Projective Geometry for Perfectoid Spaces

Gabriel Dorfsman-Hopkins

August 19, 2018

Abstract

To understand the structure of an algebraic variety we often embed it in various projective spaces. This develops the notion of projective geometry which has been an invaluable tool in algebraic geometry. We develop a perfectoid analog of projective geometry, and explore how equipping a perfectoid space with a map to a certain analog of projective space can be a powerful tool to understand its geometric and arithmetic structure. In particular, we show that maps from a perfectoid space X to the perfectoid analog of projective space correspond to line bundles on X together with some extra data, reflecting the classical theory. Along the way we give a complete classification of vector bundles on the perfectoid unit disk, and compute the Picard group of the perfectoid analog of projective space.

Contents

1 Introduction

An important dichotomy in algebraic geometry is the distinction between characteristic 0 and prime characteristic $p > 0$. Algebraic geometry provides a framework to do geometry in positive characteristic, transporting our classical intuition to a more exotic algebraic world. But in positive characteristic we are also provided with extra tools, such as the Frobenius map, making many geometric results more accessible. Therefore transporting information from positive characteristic up to characteristic 0 proves very useful as well.

In [9], Fontaine and Wintenberger produced an isomorphism which hinted at a deep correspondence between algebraic objects of each type.

Theorem 1.1 (Fontaine-Wintenberger)
There is a canonical isomorphism of topological groups between the absolute Galois groups of $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$ *and* $\mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right)$.

An algebraic geometer would perhaps ask if there is some deeper geometric correspondence, of which this is a manifestation on the level of points. Quite recently, in [27], Scholze introduced a class of geometric objects called *perfectoid spaces*, which exhibit this very correspondence. In particular, to a perfectoid space X of any characteristic, we can associate its *tilt* $X^\flat$ which is a perfectoid space of characteristic p, and furthermore X and $X^\flat$ have isomorphic étale sites. These spaces have proved useful far beyond giving a geometric framework in which to understand the Fontaine-Wintenberger isomorphism. Indeed, they have found applications in extending instances of Deligne's Weight-Monodromy Conjecture, classifying p-divisible groups, have been used in work on the geometric Langlands program, and even aid in the understanding of singularities in positive characteristic. For a survey, see [28].

This paper is inspired by the goal of understanding vector bundles on perfectoid spaces, and how they behave under the so called *tilting correspondence* of Scholze. To do so, we develop a perfectoid analog of projective geometry. We define a perfectoid analog of projective space, which we call *projectivoid space* and denote by $\mathbb{P}^{n,\mathrm{perf}}$, and show that maps from a perfectoid space X to $\mathbb{P}^{n,\mathrm{perf}}$ correspond to line bundles on X together with some extra data, giving an analog to the classical theory of maps to projective space.

To get to this point we must first understand the theory of line bundles on projectivoid space itself, and in particular, its Picard group. In his dissertation [7], Das worked toward computing Picard group of the projectivoid line, $\mathbb{P}^{1,\mathrm{perf}}$. His proof relied on having certain local trivializations of line bundles, requiring a perfectoid analog of the Quillen-Suslin theorem. Therefore, in order to begin developing the theory of so called projectivoid geometry, we must prove this first.

The Quillen-Suslin theorem says that finite dimensional vector bundles on affine n-space over a field are all trivial. Equivalently, all finite projective modules on a polynomial ring $K[T]$ are free, where T is an n-tuple of indeterminates. In rigid analytic geometry, we replace polynomial rings with rings of convergent power series called *Tate algebras*, denoted $K\langle T\rangle$, and it can be shown that over such rings the Quillen-Suslin theorem still holds, that is, all finite projective $K\langle T\rangle$-modules are free. The analog of these rings for perfectoid spaces is the ring $K\langle T^{1/p^\infty}\rangle$ of convergent power series where the indeterminates have all their pth power roots. The difficulty in extending the theorem to this *perfectoid Tate algebra* is that the ring is no longer noetherian, and so the result cannot be easily reduced to the polynomial case.

Sections 2 through 4 of this paper set up the theory of perfectoid spaces. In Section 2 we define our fundamental algebraic objects, perfectoid fields and algebras, and explore some of their algebraic properties in relating characteristic 0 and characteristic p. In Section 3 we review Huber's theory of adic spaces, which provide the geometric framework for globalizing perfectoid algebras into spaces (playing a role analogous to schemes in algebraic geometry). In Section 4 we apply Huber's theory of adic spaces to perfectoid rings and algebras, and explore the geometric properties of perfectoid spaces and their tilts.

Sections 5 through 9 constitute the author's work on the subject. In Section 5 we explore the commutative ring theoretic properties of the perfectoid Tate algebra. We compute its unit group, and prove perfectoid analogues of Weierstrass division and preparation. We also compute the Krull dimension of the perfectoid Tate algebra in positive characteristic.

In Section 6 we have our first main theorem.

Theorem 1.2 (The Quillen-Suslin Theorem for the Perfectoid Tate Algebra)
Finite projective modules on the perfectoid Tate algebra $K\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle$ *are all free. Equivalently, finite dimensional vector bundles on the perfectoid unit disk are all isomorphic to the trivial vector bundle.*

This completes Das' proof, and lays the groundwork to begin studying vector bundles on more general perfectoid spaces.

