Def^n H, K. $\phi: K \to \mathrm{Aut}(H)$

Form $H \rtimes_\phi K = \{(h,k) \mid h \in H, k \in K\}$

$(a,b)(c,d) = (a\, b\cdot c, bd)$

Recall

* $H \rtimes K \simeq H \times K \iff \phi \equiv 1$.
* $h \in H$, $k \in K$ $\leq H \rtimes K$

Then $\underbrace{khk^{-1}}_{\in H} = k \cdot h$

Prop If $\phi \neq 1$

Then $H \rtimes K$ not abelian

Pf/ $\phi \neq 1$ $\exists\, k \in K$, $h \in H$ w/ $k \cdot h \neq h$

In $H \rtimes K$

$khk^{-1} \neq h$

i.e. $kh \neq hk$ ∅

Remarks: Get a vast source of new interesting groups. Before

1) Cyclic gps & products
2) Groups from geometry
   * D_{2n}
   * $GL_n(F)$, $SL_n(F)$, T, $\bar{T}$
3) Permutations
   * S_n, A_n
4) Multiplication Tables (small gps).
   * Q_8

W/ semidirect product get interesting & new groups to study.

Semidirect Products as a tool for classification

Outline of argument

G a group.

* Use Sylow's thms to produce $P \trianglelefteq G$, $Q \leq G$
* Use counting $PQ = G$
* Use Lagrange $\Rightarrow P \cap Q = 1$

Then

$G \simeq P \rtimes_\phi Q$

Reduced

(All such G)

$\Updownarrow$

(Maps $\phi: Q \to \mathrm{Aut}(P)$)

↑ Easier in practice. ∅

First let's look @ a bunch of examples!

Examples

① $\mathbb{Z}_2$ extensions of Abelian groups.

A - abelian group.

$\phi: \mathbb{Z}_2 \to \mathrm{Aut}\, A$

$1 \mapsto id$

$x \mapsto \iota$

inversion $\iota(a) = a^{-1}$

$\iota(ab) = (ab)^{-1} = b^{-1}a^{-1}$
$= a^{-1}b^{-1}$
$= \iota(a)\iota(b)$.

& $\iota(\iota(a)) = \iota(a^{-1}) = (a^{-1})^{-1} = 1$

Form $A \rtimes \mathbb{Z}_2$

② Do this for $A = \mathbb{Z}_n$

Claim

$\mathbb{Z}_n \rtimes \mathbb{Z}_2 \simeq D_{2n}$

Pf/ $\mathbb{Z}_n = \langle x \rangle$ $\mathbb{Z}_2 = \langle y \rangle$

$D_{2n} \xrightarrow{\sim} \mathbb{Z}_n \rtimes \mathbb{Z}_2$

$r \mapsto x$

$s \mapsto y$

$\phi(sr^i) = yx^i$

Hom: $x^n = 1$

$y^2 = 1$

$yxy^{-1} = y \cdot x = x^{-1}$ ∅

③ A = abelian (Generalized extns)

$\mathbb{Z}_{2n} \to \mathrm{Aut}\, A$

$\langle x \rangle$, $x \mapsto \iota$

(action $x^i \cdot a = \begin{cases} a & i = \text{even} \\ a^{-1} & i = \text{odd} \end{cases}$)

Form $G = A \rtimes \mathbb{Z}_{2n}$

For any $a \in A$

$xax^{-1} = a^{-1}$

$x^2 a x^{-2} = a$

x^2 commutes w/ a $\forall x^i$

$\Rightarrow x^2 \in Z(G)$.

④ Do this w/ $A = \mathbb{Z}_3$, $\mathbb{Z}_{2n} = \mathbb{Z}_4$

Get $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ nonabelian order 12

Seen 2 already D_{12}, A_4.

Prop: A_4, D_{12}, $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ are nonisomorphic

Pf/ **Sylow 2 subs:**

A_4: $\langle (12)(34), (13)(24) \rangle \trianglelefteq A_4$, $\cong V_4$.

D_{12}: $n_2 = 3$ $(12 = 2^2 \cdot 3)$

$P = \{1, r^3, s, sr^3\} = V_4$ HW4 #5

~~$\mathbb{Z}_4 \rtimes \mathbb{Z}_3 \simeq G$~~

~~$\mathrm{Syl}_2 = \{\mathbb{Z}_4 \trianglelefteq G\}$~~

~~Only 1 $\neq D_{12}$~~

~~Not V_4 $\neq A_4$.~~ ∅

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

Show $\mathrm{Syl}_2 \supseteq \{\mathbb{Z}_4\}$

(not A_4)

(D_4) ∅

⑤ **Inner Semidirect Products**

$G \circlearrowright G$ by conjugations

Giving $G \to \mathrm{Aut}(G)$.

