex
$$\begin{tikzcd} \phi: \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/4\mathbb{Z} \\ 1 \arrow[r, mapsto] & 2 \end{tikzcd}$$

$$Z_2 \simeq \Gamma_\phi = \{(0,0), (1,2)\} \leq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

$$Z_2 \simeq \mathbb{Z}/2\mathbb{Z} = \{(0,0), (1,0)\} \leq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

$$\frac{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \simeq \mathbb{Z}/4\mathbb{Z}$$

Let $H = \langle (0,2) \rangle = \{(0,0),(0,2)\}$
$\le \mathbb{Z}/2 \times \mathbb{Z}/4\mathbb{Z}.$

Notice $H \simeq Z_2$

$$\frac{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}{H} = Z_2 \times Z_2$$

(pf/ $(a,b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

$2a = 0 \in \mathbb{Z}/2\mathbb{Z}$
$2b \in \langle 2 \rangle$
$\Rightarrow |\overline{(a,b)}| \le 2$ in $\frac{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}{H}$ $\emptyset$

Moral Although normal subgroups can be iso the quotients might not be.

If $H_1, H_2 \trianglelefteq G$ & $H_1 \simeq H_2$
not necessarily true
$G/H_1 \simeq G/H_2$. ←

HW 8 #8

If H_1, H_2 are finite $\trianglelefteq$ factors of G

($\exists\ K_1, K_2 \trianglelefteq G$ s.t. $H_1 \times K_1 \simeq G \simeq H_2 \times K_2$)

then true.

Question: Give $H \trianglelefteq G$. When does it have a "orthogonal complement" i.e. some $K \trianglelefteq G$ w/ $H \times K \simeq G$.

"recognizing direct products internally"

Example

Subs of order 3 in $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

- $\langle (1,0) \rangle = \{(0,0), (1,0), (2,0)\}$
- $\langle (0,1) \rangle = \{(0,0), (0,1), (0,2)\}$
- $\langle (1,1) \rangle = \{(0,0), (1,1), (2,2)\} = \Delta$
- $\langle (1,2) \rangle = \{(0,0), (1,2), (2,1)\} = \Gamma_{y=2x}$

Prop / Exercise

$\mathbb{Z}_p \times \mathbb{Z}_p$ has $p+1$ subgroups of order p.

$Z_p \times 1$ & $1 \times Z_p$

$$\begin{aligned} \phi_a : Z_p &\longrightarrow Z_p \\ x &\longmapsto x^a \end{aligned} \qquad \underline{a = 1, \dots, p-1}$$

Γ_{ϕ_a}

Philosophy

If we understand factors G & H we pretty much understand $G \times H$.

Prop $G_1 \simeq G_2$ & $H_1 \simeq H_2$

Then $G_1 \times H_1 \simeq G_2 \times H_2$

Pf (cf HW6 1(b))

$\phi : G_1 \xrightarrow{\sim} G_2$

$\psi : H_1 \xrightarrow{\sim} H_2$

$$\begin{aligned} \phi \times \psi : G_1 \times H_1 &\xrightarrow{\simeq} G_2 \times H_2 \\ (g, h) &\longmapsto (\phi(g), \psi(h)) \end{aligned}$$

① $\phi \times \psi$ homom.

② $(\phi \times \psi)^{-1} = \phi^{-1} \times \psi^{-1}$ $\varnothing$

Facts ($\exists$ bijections...)

$$\left\{ \begin{array}{l} \phi_1 : K \longrightarrow G_1 \\ \phi_2 : K \longrightarrow G_2 \end{array} \right\} \longleftrightarrow \left\{ \phi : K \longrightarrow G_1 \times G_2 \right\}$$

$$(\phi_1, \phi_2) \longrightarrow \phi_1 \times \phi_2 : k \longmapsto (\phi_1(k), \phi_2(k))$$

$$\phi_i = \pi_i \circ \phi \longleftarrow\!\shortmid\; \phi$$

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[dr, dashed, "\phi_i"'] & G_1 \times G_2 \arrow[d, "\pi_i"] \\ & G_i \end{tikzcd}$$

$$\{\psi : G_1 \times G_2 \to H\} \longleftrightarrow \left\{ \begin{array}{l} \psi_1 : G_1 \longrightarrow H \\ \psi_2 : G_2 \longrightarrow H \end{array} \right\}$$

$$\psi \longmapsto \begin{tikzcd} G_i \arrow[r, hook, "i"] \arrow[rr, bend right, "\psi_i"'] & G_1 \times G_2 \arrow[r, "\psi"] & H \end{tikzcd}$$

$$\psi(g_1, g_2) \longleftarrow\!\shortmid\; (\psi_1, \psi_2)$$

$$= \psi_1(g_1) \cdot \psi_2(g_2)$$

$\varnothing$

If we want to understand or classify groups, we can first try to decompose them into "direct product factors".

