Table of Stuff we know

$|G| = p$

$G \simeq \mathbb{Z}_p$

$|G| = p^2$

$G \simeq \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$

$|G| = pq \quad p < q$

$\exists\, Q \trianglelefteq G \quad |Q| = q$

$\exists\, P \leq G \quad |P| = p$

* $P \trianglelefteq G \Rightarrow G \simeq \mathbb{Z}_{pq}$

* $p \nmid q-1 \Rightarrow$

* Left: $p \mid q-1$ & $P \ntrianglelefteq G$

$|G| = 45 \Rightarrow G$ abelian.

$|G| = 30$

$\Rightarrow \exists\, H \trianglelefteq G$ w/ $H \simeq \mathbb{Z}_{15}$

$\exists\, P \trianglelefteq G$ order 3

$Q \trianglelefteq G$ order 5

$|G| = 12$

Either ① $\exists\, P \trianglelefteq G \quad |P| = 3$

or ② $G \simeq A_4$

$|G| = p^2 q \quad p \neq q$

$\Rightarrow \exists\, H \trianglelefteq G$ order p^2 or q.

Groups of order 30

Prop $|G| = 30$ then $\exists\, H \trianglelefteq G$ w/ $H \simeq \mathbb{Z}_{15}$

Pf/ Claim

Suffice to produce $H \leq G$ w/ $|H| = 15$

Pf/ 1) $|G:H| = \frac{|G|}{|H|} = \frac{30}{15} = 2$

$\Rightarrow H \trianglelefteq G$

2) $|H| = 15 = \underset{p}{3} \cdot \underset{q}{5}$

$3 = p \nmid q-1 = 4$

$\Rightarrow H \simeq \mathbb{Z}_{3 \cdot 5} = \mathbb{Z}_{15}$ $\varnothing$

$|G| = 30 = 2 \cdot 3 \cdot 5$

$\Rightarrow P \in Syl_3(G) \leftarrow$ order 3

$Q \in Syl_5(G) \leftarrow$ order 5

$|PQ| = \frac{|P| \cdot |Q|}{|P \cap Q|} = \frac{3 \cdot 5}{1} = 15.$

$PQ = \{xy \mid x \in P, y \in Q\}$

Ok so $PQ = H$

If $P \trianglelefteq G$ or $Q \trianglelefteq G$

$\Rightarrow PQ \leq G$ & done

Suppose P, Q not normal

$n_3 = \{\cancel{1}, \cancel{4}, \cancel{7}, 10, \dots\}$

$n_5 = \{\cancel{1}, 6, \cancel{11}, \cancel{16}, \dots\}$

$n_3 \mid 10 \leftarrow \quad n_5 \mid 6 \leftarrow$

$|G| = 30$

10 subs order 3 P_i

All $P_i \ni x \neq$ are distinct order 3.

$\Rightarrow 20$ elts order 3

6 subs order 5

get $6 \cdot 4 = 24$ elts order 5

$\Rightarrow 44$ different elts in $G \leftarrow$ order 30 ↯

Remark

$\underset{3}{P}$ & $\underset{5}{Q}$ both Normal in G.

Pf/ $P \trianglelefteq PQ \trianglelefteq G$

$n_3 = \{1, 4, 7, \dots\}$

$n_3 \mid 5 \Rightarrow n_3 = 1$

$gPg^{-1} \leq PQ$ order 3

$\Rightarrow gPg^{-1} = P \quad \forall\, g \in G.$

$\Rightarrow P \trianglelefteq G.$ $\varnothing$

Groups of order 12

Prop $|G| = 12 = 2^2 \cdot 3$

Either ① G has a normal subgroup of order 3

② $G \simeq A_4 \trianglelefteq S_4$

Pf/ $n_3 = \{1, 4, \cancel{7}, \cancel{10}, \dots\}$

$n_3 \mid 4$

$n_3 = 1$ or 4

Case ①

Asume $n_3 = 4$

$Syl_3(G) = \{P_1, P_2, P_3, P_4\}$

$g \in G: \; gP_ig^{-1} = P_j$

$G \circlearrowright Syl_3(G)$ by conj

4 elts

$G \longrightarrow S_4$

Claim 1 $\quad Ker = 1 \trianglelefteq G$

Pf/ $gP_ig^{-1} = P_i \Leftrightarrow g \in N_G(P_i)$

$K = Ker = N_G(P_1) \cap N_G(P_2) \cap N_G(P_3) \cap N_G(P_4)$

$\Rightarrow K \leq N_G(P_1)$

& $|G : N_G(P_1)| = 4 = n_3$

$\underset{\text{order } 3}{P_1} \leq \underset{\text{order } 3}{N_G(P_1)} \leq G \quad \leftarrow \frac{\overset{12}{|G|}}{|N_G(P_1)|} = 4$

i.e. $P_1 = N_G(P_1)$

Also true for P_2, P_3, P_4

$\Rightarrow K = P_1 \cap P_2 \cap P_3 \cap P_4 = 1$ $\varnothing$

So $G \leq S_4$ order 12

$A_4 \leq S_4$ order 12

G has 4 Sylow p-subs $P_1, \dots, P_4$.

$\Rightarrow 2 \cdot 4 = 8$ elts order 3

$\sigma \in S_4$ has order 3

$\sigma = (a\ b\ c)$

$\Rightarrow sgn(\sigma) = 1$

So G & A_4 share 8 elements.

$|G \cap A_4| \geq 8$ & divides k

$\Rightarrow = 12$

$\Rightarrow G = A_4$ $\varnothing$

Rmk $45 = 3^2 \cdot 5$

$12 = 2^2 \cdot 3$

Groups order $p^2 q$ w/ $p \neq q$.

Prop

$|G| = p^2 q \quad p \neq q$ primes.

Then G has a normal Subgroup of order p^2 or q.

Pf/ $P \in Syl_p(G)$ order p^2

$Q \in Syl_q(G)$ order p.

Case 1 $\quad p > q \Rightarrow P \trianglelefteq G$

Pf/ $n_p \mid q$

& $n_p = 1 + kp \overset{k \neq 0}{>} p > q$ ↯

So $k = 0 \quad n_p = 1$ $\varnothing$

Case 2 $\quad q > p$.

If $n_q = 1$ done ✓

Else $n_q = 1 + kq > q > p$

& $n_q \mid p^2$ so

$n_q = \{\cancel{1}, \cancel{p}, p^2\}$

$n_q = p^2 = 1 + kq$

So $kq = p^2 - 1 = (p-1)(p+1)$

So $q > p$ divides $\cancel{p-1}$ or $p+1$

$q = p+1$

$p = 2$ & $q = 3$

so $p^2 q = 12$

Either $Q \trianglelefteq G$

or $G \simeq A_4 \trianglerighteq V_4$

$\varnothing \quad \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \simeq V_4$

Groups of order 60

Prop

$|G| = 60 = 2^2 \cdot 3 \cdot 5.$

Suppose $n_5 > 1$

Then G is simple.

Pf next time.

Corollary

A_5 is simple.

Pf/ $\langle (1\ 2\ 3\ 4\ 5) \rangle \neq \langle (1\ 2\ 3\ 5\ 4) \rangle$

Done $\varnothing$