Cayley's Thm

$|G|=n$. Then $\exists$ injective hom $\phi: G \longrightarrow S_n$.

So $G \simeq \phi(G) \leq S_n$

Tool Groups acting on themselves via multiplication.

$G \circlearrowleft G$ via $g \cdot a = ga$

$|G|=n$

$G=\{g_1, g_2, \dots, g_n\} \longrightarrow \{1,2,\dots,n\}=A$

1_G

$G \circlearrowleft G \rightsquigarrow G \circlearrowleft A$

$g \cdot g_i = g_j \rightsquigarrow g \cdot i = j$

$G \xrightarrow{\phi} S_A = S_n$

$g \longmapsto \sigma_g$

Lemma $\phi: G \longrightarrow S_n$

$g \longmapsto \sigma_g$

$g \cdot g_i = g_j \iff \sigma(i)=j$

Le ϕ is inj.

Pf/ $\phi(g)=\sigma_g \in S_n$

$g \in \ker\phi$

$\sigma_g(1)=1$

$\Downarrow$

$g \cdot g_1 = g_1$

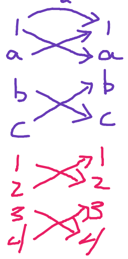
$\Rightarrow g = 1$ $\boxtimes$

Ex/ $V_4=\{1,a,b,c\} \longleftrightarrow \{1,2,3,4\}$

$a^2=b^2=c^2=1$

$ab=c$

σ_a



1) $1 \cdot a = a \leftarrow \sigma_a(1)=2$

2) $a \cdot a = 1 \leftarrow \sigma_a(2)=1$

3) $b \cdot a = c \leftarrow \sigma_a(3)=4$

4) $c \cdot a = b \leftarrow \sigma_a(4)=3$

$\sigma_a=(12)(34) \in S_4$

$\sigma_b=(13)(24)$

$\sigma_c=(14)(23)$

$\sigma_1=(1)$

So $\phi: V_4 \longrightarrow S_4$

$V_4 \simeq \phi(V_4)$

$=\{(1),(12)(34),(13)(24),(14)(23)\} \leq S_4$

Def $G \circlearrowleft A$ is transitive if it has only one orbit. i.e. $a,b\in A$, $\exists\, g$ w/ $b = g\cdot a$

Contruction $H \leq G$. $A = G/H = \{xH\}$ ← Not always a group.

Def $G \circlearrowleft A$

$g \cdot xH = gxH \in A$

1) $H=\{1\}$, this is $G \circlearrowleft G$ by mult

2) $|G:H|=n \Rightarrow$
$A=\{x_1H, x_2H, \dots, x_nH\}$

get $G \longrightarrow S_n$ $\quad \sigma_g(i)=j \iff g\cdot x_iH = x_jH$

Ex/ $G=D_8$ $\quad H=\{1,s\}$

$A=\{\underset{1}{H}, \underset{2}{rH}, \underset{3}{r^2H}, \underset{4}{r^3H}\}$

$\sigma_s=(24)$

1) $s\cdot H=H$ $\quad \sigma_s(1)=1$

2) $s\cdot rH = r^{-1}sH = r^3H$ $\quad \sigma_s(2)=4$

3) $sr^2H=r^2H$ $\quad \sigma_s(3)=3$

4) $sr^3H=rH$ $\quad \sigma_s(4)=2$

$\sigma_r=(1\ 2\ 3\ 4)$

$\phi: D_8 \longrightarrow S_4$

$\sigma_{r^2s}=\sigma_r\sigma_r\sigma_s$

. . . . . .

Thm $H\leq G$. $A=G/H$

$G \circlearrowleft A$

1) G acts transitively

2) $G_H=H$

3) $\pi_H: G \longrightarrow S_A$ perm rep

$\ker\pi_H = \bigcap_{x\in G} xHx^{-1}$

This is the largest normal subgroup of G in H

Pf/ ① $xH, yH \in A$

$g=yx^{-1}$

$gH = yx^{-1}xH = yH$ ✓

② $G_H=\{g\in G \mid g\cdot H=H\}$

$=H$

③ $\ker\pi_H=\{g \mid g\cdot xH = xH\ \forall x\}$

$=\{g \mid x^{-1}gxH=H\ \forall x\}$

$=\{g \mid x^{-1}gx\in H\ \forall x\}$

$=\{g \mid g\in xHx^{-1}\ \forall x\}$

$=\bigcap_{x\in G} xHx^{-1}$

1) Normal ✓

2) $g\in\ker\pi_H$
$\Rightarrow g\cdot H=H \Rightarrow g\in G_H=H$

$\ker\pi_H\leq H$

$N\leq H\leq G$ normal

$\forall x$

$N=xNx^{-1}\leq xHx^{-1}$

$N\leq\bigcap_x xHx^{-1}=\ker\pi_H$ $\varnothing$

Corollary $|G|=n$

$G \hookrightarrow S_n$.

Pf/ Let $H=\{1\}$

$G\longrightarrow S_n$

w/ $\ker\leq H=1$. $\varnothing$

Thm $|G|=n$. $p\mid n$ smallest prime dividing n. if $H\leq G$ w/ $|G:H|=p$, then $H\trianglelefteq G$

Pf/ $H\leq G$ index p.

$A=G/H=\{x_1H,\dots,x_pH\}$

Action $G\circlearrowleft A$

$\pi_H: G\longrightarrow S_p$

$K=\ker\pi_H\leq H$

$|H:K|=k \leftarrow 1$ done

so $|G:K|=|G:H||H:K|=p\cdot k$

$$\begin{tikzcd} G \arrow[r] \arrow[d] & S_p \\ G/K \arrow[ur, hook, "\text{injective}"'] & \end{tikzcd}$$

order G/K divides $|S_p|$

$\cancel{p}k \mid p! = \cancel{p}(p-1)\dots$

$k \mid (p-1)!$

All prime factors of $k < p$.

$\Rightarrow k=1$

So $|H:K|=1$

$\Rightarrow H=K=\ker\pi_H\trianglelefteq G$ $\varnothing$