Jan19

Tuesday, January 19, 2021 1:54 PM

Math 113

Office Hours:

5:00 - 6:30 PM PT
or by appt.

GSI: TBA

HW due Fridays @ Noon

4 takehomes. Due Mondays @ 5PM.
(don't replace HW).

Try to be flexible w/ extensions.

Once solutions are posted won't accept.

No grainy Photos as homework!!!

Tour of websites: Discord & course page.

Philosophy & Motivation

Discovering Algebra numerically

* The notion of unity....

1

* The notion of distinction...

.... → differentiate & count.

$\{1, 2, 3, 4, \dots\} = \mathbb{N}$ = natural numbers.

Get addition $+$ $(+, \times)$

Get multiplication $\times$

Relationship $(a+b)\times c = a\times c + b\times c$

↑ distributive.

* Removal

⇝ subtraction.

what is $2-3$

get $\mathbb{Z}$ = integers

$= \{, \dots, -2, -1, 0, 1, 2, 3, \dots\}.$

$(\mathbb{Z}, +, \times, -)$

Nice properties for $+$

*i) Identity: 0 $\quad a + 0 = a$ any a

*ii) Inverse: $a + (-a) = 0$ any a.

* Division $\quad \boxed{2x = 3 \qquad x = ?}$

Formally add fractions to $\mathbb{Z}$.

Get $\mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \ b \neq 0 \}$

Hold for ×

*1) id = 1

*2) inverse of a is $\frac{1}{a}$

⇣ geometry

$\mathbb{R}$

⇣ Allow neg square roots

$\mathbb{C}$

Observe

At each step + & × gained formal properties which we can study.

Got 3 structures

$(\mathbb{Z}, +) \leftarrow$ Groups $\quad$ Set w/ + satisfying *1 *2

$(\mathbb{Z}, +, \times) \leftarrow$ Ring. Group + a distributive law.

$(\mathbb{Q}, +, \times) \leftarrow$ Field + & × satisfy $\underset{\substack{\uparrow \\ \text{id}}}{*1}$ & $\underset{\substack{\uparrow \\ \text{inverse.}}}{*2}$

Example

Add, subtract, multiply polynomials.

$$(x^2+5) + (2x-7) = x^2+2x-2$$

$$(x^2+5)(2x-7) = \ldots$$

Like $\mathbb{Z}$, sometimes divide.

$$\frac{(x^2-1)}{(x-1)} = (x+1)$$

Not always $\frac{(x^2-1)}{x^3-13} = ??$

$\dashrightarrow (\mathbb{Z}[x], +, \times)$ a ring.

↑

Not all examples are numerical.

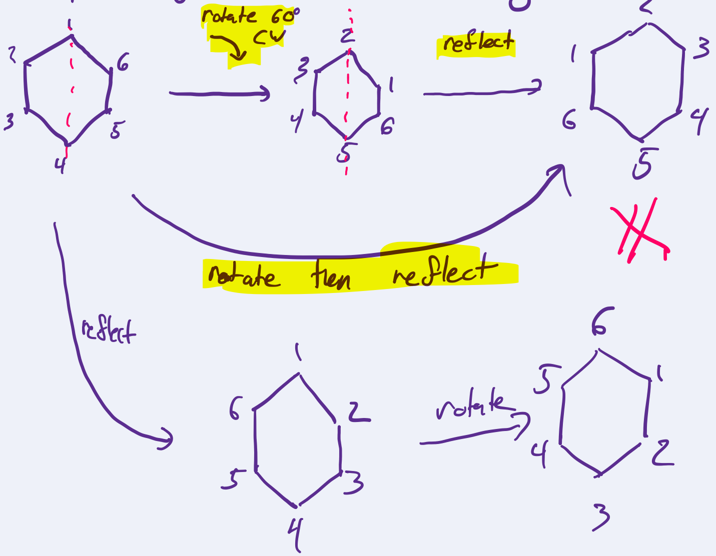
Remark

Most of our time on groups. ↙ one operation.

Turns out, numerical model is most simples ↖ $(\mathbb{Z}, +)$.

Symmetries

Example Symmetries of hexagon.



"Feels groupy"

Numbers ⟷ Symmetries of Hexagon.

Add $x+y$ ⟷ Compose $(f \circ g)$

"Nice Properties"

*1) Identity: Do nothing. ✓

*2) Inverse: Undo symmetry. ✓

You've probably seen this before... and it felt algebraic.

Linear Algebra.

Symmetries of $\mathbb{R}^n$ ⟷ $n \times n$ matrices over $\mathbb{R}$ & $\det \neq 0$

Composing symmetries ⟷ Multiplying Matrices.

Observe

$a, b \in \mathbb{Z}$

$a+b=b+a$ } commutative

M, N matrices often $MN \neq NM$.

Remark Although the axioms we will define will be abstract, in a sense they will be "obviously true" for symmetries.

Youtube video 3 blue 1 brown.
"groups & the monster"
I'll put it on course website.

A rigorous perspective

All the objects were a set with some extra structure

(A way to take 2 objects & get a third.

collection of objects

Satisfying nice properties

Example $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, \ldots\}$

$$+ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$(a, b) \longmapsto a + b.$$

Logic Notation

* P, Q be logical statements

$P \implies Q$ .... "P implies Q"

ex/ $(x=2) \implies x$ is even

* $P \iff Q$. P if & only if Q.

$$\left.\begin{array}{l} * \ P \implies Q \\ * \ Q \implies P \end{array}\right\}$$ Both hold.