Next theorem says this works for finite abelian groups.

Theorem. (Fundamental theorem of finite abelian groups)

Let G be a finite abelian group.

Then

① $G \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$ s.t.

(invariant factor decomposition)

- (i) $n_i \geq 2$
- (ii) $n_{i+1} \mid n_i$

invariant factors

② The sequence $(n_1, n_2, \dots, n_s)$ is unique.

Ex G finite ab order 12

$G \simeq \mathbb{Z}_{12}$ or $G \simeq \mathbb{Z}_6 \times \mathbb{Z}_2$

Concerned

HW4 #5c

$\mathbb{Z}_4 \times \mathbb{Z}_3 \longleftarrow \simeq \mathbb{Z}_{12}$

$$\underbrace{\mathbb{Z}_3\times\mathbb{Z}_2}_{\mathbb{Z}_6}\times\mathbb{Z}_2 \underset{\substack{\uparrow\\ \text{HW 4 \#5c}}}{\longleftarrow} \simeq \mathbb{Z}_6\times\mathbb{Z}_2$$

Arguing as in red, easy to see that any product of cyclic groups can be put into some invariant factor form (①)

Consequences

① Classifies all finite abelian groups in terms of numerical/combinatorial data.

$$\underline{\left\{\begin{array}{l}\text{Abelian groups}\\ \text{of order}\\ \quad n\end{array}\right\}}\Big/\sim \quad\longleftrightarrow\quad \underset{\uparrow}{\left\{\begin{array}{l} n_1, n_2, \dots, n_s \quad \text{s.t.}\\ \text{① } n_i \geq 2\\ \text{② } n_{i+1} \mid n_i\\ \text{③ } n_1 n_2 \cdots n_s = n\end{array}\right\}}$$

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② Let $G \simeq \mathbb{Z}_{n_1}\times\dots\times\mathbb{Z}_{n_s}$ as in thm.

Let $p \mid |G|$.

Then $p \mid n_1$

Pf $\underset{\substack{\uparrow\\ =|G|}}{p \mid n_1 n_2 \cdots n_s}$ so $p \mid n_i$ some i.

But $n_i \mid n_{i-1} \mid n_{i-2} \mid \cdots \mid n_1$ $\varnothing$

③ If $|G| = p_1 p_2 \cdots p_r$ $\underline{\text{abelian}}$ w/ $p_i \neq p_j$ is a product of distinct primes (call this type of # square free)

Then G is cyclic.

Pf/ $G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$ as in Thm.

Then $p_i \mid n_1$ $\forall i$

so $n = \prod p_i \mid n_1$ ∅

so $n = n_1$

so $G = \mathbb{Z}_n$.

Example

Abelian groups of order $180 = 2^2 \cdot 3^2 \cdot 5$

$G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$

Know $2 \cdot 3 \cdot 5 \mid n_1$

① $n_1 = 2^2 \cdot 3^2 \cdot 5 = 180$

② $n_1 = 2 \cdot 3^2 \cdot 5 = 90$

③ $n_1 = 2^2 \cdot 3 \cdot 5 = 60$

④ $n_1 = 2 \cdot 3 \cdot 5 = 30$

① $G \simeq Z_{180}$

② $G \simeq Z_{90} \times Z_{n_2} \times \cdots \times Z_{n_s}$

& $90 n_2 n_3 \cdots n_s = 180$

$n_2 \cdots n_s = 2$

$\rightsquigarrow n_2 = 2$

$G \simeq Z_{90} \times Z_2$

③ $60 n_2 \cdots n_s = 180$

$n_2 \cdots n_s = 3$

$n_2 = 3$

$G \simeq Z_{60} \times Z_3$

④ $n_2 \cdots n_s = 6$

$n_2 = 6$ or 3 or 2.

$G = Z_{30} \times G_1$ $\qquad |G_1| = 6 \leftarrow$

$G_1 = Z_{n_2} \times \cdots \times Z_{n_s} \leftarrow$

But 2 & 3 $| n_2$

So $n_2 = 6$

$G = Z_{30} \times Z_6$

Classification: Ab. gps of order 180.

Inv factors	Grps	Largest order of elt
(180)	$\mathbb{Z}_{180}$	180
$(90, 2)$	$\mathbb{Z}_{90} \times \mathbb{Z}_2$	90
$(60, 3)$	$\mathbb{Z}_{60} \times \mathbb{Z}_3$	60
$(30, 6)$	$\mathbb{Z}_{30} \times \mathbb{Z}_6$	30

Invariant factor decompositions for abelian p-groups.

