Table of Stuff

$|G|=p$
$\Rightarrow G\simeq \mathbb{Z}_p$

$|G|=p^2$
$\Rightarrow G\simeq \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p\times\mathbb{Z}_p$

$|G|=pq$ $p<q$
$P,Q\le G$ $|P|=p$, $|Q|=q$
* $Q\trianglelefteq G$ ←
* If $P\trianglelefteq G \Rightarrow G\simeq \mathbb{Z}_{pq}$
* If $p\nmid q-1 \Rightarrow P\trianglelefteq G$

Left: $p\mid q-1$ & $P\ntrianglelefteq G$ ←

$|G|=30$
$\exists\ H\trianglelefteq G$ $H\simeq\mathbb{Z}_{15}$

$|G|=12$
Either * $\exists\ H\trianglelefteq G$ $|H|=3$
or * $G\simeq A_4$

$|G|=p^2q$ $p\neq q$
$P,Q\le G$ $|P|=p^2$ $|Q|=q$
* $p>q \Rightarrow P\trianglelefteq G$
* $q>p \Rightarrow$ either * $Q\trianglelefteq G$
or * $G\simeq A_4$

$|G|=60$
* $n_5>1 \Rightarrow G$ simple. $\Rightarrow G\simeq A_5$

Lemma A $|G|<\infty$
Let $H\trianglelefteq G$
$P\trianglelefteq H$ & $P\in Syl_p(G)$
$\Rightarrow P\trianglelefteq G$.
Pf/ Exercise.

Groups of order 60

Thm $|G|=60=2^2\cdot3\cdot5$
$n_5>1$. $\Rightarrow G$ simple.

Pf/ Suppose $\exists\ H\trianglelefteq G$
w/ $H\neq1$ & $H\neq G$.
$|H|=\{\cancel{1},\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},\cancel{10},\cancel{12},\cancel{15},\cancel{20},\cancel{30},\cancel{60}\}$

No normal subs order 5. ($n_5>1$) by Sylow

Lemma $5\nmid|H|$ ←
Pf/ $5\mid|H| \Rightarrow \exists\ P\in Syl_5(H)$
If $P\trianglelefteq H \Rightarrow P\trianglelefteq G$ by lemma
↯ so $P\ntrianglelefteq H$
$n_5(H)=\{1,6,11,\dots\}$
spose $=6$
$\Rightarrow 6$ gps order 5
each 4 elts order 5
$\Rightarrow$ 24 elts order 5 in H
$\Rightarrow |H|\ge25$
$\Rightarrow |H|=30$ $|Q|=5$
Get $Q\trianglelefteq\mathbb{Z}_{15}\trianglelefteq H$
Abelian gps have 1 Sylow p sub for all p
lemma $\Rightarrow Q\trianglelefteq H$
Lemma $\Rightarrow Q\trianglelefteq G$ ↯
∅

Spose $|H|=6$ or 12
$H=6=2\cdot3$
$Q\trianglelefteq H$ Lemma
$|Q|=3$ $\Rightarrow Q\trianglelefteq G$

$|H|=12$
either $Q\trianglelefteq H$ order 3
or $Q\trianglelefteq H$ order 4
$\underset{A_4}{\shortparallel}$

So suffice to show
$|H|\neq 2,3,4$.

To find contradiction
assume $|H|=2,3,4$
$\Rightarrow \bar G=G/H$
& $|\bar G|=30,20,15$

Claim In each case
$\bar P\trianglelefteq\bar G$ order 5

Pf/ $|\bar G|=30$
$P\trianglelefteq\mathbb{Z}_{15}\trianglelefteq\bar G$
↑Lemma. $P\trianglelefteq\bar G$ ✓

$|\bar G|=20=2^2\cdot5$
$p^2\cdot q$ $q>p$
$\Rightarrow Q\trianglelefteq\bar G$
↑order 5

$|\bar G|=15$. $\Rightarrow 3\cdot5$
$p=3\nmid4=5-1=q-1$
$G\simeq\mathbb{Z}_{15}\trianglerighteq P$ ← order 5
∅

Use 4th iso
$\bar P\trianglelefteq\bar G \longleftrightarrow H\le P\trianglelefteq G$
$P/H \longleftarrow P$
$\Rightarrow$ Get $P\trianglelefteq G$ w/ $\frac{|P|}{|H|}=|\bar P|=3$
$\Rightarrow 5\mid|P|$ ↯

Therefor A_5 simple.

