Lemma 1

H, K groups coprime order.

$\Rightarrow Aut(H \times K) \simeq Aut H \times Aut K$

Pf/ HW.

Lemma 2

H, K, L 3 groups.

$K \longrightarrow Aut\, H$

$K \curvearrowright L$ trivially.

Get an action $K \curvearrowright L \times H$

$k * (l, h) = (l, k \cdot h)$

$(L \times H) \rtimes_* K \xrightarrow{\sim} L \times (H \rtimes_\phi K)$

$((l,h),k) \longmapsto (l, (h,k))$

Pf/ Exercise

Groups of order 30

$|G| = 30$

$\exists\ H \trianglelefteq G \quad H \simeq \mathbb{Z}_{15}$

Cauchy $\Rightarrow \exists\ K \leq G$ w/ $|K| = 2$.

Then $H \cap K = 1$

$HK = G$

$\Rightarrow G \simeq H \rtimes K$

$\simeq \mathbb{Z}_{15} \rtimes_\phi \mathbb{Z}_2$

classify all ϕ.

Maps

$\phi : \mathbb{Z}_2 \longrightarrow Aut(\mathbb{Z}_{15})$

$Aut(\mathbb{Z}_{15}) = Aut(\mathbb{Z}_3 \times \mathbb{Z}_5)$

$= Aut\,\mathbb{Z}_3 \times Aut\,\mathbb{Z}_5$

$\simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.

Study maps

$\phi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/4$

$\langle x \rangle \longmapsto (a, b)$

double to 0

1) $(0,0)^{\phi_1}$ 3) $(0,2)^{\phi_3}$

2) $(1,0)^{\phi_2}$ 4) $(1,2)^{\phi_4}$

ϕ_1 = trivial map

$\mathbb{Z}_{15} \rtimes_{\phi_1} \mathbb{Z}_2 = \mathbb{Z}_{15} \times \mathbb{Z}_2$

$= \mathbb{Z}_{30}$.

$\mathbb{Z}_{15} = \mathbb{Z}_3 \times \mathbb{Z}_5 = \langle x \rangle \times \langle y \rangle$.

ϕ_2 $(1,0)$: $x \overset{\iota}{\longmapsto} x^{-1}$, $y \longmapsto y$

ϕ_3 $(0,2)$: $x \longmapsto x$, $y \overset{\iota}{\longmapsto} y^{-1}$

ϕ_4 $(1,2)$: $x \overset{\iota}{\longmapsto} x^{-1}$, $y \overset{\iota}{\longmapsto} y^{-1}$

What are these?

$\phi_4 : x \longmapsto \iota$

$\mathbb{Z}_{15} \rtimes_{\phi_4} \mathbb{Z}_2 \simeq D_{30}$

$\phi_2 : x \longmapsto (\iota, id)$

$(\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes_{\phi_2} \mathbb{Z}_2$

$= (\mathbb{Z}_5 \times \mathbb{Z}_3) \rtimes_{\phi_2} \mathbb{Z}_2$

$= \mathbb{Z}_5 \times (\mathbb{Z}_3 \rtimes_\iota \mathbb{Z}_2)$ Lemma 2

$= \mathbb{Z}_5 \times D_6$

$\phi_3 : x \longmapsto (id, \iota)$

$(\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes_{\phi_3} \mathbb{Z}_2$

$= \mathbb{Z}_3 \times (\mathbb{Z}_5 \rtimes_\iota \mathbb{Z}_2)$

$= \mathbb{Z}_3 \times D_{10}$

Easy to check

$|Z(D_{30})| = 1$

$|Z(\mathbb{Z}_5 \times D_6)| = 5$

$|Z(\mathbb{Z}_3 \times D_{10})| = 3$

All different

Groups of order p^3

Tool: $x \longmapsto x^p$.

Rmk: not generally a homom if G not abelian

Difference btwn $(xy)^p$ & $x^p y^p$

Lemma 3 G any gp.

Suppose $x, y \in G$ s.t.