Ex

$|G| = p^3$

$G = \mathbb{Z}_{p^3}$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ $\quad \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

$(3) \qquad (2, 1) \qquad (1, 1, 1)$

These are partitions of 3.

$$\{\text{Partitions of } 3\} \Longleftrightarrow \{\text{Abelian groups of order } p^3\}$$

More Generally $\quad |G| = p^\alpha \quad G$ abelian

Fundamental Theorem

$$G \simeq Z_{p^{\beta_1}} \times Z_{p^{\beta_2}} \times \dots \times Z_{p^{\beta_s}}$$

① $p^{\beta_i} \geq 2 \Longleftrightarrow \beta_i \geq 1$

② $p^{\beta_{i+1}} \mid p^{\beta_i} \Longleftrightarrow \beta_i \leq \beta_{i+1}$

③ $\prod p^{\beta_i} = p^{\alpha} \Longleftrightarrow \sum \underset{\uparrow}{\beta_i} = \alpha$

This is exactly the data of a partition of alpha.

Invariant factor decomps for abelian p-groups are classified by partitions

If $G = p^{\alpha}$ $\leftarrow$

Inv Factors of p^{α}	Partitions of α
$(n_1, \dots, n_s)$	$(\beta_1, \dots, \beta_s)$
① $n_i \geq 2$	① $\beta_i \geq 1$
② $n_{i+1} \mid n_i$	② $\beta_{i+1} \leq \beta_i$
③ $\prod n_i = p^{\alpha}$	③ $\sum \beta_i = \alpha$

$$n_i = p^{\beta_i} \xrightarrow{\log_p} \log_p n_i = \beta_i$$

$$n_i = p^{\beta_i} \xleftarrow{p^x} \beta_i$$

Example

Abelian groups of order p^5

Partitions	Group
5	$\mathbb{Z}_{p^5}$
$4+1$	$\mathbb{Z}_{p^4} \times \mathbb{Z}_p$
$3+2$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$
$3+1+1$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p \times \mathbb{Z}_p$
$2+2+1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p$
$2+1+1+1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
$1+1+1+1+1$	$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

Useful both in proving the fundamental theorem & in computing all possibilities for larger #s to reduce to prime case where its just partitions!

Ex HW4 5(c)

$$\mathbb{Z}_{180} = \underset{2}{\underline{\mathbb{Z}_{2^2}}} \times \underset{2}{\underline{\mathbb{Z}_{3^2}}} \times \underset{1}{\underline{\mathbb{Z}_5}}$$

$\nearrow$

$$\mathbb{Z}_{90} \times \mathbb{Z}_2 = (\mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5) \times \mathbb{Z}_2$$

$$= \underset{1+1}{\underline{(\mathbb{Z}_2 \times \mathbb{Z}_2)}} \times \underset{2}{\underline{\mathbb{Z}_{3^2}}} \times \underset{1}{\underline{\mathbb{Z}_5}}$$

$\nearrow$

Same Sylow 3, 5 subs.

Different Sylow 2-subs

$$2, \cdot 2, \cdot 1 = 4$$

$$\left.\begin{array}{l} \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\ \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \end{array}\right\} \begin{array}{l} \mathbb{Z}_{30} \times \mathbb{Z}_6 \\ \\ \mathbb{Z}_{60} \times \mathbb{Z}_3 \end{array}$$

Connect

Theorem (Fundamental Theorem: Elementary Factor Form)

G abelian group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$

① $G \simeq A_1 \times A_2 \times \cdots \times A_s$ w/

$$|A_i| = p_i^{\alpha_i}$$

(**Rmk** An abelian group is isomorphic to a product of its Sylow subgroups)

② If $|A| = p^\alpha$

$$A \simeq \mathbb{Z}_{p_1^{\beta_1}} \times \dots \times \mathbb{Z}_{p_r^{\beta_r}} \qquad \beta_1 \geq \dots \geq \beta_r \geq 1, \quad \sum \beta_i = \alpha$$

③ This decomposition is unique

Ex Groups of order 1600

$= 2^3 \cdot 3^2 \cdot 5^2$

Thm $$G \simeq \underset{\text{order } 8}{\underset{\uparrow}{A_2}} \times \underset{\text{order } 9}{\underset{\uparrow}{A_3}} \times \underset{\text{order } 25}{\underset{\nwarrow}{A_5}}$$

Classify possibilities for $A_2, A_3, A_5 \dots$ partitions

	order	part	gp
		3	$\mathbb{Z}_8$
A_2	$2^3 = 8$	2 + 1	$\mathbb{Z}_4 \times \mathbb{Z}_2 \longleftarrow A_2$
		1 + 1 + 1	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$A_3 \quad 3^2 = 9 \quad \overset{2}{1 + 1} \quad \overset{Z_9}{Z_3 \times Z_3} \longleftarrow A_3$

$A_5 \quad 5^2 = 25 \quad \overset{2}{1 + 1} \quad \overset{Z_{25} \longleftarrow A_5}{Z_5 \times Z_5}$

$$G = (Z_4 \times Z_2) \times (Z_3 \times Z_3) \times Z_{25}$$