Theorem
$|G|=60$ & G simple.
$\Rightarrow G\simeq A_5$.

Lemma B
$|G|=60$. G simple.
Suppose $\exists\ N\le G$ s.t.
$|G:N|=5$.
Then $G\simeq A_5$.

Pf of Lemma B
$N\le G$ w/ 5 cosets
$\{g_1N,g_2N,g_3N,g_4N,g_5N\}$
$=G/N$ (Transitive)
$G\curvearrowright G/N$ left mult.
$g * g_iN=gg_iN$
Perm rep
$\phi: G\longrightarrow S_5$
1) ϕ is injective
Pf/ $\ker\phi\trianglelefteq G$ ← simple
so $\ker\phi$ is 1 or $\cancel{G}$
∅

$G\simeq\phi(G)\le S_5$
order 60.
Identify $G\le S_5$
Notice $A_5\trianglelefteq S_5$
So use 2nd isom thm

$$\begin{array}{ccc} & S_5 & \\ & | & \\ & G\cdot A_5 & \\ G & & A_5 \\ & G\cap A_5 & \end{array}$$

Claim
$G=A_5$ ←
or
$G\cdot A_5=S_5$

Pf/ $A_5\lneq G\cdot A_5\le S_5$
A_5: 30; S_5: 60
$G\cdot A_5$: 30 .. 60 $G\neq A_5$
∅

Assume $G\cdot A_5=S_5$ ←
(else done)

$$\begin{array}{ccc} & S_5 & \\ G & & A_5 \\ & G\cap A_5 & \end{array}$$

2nd iso thm
$G\cap A_5$ index 2 in G
$\Rightarrow G\cap A_5\trianglelefteq G$ ↯
G simple ∅

To prove thm, suffices
to produce $N\le G$
of index 5

Proof of thm
$|G|=2^2\cdot3\cdot5$
$n_2=\{\cancel{1},\cancel{3},5,\cancel{7},\cancel{9},\cancel{11},\cancel{13},15\}$
G simple

If $n_2=5$
$P\in Syl_5(G)$
Sylows $\Rightarrow |G:N_G(P)|=n_2$
$=5$
So done by lemma!

$n_2=15$ $\exists$
Claim $P\neq Q\in Syl_2(G)$
$P\cap Q\neq1$
Pf/ Suppose not
Then 15 subgroups order 4 intersection is trivial.
Get $3\cdot15=45$ dist elts order 2 or 4
B/c G simple
$n_5>1 \Rightarrow n_5\ge6$
Each gives 4 elts order 5 $\Rightarrow$ 24 more
$45+24=69>60$ ↯

Fix $P\neq Q\in Syl_2(G)$
& $|P\cap Q|=2$
$G\neq N_G(P\cap Q)\supseteq P,Q$
$\shortparallel$
M
$4\mid|M|\mid60$
$\neq$ $\neq$
$\Rightarrow |M|=12$
$\Rightarrow |G:M|=\frac{60}{12}=5$ done Lemma B

$n_2=3$ $P\in Syl_2(G)$
$\Rightarrow \exists\ H=N_G(P)$
& $|G:H|=3$
$G\curvearrowright(G/H)$ w/ kernel K

$$\begin{array}{ccc} G & \longrightarrow & S_3 \\ \downarrow & \nearrow & \\ G/K & & \end{array}$$

Since G simple $K=1$
$\Rightarrow |G|=60\mid6$ ↯

RMK showed no $H\le G$
w/ $|G:H|\le4$ ∅