Correction

Recall: $n_2 = \{3, \underline{5}, \underline{15}\}$ done

No subs of index 3

Produce $M \leq G$

$4 \mid |M| \mid 60$, $\neq$ $\neq$

$|M| = 12$ or 20

Theorem $n \geq 5$

A_n is simple

Rmk

* Gives our second ∞ family of finite simple groups (other $\mathbb{Z}_p$)
* $A_2 = \{1\}$

Simp $\rightarrow A_3 \simeq \mathbb{Z}_3$

Not simp $\rightarrow A_4 \trianglerighteq \langle (12)(34), (13)(42) \rangle$

Simp $\rightarrow A_n$

Proof of Thm

Induction on n

Base Case: $n=5$

A_5 simple ✓

Inductive Step

Assume A_{n-1} simple.

$G = A_n$

Notice $G \circlearrowleft \{1,2,3,\dots,n\}$

Let G_i = Stabilizer of i

$= \{\sigma \in G \mid \sigma(i) = i\}$

G_i = permutations of $\{1,2,3,\dots,i-1,i+1,\dots,n\}$ which are even.

$\Rightarrow G_i \simeq A_{n-1}$

$\Rightarrow G_i$ is simple

Assume that $\exists H \trianglelefteq G$ s.t. $H \neq 1$, $H \neq G$.

Case 1:

$\exists \underset{\neq 1}{\tau} \in H$ s.t. $\tau(i) = i$

$\rightsquigarrow \tau \in H \cap G_i$

$H \trianglelefteq G$
$\quad \vee| \quad \vee|$
$H \cap G_i \trianglelefteq G_i$

Pf $h \in H \cap G_i$, $g \in G_i$, $ghg^{-1} \in H$, $\in G_i$ $\varnothing$

$\Rightarrow$ b/c G_i simple

$H \cap G_i = 1$ or G_i

$\tau \in H \cap G_i$ & $\neq 1$

$\Rightarrow H \cap G_i = G_i$

$\Rightarrow G_i \leq H$.

Lemma $\sigma G_i \sigma^{-1} = G_{\sigma(i)}$.

Pf/ HW $\varnothing$

Since $H \trianglelefteq G$

$\Rightarrow \sigma G_i \sigma^{-1} \leq H$

$= G_{\sigma(i)}$

$\Rightarrow G_j \leq H \quad \forall j: 1, \dots, n$

$G = \langle G_1, G_2, \dots, G_n \rangle \leq H$

$\sigma \in G$, $\sigma = (i_1\, j_1)(i_2\, j_2)\cdots$

$k \neq i_1, j_1$

$(i_1\, j_1) \in G_k$ $\varnothing$

$\Rightarrow H = G$

So we can conclude $\forall \underset{1}{\tau} \in H$, & $\forall i: 1, \dots, n$

$\tau(i) \neq i$

Therefore... if $\tau_1 \cdot \tau_2 \in H$.

& $\tau_1(i) = \tau_2(i)$ some i

$\Rightarrow \tau_2^{-1}\tau_1(i) = i$ (*)

$\Rightarrow \tau_2^{-1}\tau_1 = 1 \Rightarrow \tau_2 = \tau_1$

2 cases left

Case 2

$\tau \in H$ w/ a 3-cycle

$\tau: (a_1\, a_2\, a_3)(b_1\, b_2 \cdots)$

Find $\sigma \in G: A_5$

$\sigma(a_1) = a_1$

$\sigma(a_2) = a_2$

$\sigma(a_3) = x \neq a_3$

$\sigma\tau\sigma^{-1} = (\sigma(a_1)\, \sigma(a_2)\, \sigma(a_3))(\sigma(b_1) \cdots)$

