Thm
There is a bijection

$\left\{\begin{matrix}\text{finite ab}\\ \text{groups}\\ \text{order } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{sequences } (n_1, n_2, \ldots, n_s)\\ \text{s.t. } ① \; n_i \geq 2\\ ② \; n_{i+1} \mid n_i\\ ③ \; n_1 \cdots n_s = n\end{matrix}\right\}$ ← call these invariant factor sequences

Correspondence ⇐
$(n_1, n_2, \ldots, n_s)$ on RHS.

$A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$

→ Fund. thm for f. ab. gps
$\Rightarrow A \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$
w/ $(n_1, \ldots, n_s)$ sats ①-③

Def^n A finite ab. gp is type $(n_1, n_2, \ldots, n_s)$ if it's isom to $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$ sat ①-③
↳ Called invariant factor decomp.

Recall
$(n_1, n_2, \ldots, n_s)$ a sequence of invariant factors w/ product n ⇒
All primes dividing n divide n_1.

Classifying Groups of order 180

Rmk equivalent to classif. sequences $(n_1, \ldots, n_s)$ of inv factors w/ product 180

$180 = 2^2 \cdot 3^2 \cdot 5$

So rmk is $2, 3, 5 \mid n_1$

So...

$n_1 = 2^2 \cdot 3^2 \cdot 5 = 180$ ①

$n_1 = 2 \cdot 3^2 \cdot 5 = 90$ ②

$n_1 = 2^2 \cdot 3 \cdot 5 = 60$ ③

$n_1 = 2 \cdot 3 \cdot 5 = 30$ ④

Case 1
$n_1 = 180$
$n_1 \cdots n_s = 180$
So $s = 1$ (180).

Case 2
$n_1 = 90$
$n_1 n_2 \cdots n_s = 180$
$n_2 \cdots n_s = 2$ (90, 2)
$\Rightarrow n_2 = 2$

Case 3
$n_1 = 60$
$n_1 n_2 \cdots n_s = 180$
$n_2 \cdots n_s = 3$
$n_2 \geq 2$ & $n_2 \mid 3 \Rightarrow n_2 = 3$
done (60, 3)

Case 4
$n_1 = 30$
$n_1 \cdot n_2 \cdots n_s = 180$
$n_2 \cdots n_s = 6$
$n_2 = 6$ or ~~3~~ or ~~2~~
↳ (30, 6)
↳ $n_3 \cdots n_s = 2$
$\Rightarrow n_3 = 2 \rightsquigarrow$ ~~(30, 3, 2)~~
But $2 \nmid 3$

Classification

Gps order 180 (Abelian)

Inv Factor	Gp
(180)	$\mathbb{Z}_{180}$
(90, 2)	$\mathbb{Z}_{90} \times \mathbb{Z}_2$ ←
(60, 3)	$\mathbb{Z}_{60} \times \mathbb{Z}_3$
(30, 6)	$\mathbb{Z}_{30} \times \mathbb{Z}_6$

Rmk You may wonder... what about

$\mathbb{Z}_{45} \times \mathbb{Z}_2 \times \mathbb{Z}_2$?

Order 180:
But in HW 4 showed $(m, n) = 1$
$\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{m \cdot n}$

$\mathbb{Z}_{45} \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_{90} \times \mathbb{Z}_2$

Thm
G abelian group of order $n = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. Then
① $G \simeq A_1 \times \cdots \times A_n$ where A_i abelian $|A_i| = p_i^{\alpha_i}$

Rmk Abelian groups factor as products of their Sylow subgroups.

② A abelian of order p^α.
$\Rightarrow A \simeq \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \times \cdots \times \mathbb{Z}_{p^{\beta_t}}$
w/ $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t$ & $\sum \beta_i = \alpha$

③ This decomp is unique.

Ex/
$\mathbb{Z}_{180} = \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$ ←

$\mathbb{Z}_{90} \times \mathbb{Z}_2 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

Def^n The $(p_i^{\beta_j})$ are called the elementary factors of G.
The decomp
$G = \mathbb{Z}_{p_1^{\beta_{1k}}} \cdots \times \mathbb{Z}_{p_i^{\beta_{n1}}} \cdots \times \mathbb{Z}_{p_k^{\beta_{tk}}}$
is the elementary factor decomposition.