* $x[x,y] = [x,y]x$
* $y[x,y] = [x,y]y$

Then $(xy)^n = x^n y^n [y,x]^{\frac{n(n-1)}{2}}$

Pf/ Induction

Base $n=1$

$xy = xy$ ✓

General

$(xy)^n = (xy)^{n-1}(xy)$

$= x^{n-1} y^{n-1} [y,x]^{\frac{(n-1)(n-2)}{2}} xy$

$= x^{n-1} y^{n-1} x y [y,x]^{\frac{(n-1)(n-2)}{2}}$

$= x^{n-1} (x y^{n-1} [y,x]^{n-1}) y [y,x]^{\frac{(n-1)(n-2)}{2}}$

$= x^n y^n [y,x]^{\frac{n(n-1)}{2}}$ $\varnothing$

Lemma 4

G nonabelian $|G| = p^3$

① $G/Z(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

② $Z(G) \simeq \mathbb{Z}_p$. ✓

Pf/ Class equation

$Z(G) \neq 1$

$|Z(G)| = p, \cancel{p^2}, \cancel{p^3}$ not abelian

$|G/Z(G)| = p \Rightarrow$ cyclic

$\Rightarrow G$ abelian ↯

So $|Z(G)| = p$

So $|G/Z(G)| = p^2$

So $G/Z(G) = \cancel{\mathbb{Z}_{p^2}}$ ← G not abelian

$\mathbb{Z}_p \times \mathbb{Z}_p$

$\varnothing$

Lemma 5

G nonab order p^3

$[G,G] \leq Z(G)$.

Pf/ $G/Z(G)$ ← abelian.

$\Rightarrow [G,G] \leq Z(G)$

(Fact about commutators)

$\varnothing$

Lemma 6

G non ab order p^3

$x, y \in G$.

$[x,y]^p = 1$

Pf/ $[x,y] \leq Z(G) \simeq \mathbb{Z}_p$

$\varnothing$ ⑤ ④

Lemma 7

G nonabelian order p^3 for p odd prime.

$\phi : G \longrightarrow G$

$x \longmapsto x^p$

is a homomorphism

& $\operatorname{im}\phi = G^p \leq Z(G)$.

Pf/ $(xy)^p \overset{⑤⑥}{=} x^p y^p [y,x]^{\frac{p(p-1)}{2}}$

$\underset{p \text{ odd}}{=} x^p y^p ([y,x]^p)^{\frac{p-1}{2}}$

$\overset{⑥}{=} x^p y^p$ ✓

Why is $G^p \leq Z(G)$

$x \in G^p \rightsquigarrow x = y^p$

Look @ $\bar{y} \in G/Z(G)$

$\overset{④}{\simeq} \mathbb{Z}_p \times \mathbb{Z}_p$

$\Rightarrow \bar{y}^p = \bar{1}$

$\Rightarrow y^p \in Z(G)$

$\| \; x$

$\varnothing$

Groups order p^3

(p odd) (nonabelian)

Let $G_p = \ker\phi$

$= \{x \in G \mid x^p = 1\}$

$G/G_p \simeq G^p \leq Z(G)$ ↑ order p

p^3

$|G| / |G_p| \;\big|\; p$

$|G_p| = p^2$ or p^3

① $\exists\ x \notin G_p$, $|x| = p^2$

② $x \neq 1$, $|x| = p$

Case 1: $|x| = p^2$

$H = \langle x \rangle$.

$[G:H] = p$ ← smallest prime div $|G|$.

$\Rightarrow H \trianglelefteq G$

$(x^p)^p = 1 \Rightarrow x^p \in G_p$

$\underset{p}{\langle x^p \rangle} \leq \underset{p}{G_p \cap H} \not\leq \underset{p^2}{H}$

$|G_p| = p^2$ $\quad y \in G_p \setminus \langle x^p \rangle$

$K = \langle y \rangle$

Notice $H \cap K = 1$

$HK = G$

$G \simeq H \rtimes K$.

Find

$$\begin{array}{ccc} \phi_i : K & \longrightarrow & Aut\,H \\ \wr\!\! \mid & y \longmapsto \gamma^i & \wr\!\! \mid \\ \mathbb{Z}_p & & \mathbb{Z}_{p(p-1)} \\ \wr\!\! \mid & & \vee\!\mid \\ \langle y \rangle & & \langle \gamma \rangle \end{array}$$

unique ← order p

Let $G_i = H \rtimes_{\phi_i} K$

$i = 0$ $\quad G_0 = H \times K = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$

else $G_i \simeq G_j$

By HW

$\exists$ unique nontrivial $\mathbb{Z}_{p^2} \rtimes_\phi \mathbb{Z}_p$.

Rmk

$\gamma : \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_{p^2}$

$x \longmapsto x^{1+p}$.