Get $G \rtimes G$

⑥ **Holomorphs**

H any group

$K = \mathrm{Aut}\, H$

$id: K \to \mathrm{Aut}\, H$

Form $H \rtimes K$

$= H \rtimes \mathrm{Aut}(H) =: \mathrm{Hol}(H)$

HW: $\mathrm{Hol}(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_4$.

Remark

Every automorphism is conjugation in some group.

⚠ Not saying every automorphism is inner.

G gp. $\mathfrak{G} = G \rtimes \mathrm{Aut}\, G$

$G, \mathrm{Aut}\, G \leq \mathfrak{G}$

$\phi \in \mathrm{Aut}\, G$

in G: $g \mapsto \phi(g)$ | In $\mathfrak{G}$: $g \mapsto \phi g \phi^{-1}$

Groups of order pq $p<q$

$|G| = pq$, $P \in \mathrm{Syl}_p$, $Q \in \mathrm{Syl}_q$.

Sylow $\Rightarrow Q \trianglelefteq G$.

Lagrange $\Rightarrow P \cap Q = 1$

& $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{p \cdot q}{1} = |G|$.

$\Rightarrow PQ = G$.

Thm from last time

$\Rightarrow G \simeq Q \rtimes P$.

Notice

$P \trianglelefteq G \Rightarrow G \simeq Q \times P = \mathbb{Z}_q \times \mathbb{Z}_p = \mathbb{Z}_{pq}$

Groups $Q \rtimes P$ are given by maps

$P \longrightarrow \mathrm{Aut}(Q)$

$\|$

$\mathbb{Z}_p \xrightarrow{\phi} \mathrm{Aut}(\mathbb{Z}_q) \simeq (\mathbb{Z}/q\mathbb{Z})^\times$ ↑ order $q-1$

If $p \nmid q-1$

$\Rightarrow \phi$ trivial

$\Rightarrow G \simeq \mathbb{Z}_q \times \mathbb{Z}_p \simeq \mathbb{Z}_{pq}$.

Assume $p \mid q-1$

Fact p prime

$\Rightarrow (\mathbb{Z}/p\mathbb{Z})^\times \simeq \mathbb{Z}_{p-1}$

Reduced to $p \mid q-1$

Classifying maps

$\mathbb{Z}_p = \langle x \rangle \longrightarrow \mathbb{Z}_{q-1} \geq H = \langle \gamma \rangle$ $\exists!$ sub order p

$\phi_i(x) = \gamma^i$ $i = 0, \dots, p-1$

p maps

Get p groups

$G_i = \mathbb{Z}_q \rtimes_{\phi_i} \mathbb{Z}_p$.

rmk $\phi_0 \equiv 1$

so $G_0 = \mathbb{Z}_q \times \mathbb{Z}_p = \mathbb{Z}_{pq}$

$p-1$ nontrivial maps

G_i - nonabelian.

Claim: $i, j \neq 0$

$G_i \simeq G_j$.

Consequences:

≤ 2 gps order pq $\forall$ $p<q$ pr.

1 if $p \nmid q-1$

2 if $p \mid q-1$

↳ $\mathbb{Z}_{pq}$ or $\mathbb{Z}_q \rtimes \mathbb{Z}_p$.

Table of Stuff

$|G| = p$

$\boxed{G \simeq \mathbb{Z}_p}$

$|G| = p^2$

$\boxed{G \simeq \mathbb{Z}_{p^2} \text{ or } G \simeq \mathbb{Z}_p \times \mathbb{Z}_p}$

Key: yellow box = Complete classification

$|G| = pq$ $P \in Syl_p$ $Q \in Syl_q$

* G abelian $\Rightarrow G \simeq \mathbb{Z}_{pq}$
* $Q \trianglelefteq G$
* $P \trianglelefteq G \Rightarrow G$ abelian
* $p \nmid q-1 \Rightarrow P \trianglelefteq G$

$|G| = 30$

* $\exists\, H \trianglelefteq G$ w/ $H \simeq \mathbb{Z}_{15}$
* Abelian groups: $\mathbb{Z}_{30}$

$|G| = p^2 q$ $p \neq q$ $P \in Syl_p$, $Q \in Syl_q$

* $p > q \Rightarrow P \trianglelefteq G$
* $q > p \Rightarrow$ Either $\to Q \trianglelefteq G$ / $\to G \simeq A_4$
* Abelian: $\mathbb{Z}_{p^2 q}$, $\mathbb{Z}_{pq} \times \mathbb{Z}_p$

$|G| = 60$

* $n_5 > 1 \Rightarrow G$ simple $\Rightarrow G \simeq A_5$
* Abelian groups: $\mathbb{Z}_{60}$ $\mathbb{Z}_{30} \times \mathbb{Z}_2$

$|G| = 12$ $P \in Syl_2$ $Q \in Syl_3$

* Either $\to Q \trianglelefteq G$ / $\to G \simeq A_4$
* Abelian Groups: $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$
* Nonabelian: A_4, D_{12}, $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$