PROJECTIVE GEOMETRY FOR PERFECTOID SPACES

GABRIEL DORFSMAN-HOPKINS
UNIVERSITY OF WASHINGTON MATHEMATICS

SUMMARY

To understand the structure of an algebraic variety we often embed it in various projective spaces. This develops the notion of *projective geometry* which has been an invaluable tool. Motivated by [1], we begin to develop a perfectoid analog of projective geometry, and explore how equipping a perfectoid space with a map to a certain analog of projective space can be a powerful tool to understand its geometric and arithmetic structure. Along the way we do the following.

1. Give a complete classification of vector bundles on the perfectoid closed unit disk.
2. Compute the Picard group of the perfectoid analog of projective space (*projectivoid space*).
3. Compute the cohomology of all line bundles on projectivoid space.
4. Compute the functor of points of projectivoid space.
5. Use *projectivoid geometry* to compare the Picard groups of perfectoid spaces and their tilts.

PRELIMINARIES

An initial motivation for perfectoid spaces is the following isomorphism of Fontaine and Wintenberger connecting Galois theory in positive and mixed characteristics.

Theorem 1 (Fontaine-Wintenberger). *There is a canonical isomorphism of absolute Galois groups*

$$\mathrm{Gal}\left(\mathbb{Q}_p\left(p^{1/p^\infty}\right)\right) \cong \mathrm{Gal}\left(\mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right)\right).$$

In [2], Scholze introduced a class of algebro-geometric objects called *perfectoid spaces*, which exhibit this very correspondence. To any perfectoid space S one can functorially construct its *tilt*: a homeomorphic perfectoid space $S^\flat$ in characteristic p.

Theorem 2 (Scholze). *The functor $X \mapsto X^\flat$ is an equivalence of categories of perfectoid spaces over S and $S^\flat$, inducing an equivalence of étale sites:*

$$S_{\acute{e}t} \xrightarrow{\sim} S^\flat_{\acute{e}t}.$$

Letting S be the perfectoid space associated to $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$ we recover Theorem 1.

EXAMPLES

Much like varieties, schemes, and rigid analytic spaces, perfectoid spaces are locally spectra of perfectoid algebras. (We use adic spectra, see [3].)

Example 1 (Closed unit disk). $\mathbb{D}^{n,perf}_K$ *is the space associated to the perfectoid Tate algebra:*

$$K\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle = \widehat{\bigcup_i K\left\langle T_1^{1/p^i}, \cdots, T_n^{1/p^i}\right\rangle}$$

Example 2 (Projectivoid Space). *The perfectoid analog of projective space, $\mathbb{P}^{n,perf}_K$ can be constructed by glueing together closed perfectoid disks along their boundaries in the usual way. It also arises as an inverse limit along $[T_0 : \cdots : T_n] \mapsto [T_0^p : \cdots : T_n^p]$,*

$$\mathbb{P}^{n,perf}_K \sim \varprojlim\left(\cdots \longrightarrow \mathbb{P}^n_K \longrightarrow \mathbb{P}^n_K\cdot\right)$$

Lemma 1. *Let K be a perfectoid field with tilt $K^\flat$.*

$$\left(\mathbb{D}^{n,perf}_K\right)^\flat \cong \mathbb{D}^{n,perf}_{K^\flat}$$
$$\left(\mathbb{P}^{n,perf}_K\right)^\flat \cong \mathbb{P}^{n,perf}_{K^\flat}.$$

REFERENCES

[1] S. Das. *Vector Bundles on Perfectoid Spaces*. PhD thesis, University of California, San Diego, 2016.

[2] P. Scholze. Perfectoid spaces. 116(1):245–313, 2012.

[3] R. Huber. A generalization of formal schemes and rigid analytic varieties. 217(4):513–551.

CONTACT INFORMATION

Web www.math.washington.edu/~ gdh2
Email gdh2@uw.edu
Find the paper here
www.math.washington.edu/~ gdh2/projectivoidgeometry.pdf

LINE BUNDLES ON PROJECTIVOID SPACE

Since the building blocks for projectivoid space are closed perfectoid disks, we begin by establishing a perfectoid analog of the Quillen-Suslin Theorem.