* $\forall$ ← for all
* $\exists$ ← there exists
* $\exists!$ ← there exists a unique...

Sets

Defn

Loosely speaking a set is a collection of objects.

We define them using $\{ \ \}$.

EX/ $S = \{1, 2, 7\}$

$2 \in S$

$11 \notin S$

Notation

$\in$ ⟷ is an element of

$\notin$ ⟷ is not an elt of.

Often specify sets as follows

$S = \{ \text{Notation for elts} \mid \text{Properties specifying set} \}$

↑ such that.

*Ex/ $S = \{ x \in \mathbb{Z} \mid 2 \text{ divides } x \} = \{ \dots, -4, -2, 0, 2, 4, 6, \dots \}$

Notation S, T sets

$T \subseteq S \longleftrightarrow T$ is a subset of S.

$$\Longleftrightarrow (x \in T \Longrightarrow x \in S)$$

Ex/ $S \subseteq \mathbb{Z}$

Def^{n} (operations on sets). S, T sets.

Suppose $T \subseteq S$.

$S \setminus T$ = the complement of T in S

$$= \{ x \in S \mid x \notin T \}.$$

Ex $\mathbb{Z} \setminus S = \{ \dots, -1, 1, 3, 5, 7, \dots \}$.

Def^{n} (More Operations) S, T sets.

* Intersection

$$S \cap T = \{ x \mid x \in S \text{ and } x \in T \}$$

* Union

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\},$$

* Cartersian Product

$$S \times T = \{(x,y) \mid x \in S \ \& \ y \in T\}$$

Exercise

Let $T \subseteq S$. Describe the following sets. Prove you are correct.

1) $T \cap S$
2) $T \cup S$
3) $S \cap (S \setminus T)$
4) $S \cup (S \setminus T)$

Def^n / Notation (The empty set).

$\emptyset = \{\}$ = the empty set
= the set with no elements.

Functions

Def^n / Notation

Map.

Let S, T be sets. A (function) from S to T is a rule assigning to each element of S a unique element of T.

We use the following notation:

$$\begin{aligned} f : S &\longrightarrow T \\ x &\underset{\uparrow \text{ maps to.}}{\longmapsto} f(x) \end{aligned}$$

Or

$$S \xrightarrow{f} T$$

Examples

* $$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{N} \\ a &\longmapsto a^2 \end{aligned}$$

$(f(a) = a^2)$

* Addition is a function

Def^n

S, T sets. $f : S \longrightarrow T$ a map.

i) S is called domain of f. (or source).

2) T is called codomain of f.
(or target).

3) $S = T$ & $f(x) = x$ $\forall$ $x \in S$.
f is called the identity on S

$$f = \mathrm{id}_S$$

4) Suppose for all $x, y \in S$.

$$f(x) = f(y) \Longrightarrow x = y$$

equiv $\left(f(x) \neq f(y) \Longleftarrow x \neq y \right)$

we call f injective (one-to-one).

5) If for any $y \in T$, there exists some $x \in S$ s.t. $f(x) = y$,
f is called surjective (onto)

6) If f is injective & surjective it is called bijective.

Def^n (Composition) S, T, R sets.

$$f : S \longrightarrow T,$$

$g : T \longrightarrow R$.

The composition of g w/ f is

$$\begin{aligned} g \circ f : S &\longrightarrow R \\ x &\longmapsto g(f(x)) \end{aligned}$$

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[rr, bend right=40, "g \circ f"'] & T \arrow[r, "g"] & R \\
x \arrow[r, mapsto] & f(x) \arrow[r, mapsto] & g(f(x))
\end{tikzcd}$$

Def^n: Let $f : S \longrightarrow T$.

A function $g : T \longrightarrow S$ is

① A left inverse if $g \circ f = id_S$.

$$\begin{tikzcd}
S \arrow[r, "f"] & T \arrow[r, "g"] & S
\end{tikzcd}$$

$$s \longmapsto g(f(s)) = s$$

② A right inverse if $f \circ g = id_T$

$$\begin{tikzcd}
T \arrow[r, "g"] & S \arrow[r, "f"] & T
\end{tikzcd}$$

$$t \longmapsto f(g(t)) = t$$

③ An inverse if it is both a right & a left inverse.

Theorem (HW) $f: S \longrightarrow T$ is a function between nonempty sets.

① f is injective $\Longleftrightarrow$ $\exists$ a left inverse.

② f is surjective $\Longleftrightarrow$ $\exists$ a right inverse.

③ f is bijective $\Longleftrightarrow$ $\exists$ an inverse.

④ If an inverse to f exists, it is unique.

Example of part of pf

① (i) f inj. $\longrightarrow$ $\exists$ a left inverse.

(ii) $\exists$ left inverse $\Longrightarrow$ f injective.

↑

$g: T \longrightarrow S$ a left inverse.

Fix $x, y \in S$ s.t. $f(x) = f(y)$.

Then $\underset{\substack{\parallel \\ x}}{g(f(x))} = \underset{\substack{\parallel \\ y}}{g(f(y))}$

So $x = y$. Thus f injective.

$\varnothing$

$\boxed{\text{Rest HW}}$