$\Rightarrow \gamma \in Aut(\mathbb{Z}_{p^2})$ order p

$\Rightarrow$

$G = \langle x, y \mid x^{p^2} = y^p = 1,\ yxy^{-1} = x^{1+p} \rangle$

Point out

$p = 2$ get D_8.

Case 2

$G_p = G$ $\quad \forall x$ have $x^p = 1$.

$x \in G \quad x \neq 1$

$y \in G \setminus \langle x \rangle$

$H = \langle x, y \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

Pick $z \in G \setminus H$

$K = \langle z \rangle$.

$H \trianglelefteq G$. $HK = G$, $H \cap K = 1$

So $G \simeq H \rtimes K$

Classified by

$K \longrightarrow Aut\,H$

nontriv

$\mathbb{Z}_p \longrightarrow GL_2\mathbb{F}_p$

$z \longmapsto \gamma$ ← order p

Recall HW 9

$T = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle \leq GL_2\mathbb{F}_p$

is sylow p sub.

But $\langle \gamma \rangle$ is too

$\Rightarrow \exists\ \alpha \in GL_2\mathbb{F}_p$

s.t.

$\alpha T \alpha^{-1} = \langle \gamma \rangle$.

& $\alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \alpha^{-1} = \gamma^k$.

Let $\psi : \mathbb{Z}_p \longrightarrow GL_2\mathbb{F}_p$

$z \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Then

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r, "\psi"] \arrow[d, "\wr"] & GL_2\mathbb{F}_p \arrow[d, "\sigma_\alpha"] \\ \mathbb{Z}_p \arrow[r, "\phi"] & GL_2\mathbb{F}_p \end{tikzcd}$$

$x \longmapsto x^k$

HW 11

$\Rightarrow H \rtimes_\psi K \simeq H \rtimes_\phi K$.

unique nontrivial

$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$

Table of Stuff

$|G| = p$

$\boxed{G \simeq \mathbb{Z}_p}$

$|G| = p^2$

$\boxed{G \simeq \mathbb{Z}_{p^2} \text{ or } \mathbb{Z}_p \times \mathbb{Z}_p}$

Yellow Box = Complete Classification

$|G| = pq$

$P \in Syl_p$, $Q \in Syl_q$

* G abelian $\Rightarrow$ $\boxed{G \simeq \mathbb{Z}_{pq}}$
* $Q \trianglelefteq G$
* If $P \trianglelefteq G$
* If $p \nmid q-1$
* If $p \mid q-1$

$\boxed{G \simeq \mathbb{Z}_{pq} \text{ or } \mathbb{Z}_q \rtimes \mathbb{Z}_p}$ only 1 of these

$|G| = 12$

$P \in Syl_2$, $Q \in Syl_3$

* Either $Q \trianglelefteq G$ or $G \simeq A_4$

$\boxed{\begin{array}{l} * \text{ Abelian: } \mathbb{Z}_{12},\ \mathbb{Z}_6 \times \mathbb{Z}_2 \\ * \text{ Non Ab: } A_4,\ D_{12},\ \mathbb{Z}_3 \rtimes \mathbb{Z}_4 \end{array}}$

$|G| = 30$

* $\exists\ H \trianglelefteq G$ w/ $H \simeq \mathbb{Z}_{15}$

$\boxed{\begin{array}{l} * \text{ Abelian: } \mathbb{Z}_{30} \\ * \text{ Nonab: } D_{30} \\ \quad \mathbb{Z}_5 \times D_6 \\ \quad \mathbb{Z}_3 \times D_{10} \end{array}}$

$|G| = p^2 q$ $\quad p \neq q$

$P \in Syl_p$, $Q \in Syl_q$

* $p > q \Rightarrow P \trianglelefteq G$
* $q > p$ either $\rightarrow Q \trianglelefteq G$ or $\rightarrow G \simeq A_4$
* Abelian: $\mathbb{Z}_{p^2 q}$, $\mathbb{Z}_{pq} \times \mathbb{Z}_p$

$|G| = 60$

* $n_5 > 1 \Rightarrow G$ simple $\Rightarrow G \simeq A_5$
* Abelian: $\mathbb{Z}_{60}$, $\mathbb{Z}_{30} \times \mathbb{Z}_2$.

$|G| = p^3$ $\quad p$ prime

* Abelian: $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
* $p \neq 2$
  Nonab: $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \rtimes \mathbb{Z}_p$