Compare inv. factors & elementary factors.

if $|A| = p^\alpha$

Inv Factors	Elem Factors
$(n_1, n_2, \ldots, n_s)$	$(\beta_1, \beta_2, \ldots, \beta_t)$
① $n_i \geq 2$	① $\beta_i \geq 1$
② $n_{i+1} \mid n_i$	② $\beta_{i+1} \leq \beta_i$
③ $n_1 \cdots n_s = n = p^\alpha$	③ $\beta_1 + \cdots + \beta_t = \alpha$

$n_i = p^{\beta_i} \longleftarrow \beta_i$
$n_i \longmapsto \beta_i = \log_p n_i$

Finding ab gps order p^α
↕
Finding decreasing sequences of pos. ints adding up to α
↕
Partitions of α

Example
Abelian groups of order p^5

Part	Gp
5	$\mathbb{Z}_{p^5}$
4, 1	$\mathbb{Z}_{p^4} \times \mathbb{Z}_p$
3, 2	$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$
3, 1, 1	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p \times \mathbb{Z}_p$
2, 2, 1	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p$
2, 1, 1, 1	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
1, 1, 1, 1, 1	$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

Example Ab Groups of order
$1800 = 2^3 \cdot 3^2 \cdot 5^2$

Thm ⇒ If G an abelian group order 1800.
$\Rightarrow G \simeq A_2 \times A_3 \times A_5$ (order 8, order 9, order 25)

	order	parts	gps
A_2	$8 = 2^3$	3 2, 1 1, 1, 1	$\mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{Z}_2$ ← $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
A_3	$9 = 3^2$	2 1, 1	$\mathbb{Z}_9$ ← $\mathbb{Z}_3 \times \mathbb{Z}_3$
A_5	$25 = 5^2$	2 1, 1	$\mathbb{Z}_{25}$ $\mathbb{Z}_5 \times \mathbb{Z}_5$ ←

$G \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

$\underbrace{\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}}_{\text{element divisors}} \longleftrightarrow \mathbb{Z}_{1800}$ ← inv factors.

How do we go back & forth?

Prop:
1) $\mathbb{Z}_n \times \mathbb{Z}_m \simeq \mathbb{Z}_{nm} \Longleftrightarrow \gcd(n, m) = 1$
2) $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ ← $p_i \neq p_j$
$\Rightarrow \mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$

Pf/ ① HW 4 6c
② Follows applying 1 repetitively.

Inv. factors ⇒ Elementary divisors.

$G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$

$n_i = p_1^{\beta_{i1}} \cdots p_n^{\beta_{in}}$ — Letting some $\beta_i = 0$ allows n fixed

Prop ⇒
$\mathbb{Z}_{n_i} = \mathbb{Z}_{p_1^{\beta_{i1}}} \times \cdots \times \mathbb{Z}_{p_n^{\beta_{in}}}$

So elementary divs are $p_j^{\beta_{ij}}$ w/ $\beta_{ij} \neq 0$

Ex/ $\mathbb{Z}_{90} \times \mathbb{Z}_2 \simeq (\mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_2)$
$90 = 2 \cdot 3^2 \cdot 5$ $\simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_9) \times (\mathbb{Z}_5)$
$2 = 2$

Rmk note we only used that we started w/ a product of cyclic groups

Groups of order 1800: Elementary divisor form

$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$
$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$
$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$
2^3	$2^2 \cdot 2$	$2 \cdot 2 \cdot 2$

Groups of order 1800: Invariant factor form

$\mathbb{Z}_{1800}$	$\mathbb{Z}_{900} \times \mathbb{Z}_2$	$\mathbb{Z}_{450} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_{360} \times \mathbb{Z}_5$	$\mathbb{Z}_{180} \times \mathbb{Z}_{10}$	$\mathbb{Z}_{90} \times \mathbb{Z}_{10} \times \mathbb{Z}_2$
$\mathbb{Z}_{600} \times \mathbb{Z}_3$	$\mathbb{Z}_{300} \times \mathbb{Z}_6$	$\mathbb{Z}_{150} \times \mathbb{Z}_6 \times \mathbb{Z}_2$
$\mathbb{Z}_{120} \times \mathbb{Z}_{15}$	$\mathbb{Z}_{60} \times \mathbb{Z}_{30}$	$\mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_2$

why longer?