Direct Products

Theorem:

$G = G_1\times G_2\times\cdots\times G_n$

① $\iota: G_i \longrightarrow G$

$g_i \longmapsto (1,\dots,1,g_i,\dots,1)$ ← i th pos.

Exhibits $G_i \trianglelefteq G$

& $G/G_i \cong G_1\times\cdots\times G_{i-1}\times G_{i+1}\times\cdots\times G_n$

② $\pi: G \twoheadrightarrow G_i$

$(g_1,\dots,g_n)\longmapsto g_i$

$\ker\pi \cong G_1\times\cdots\times G_{i-1}\times G_{i+1}\times\cdots\times G_n$

③ $G_i, G_j \le G$ $\quad i\neq j$

$x\in G_i$, $y\in G_j$

$\Rightarrow xy=yx$

Rmk

* Condions 1 & 2 sort of define what it means to be a factor a product.
* Condition 3 makes the product direct.

Pf/ $x\in G_i$, $y\in G_j$ $\quad i<j$

$x = (1,1,\dots,x,\dots,1)$ (ith), $\quad y = (1,\dots,y,\dots,1)$ (jth)

$xy = (1,\dots,x,\dots,y,\dots,1)$

$= yx$ ∅

Examples

① G, H groups.

$G\times H$.

$G \hookrightarrow G\times H$

$g \longmapsto (g,1)$

$\frac{G\times H}{G} = H$

$\pi: G\times H \longrightarrow G$

$(g,h)\longmapsto g$

$\ker\pi = \{(g,h)\mid g=1\}$

$= \{(1,h)\mid h\in H\}$

$= H$

Get cancellation

i.e. $G\times H/G = H$

$G\times H/H = G$.

Subgroups of G×G

See 2 subgroups isom to G.

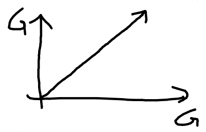
$G\simeq\{(g,1)\} = G\times 1 \trianglelefteq G\times G$

$G\simeq\{(1,g)\} = 1\times G \trianglelefteq G\times G$

$G \xrightarrow{\Delta} G\times G$

$g \longmapsto (g,g)$

diagonal map



Examples

Elementary abelian gp order p^2.

$E = \mathbb{Z}_p\times\mathbb{Z}_p$.

Prop E has $p+1$ subs order p.

Pf/ $x\in E$ $\quad x\neq 1$. $\quad |x|=p$

$\langle x\rangle$ ← sub order p

$y\notin\langle x\rangle$ The $\langle y\rangle$

& $\langle x\rangle\cap\langle y\rangle=\{1\}$

Get $p-1$ new elts each time.

Partioned non 1 elts of E into sets of size $p-1$

How many?

$\frac{p^2-1}{p-1} = p+1$ ∅

Subs order 3 in $(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/3\mathbb{Z})$ (4 by prop)

* $\langle(1,0)\rangle = \{(1,0),(2,0),(0,0)\}$
* $\langle(0,1)\rangle = \{(0,1),(0,2),(0,0)\}$
* $\langle(1,1)\rangle = \{(1,1),(2,2),(0,0)\} = \Delta$
* $\langle(1,2)\rangle = \{(1,2),(2,1),(0,0)\}$

$y=2x$ $\quad \Gamma_{\times 2}$

Def^n Graph of a hom.

$\phi: G\longrightarrow H$ a hom.

Then the graph

$\Gamma_\phi: G\longrightarrow G\times H$

$g\longmapsto(g,\phi(g))$

Finitely gen'd Abelian Groups

Def^n G is finitely gen'd if exist some finite subset $A\subseteq G$ s.t. $\langle A\rangle = G$.

Def^n $r\in\mathbb{N}$. The free abelian group of rk r is

$\mathbb{Z}^r = \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{r\text{-times}}$

$\mathbb{Z}^0=\{0\}$

Lemma $\mathbb{Z}^r$ is finitely gen'd.

Pf/ $e_i = (0,0,\dots,1,0,\dots,0)$ ← ith pos.

Then $\langle e_1,e_2,\dots,e_r\rangle = \mathbb{Z}^r$. ∅

Lemma Every finite group is finitely gen'd

Theorem (Fundamental theorem for finitely gen'd abelian groups)

Let G be a f.g. ab. group.

① $G\simeq\mathbb{Z}^r\times\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_s}$

s.t.

① $r\geq 0$

② $n_i\geq 2$

③ $n_{i+1}\mid n_i$

② The r, n_i are unique.

i.e. If $G\simeq\mathbb{Z}^\ell\times\mathbb{Z}_{m_1}\times\cdots\times\mathbb{Z}_{m_t}$

w/ (ℓ, m_i) satisfying 1-3

Then $\ell = r$

$t = s$

$m_i = n_i$

Pf/ On you! :)

Def^n r = rank of G.

= Betti number

n_i = invariant factors

Remarks

* G f.g. ab. gp. ⟷ some #s $r, n_1, n_2, \dots, n_s$ that completely determine G

Rmk Let G be a f.g. ab. gp.

G finite $\iff r=0$.

If G finite ab. gp

$\Rightarrow G\simeq\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_s}$

& $|G| = n_1 n_2\cdots n_s$

This theorem gives us a way to list all finite ab. gps of a given order.

{Abelian gps of order n} ⟷ {$n_1, n_2, \dots, n_s$ w/ ① $n_i\geq 2$ ② $n_{i+1}\mid n_i$ ③ $n_1\cdots n_s = n$}

How do we do this?

Observe.

$n_1\geq n_2\geq\cdots\geq n_s$

& $n_i\mid n_1$ $\forall$ i.

Let p prime $p\mid n$.

$p\mid n_1\cdots n_s \Rightarrow p\mid n_i\mid n_1$

$\Rightarrow p\mid n_1$

Lemma Every prime div of n divides n_1

Pf/ Above

Suppose $n = p_1p_2\cdots p_s$ a product of distinct primes. (such ints called squarefree)

$\Rightarrow p_i\mid n \Rightarrow p_i\mid n_1$ $\forall i$

$\Rightarrow n\mid n_1$.

Know $n = n_1\cdots n_s \Rightarrow n_1\mid n$

so $n = n_1$. We proved

Corollary

$n = p_1\cdots p_s$ a square free int. and G abelian gp of order n.

$\Longrightarrow G\simeq\mathbb{Z}_n$.

Rmk Extends 2 results

1) $|G| = p \Rightarrow G\simeq\mathbb{Z}_p$
2) $|G| = pq \Rightarrow G\simeq\mathbb{Z}_{pq}$ ← HW8

Example Abelian gps of order 180

$180 = 2^2\cdot 3^2\cdot 5$

n_1 must be divisible by 2, 3, 5.

Options

$2^2\cdot 3^2\cdot 5 = 180$ $\quad 2\cdot 3^2\cdot 5 = 90$ $\quad 2^2\cdot 3\cdot 5 = 60$ $\quad 2\cdot 3\cdot 5 = 30$

↳ $n_1 = 180$. $\quad n_1 = n$.

$n_1 n_2\cdots n_s = 180 \Longrightarrow s=1$

$\Longrightarrow G\simeq\mathbb{Z}_{180}$

$n_1 = 2\cdot 3^2\cdot 5 = 90$

$n_1 n_2\cdots n_s = 180$

$90\cdot x$

$\Longrightarrow n_2\cdots n_s = 2 \Longrightarrow s=2$, $n_2 = 2$

$180 = n_1\cdot n_2 = 90\cdot 2$

↳ $\mathbb{Z}_{90}\times\mathbb{Z}_2$.