Theorem A. *Finite vector bundles on $\mathbb{D}^{n,perf}$ are all trivial.*

With this in hand we can compute the Picard group of projectivoid space.

Theorem B.

$$\mathrm{Pic}\,\mathbb{P}^{n,perf} \cong \mathbb{Z}[1/p].$$

In particular, for each $d \in \mathbb{Z}[1/p]$, there is a twisting sheaf $\mathscr{O}(d)$, which corresponds to homogeneous convergent power series of degree d in $K\left\langle T_0^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle$ and its various localizations. As in the classical case, this can be exhibited explicitly through their cohomology, which we compute below.

Theorem C. *Let $X = \mathbb{P}^{n,perf}$ and $d \in \mathbb{Z}[1/p]$ so that $\mathscr{O}_X(d) \in \mathrm{Pic}\,X$ an arbitrary line bundle. Then: If $d \geq 0$,*

$$\mathrm{H}^0\left(X, \mathscr{O}_X(d)\right) = K\left\langle T_0^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle_d.$$

If $d < 0$,

$$\mathrm{H}^n\left(X, \mathscr{O}_X(d)\right) = K\left\langle T_0^{-1/p^\infty}, \cdots, T_n^{-1/p^\infty}\right\rangle_d.$$

In all other cases,

$$\mathrm{H}^r\left(X, \mathscr{O}_X(d)\right) = 0.$$

MAPS TO PROJECTIVOID SPACE

Like in classical geometry, maps to projectivoid space can be expressed in terms of globally generated line bundles.

Theorem D. *A map $X \to \mathbb{P}^{n,perf}$ corresponds to tuples $\left(\mathscr{L}_i, s_j^{(i)}, \varphi_i\right)$, where $\mathscr{L}_i$ is a line bundle on X, the $s_j^{(i)}$ are $n+1$ generating global sections of $\mathscr{L}_i$, and $\varphi_i : \mathscr{L}_{i+1}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_i$ are isomorphisms under which $\left(s_j^{(i+1)}\right)^{\otimes p} \mapsto s_j^{(i)}$.*

If K has characteristic p, X is perfect, so the pth power map on $\mathrm{Pic}\,X$ is an isomorphism. Therefore we can refine the theorem.

Corollary E. *Let X be a perfectoid space over K of positive characteristic. Fix a line bundle $\mathscr{L}$ on X together with global sections $s_0, \cdots, s_n$, which generate $\mathscr{L}$. Then there is a unique morphism $\phi : X \to \mathbb{P}^{n,perf}$ such that $\phi^*(\mathscr{O}(1)) \cong \mathscr{L}$ and $\phi^*(T_i) = s_i$.*

The tilting equivalence simplifies matters. Since $\mathrm{Hom}(X, \mathbb{P}^{n,perf}_K) = \mathrm{Hom}(X^\flat, \mathbb{P}^{n,perf}_{K^\flat})$, we have:

Corollary F. *Let X be a perfectoid space over K of any characteristic, a map to $\mathbb{P}^{n,perf}_K$ is equivalent to a single line bundle $\mathscr{L}$ on $X^\flat$ together with $n+1$ global sections generating $\mathscr{L}$.*

APPLICATIONS: UNTILTING LINE BUNDLES

Projectivoid geometry gives us a hands on way to compare line bundles on X and $X^\flat$. Using the fact that X and $X^\flat$ have the same maps to projectivoid space (over their respective base fields), and chaining this with Corollary F, we get a homomorphism

$$\mathrm{Pic}\,X^\flat \longrightarrow \varprojlim \mathrm{Pic}\,X.$$

With a geometric argument we conclude:

Theorem G. *Suppose X is a perfectoid space over K. Suppose that X has an ample line bundle and that $H^0(\overline{X}, \mathscr{O}_{\overline{X}}) = \overline{K}$. Then*

$$\mathrm{Pic}\,X^\flat \hookrightarrow \varprojlim \mathrm{Pic}\,X.$$

In particular, if $\mathrm{Pic}\,X$ has no p torsion:

$$\mathrm{Pic}\,X^\flat \hookrightarrow \mathrm{Pic}\,X.$$