$= (a_1\, a_2\, x)(\cdots$

$= \tau' \in H$ b/c $H \trianglelefteq G$

$\tau'(a_1) = a_2 = \tau(a_1)$

(*) $\tau' = \tau$

$\Downarrow$ $\tau'(a_2) = x$
$\qquad \#$
$\tau(a_2) = a_3$

All that's left

Case 3 $\forall \tau \in H$

decompose into disjoint 2-cycle. $n \geq 6$

$\tau = (a_1\, a_2)(a_3\, a_4)(a_5\, a_6)\cdots \in H$

$[(1\,2)(3\,4) \Rightarrow \tau(3) = 5]$

Let $\sigma = (a_1\, a_2)(a_3\, a_5) \in A_5 = G$

$\sigma\tau\sigma^{-1} = (a_2\, a_1)(a_5\, a_4)(a_3\, a_6)\cdots$

$= \tau' \in H \trianglelefteq G$

$\tau(a_1) = a_2 = \tau'(a_1)$

$* \Rightarrow \tau = \tau'$

$\Downarrow$ b/c $\tau(a_3) = a_4$

$\tau'(a_3) = a_6$

$\varnothing$

Next Do more classification

Tool: Direct & SemiDirect Products

Direct Products

Def$^{\underline{n}}$ $G_1, \dots, G_n$ groups.

$G_1 \times G_2 \times \cdots \times G_n$

$= \{(g_1, g_2, \dots, g_n) \mid g_i \in G_i\}$

w/ multiplication:

$(g_1, g_2, \dots, g_n)(h_1, h_2, \dots, h_n)$

$= (g_1h_1, g_2h_2, \dots, g_nh_n)$

* $G_1, G_2, \dots$ ∞ sequence of gps.

$G_1 \times G_2 \times \cdots := \{(g_1 \cdot g_2, \cdots) \mid g_i \in G_i\}$

w/ mult componentwise.

Lemma:

$G_1 \times \cdots \times G_n$ is a group w/ identity $(1_{G_1}, \dots, 1_{G_n})$

& $(g_1, \dots, g_n)^{-1} = (g_1^{-1}, \dots, g_n^{-1})$.

&

$|G_1 \times \cdots \times G_n| = |G_1| \cdot \cdots \cdot |G_n|$

Pf/ Exercise

Example

$\mathbb{Z} \times S_n \times GL_2(\mathbb{R})$

$\left(n, \sigma, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot \left(m, \tau, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right)$

$= \left(n+m, \sigma \circ \tau, \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}\right)$

$|\mathbb{Z}_5 \times S_4 \times GL_2(\mathbb{F}_2)|$

$\parallel$

$|\mathbb{Z}_5| \cdot |S_4| \cdot |GL_2 \mathbb{F}_2|$

$= 5 \cdot 24 \cdot 6$

$= 720$

Slogan

Direct products have lots of fun maps.

Prop $G_1, G_2, \dots, G_n$ be groups.

$G = G_1 \times G_2 \times \cdots \times G_n$.

1) (Inclusion of a factor).

i define

$\{(1, 1, \dots, \underset{i\text{th pos}}{g_i}, 1, \dots, 1) \mid g_i \in G_i\}$

This is a subgroup $\simeq G_i$

Identifying it w/ G_i

$G_i \trianglelefteq G$.

So $G/G_i \cong$

$G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$

$\iota_i : G_i \hookrightarrow G$

$g_i \longmapsto (1, \dots, g_i, \dots, 1)$

2) Projection onto a factor

For each i define

$\pi_i : G \longrightarrow G_i$

$(g_1, \dots, g_i, \dots, g_n) \longmapsto g_i$

is a surjective homom

& $\ker \pi_i \simeq$

$G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$

3) $G_i \neq G_j \leq G$

$x \in G_i$, $y \in G_j$

$\Rightarrow xy = yx$.

Prove pt 1

1) $\iota : G_i \longrightarrow G$

Clearly homom ✓

inj ✓

So $G_i \leq G$

$G \xrightarrow{\phi} G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$

$(g_1, \dots, g_n) \longmapsto (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$

ϕ surjective hom

$\ker \phi = \{g_j = 1 \; \forall j \neq i, \; g_i \in G_i\}$

$= \mathrm{im}(\iota)$

$\Rightarrow G_i \trianglelefteq G$

Last thing G/G_i is first isom thm $\varnothing$