In Section 7 we develop the theory of line bundles on projectivoid space, extending Das' result for $n = 1$.

Theorem 1.3 (The Picard Group of Projectivoid Space)
$\operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}} \cong \mathbb{Z}[1/p]$.

We also compute the cohomology of all line bundles on projectivoid space.

In Section 8 we compute the functor of points of projectivoid space, showing that (much like in the classical theory) it is deeply connected to the theory of line bundles on perfectoid spaces.

Theorem 1.4 (The Functor of Points of Projectivoid Space)
Let X *be a perfectoid space over a field* K. *Morphisms* $X \to \mathbb{P}^{n,\mathrm{perf}}$ *correspond to tuples* $\left(\mathscr{L}_i, s_j^{(i)}, \varphi_i\right)$, *where* $\mathscr{L}_i \in \operatorname{Pic} X$, $\left\{s_0^{(i)}, \cdots, s_n^{(i)}\right\}$ *are* $n+1$ *global sections of* $\mathscr{L}_i$ *which generate* $\mathscr{L}_i$, *and* $\varphi_i : \mathscr{L}_{i+1}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_i$ *are isomorphisms under which* $\left(s_j^{(i+1)}\right)^{\otimes p} \mapsto s_j^{(i)}$.

We also provide refinements of this theorem in characteristic p and see how it behaves under the tilting equivalence of Scholze.

In Section 9 we test out this new theory, using it to compare the Picard groups of a perfectoid space X and its tilt $X^\flat$. In particular, since the tilting equivalence builds a correspondence between maps $X \to \mathbb{P}_K^{n,\mathrm{perf}}$ and maps $X^\flat \to \mathbb{P}_{K^\flat}^{n,\mathrm{perf}}$, we can chain this together with the correspondence of line bundles and maps to projectivoid space to compare line bundles on X and $X^\flat$. The main result follows.

Theorem 1.5
Suppose X *is a perfectoid space over* K. *Suppose that* X *has an ample line bundle and that* $H^0(X_{\overline{K}}, \mathscr{O}_{X_{\overline{K}}}) = \overline{K}$. *Then there is a natural injection*

$$\theta : \operatorname{Pic} X^\flat \hookrightarrow \varprojlim_{\mathscr{L} \mapsto \mathscr{L}^p} \operatorname{Pic} X.$$

In particular, if $\operatorname{Pic} X$ *has no* p *torsion, then composing with projection onto the first coordinate gives an injection*

$$\theta_0 : \operatorname{Pic} X^\flat \hookrightarrow \operatorname{Pic} X.$$

2 Algebraic Preliminaries

Algebraic geometry is locally commutative algebra, that is, the spaces we study are locally a 'model space,' which is the prime spectrum of a commutative ring. Perfectoid spaces are similar, with local model spaces corresponding to (pairs of) certain types of rings. In this section, we develop the basic algebraic objects from which we can construct our perfectoid spaces. They will all be Banach algebras over nonarchimedean fields of residue characteristic $p > 0$.

2.1 The Fontaine-Wintenberger Isomorphism

To get a better understanding of the algebraic properties that will allow us to transition between characteristic 0 and characteristic p, we begin by analyzing the correspondence on the the level of local fields. In characteristic 0, we have the field of p-adic numbers, $\mathbb{Q}_p$, which can be represented as the set

$$\mathbb{Q}_p = \left\{ \sum_{n>>-\infty} a_n p^n : a_n \in \{0, 1, \dots, p-1\} \right\}.$$

In other words, we can represent p-adic numbers uniquely as 'Laurent series in the variable p'. On the other hand, in positive characteristic, we have the field of Laurent series in the variable t,

$$\mathbb{F}_p\left((t)\right) := \left\{ \sum_{n>>-\infty} a_n t^n : a_n \in \{0, 1, \dots, p-1\} = \mathbb{F}_p \right\}.$$

Swapping out p and t shows that these fields can be regarded as having the same formal elements, but they certainly do not have the same addition and multiplication operations (indeed, they do not have matching characteristic!). Furthermore, $\mathbb{F}_p$ has a Frobenius morphism $x \mapsto x^p$, whereas $\mathbb{Q}_p$ has no such thing.

Although an isomorphism of fields is out of the question, the 'swap out p for t' analogy actually makes sense on the level of algebraic extensions. For example, for $p \neq 2$, we can compare the splitting field of $x^2 - p$ over $\mathbb{Q}_p$ to the splitting field of $x^2 - t$ over $\mathbb{F}_p\left((t)\right)$. By restricting ramification, and passing to so called 'deeply ramified' extensions, we can eliminate wild ramification and this correspondence becomes a bijection. Explicitly, we pass to the infinite extensions

$$K := \mathbb{Q}_p\left(p^{1/p^\infty}\right) = \bigcup_{n \geq 0} \widehat{\mathbb{Q}_p\left(p^{1/p^n}\right)},$$

and

$$K^\flat := \mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right) = \bigcup_{n \geq 0} \widehat{\mathbb{F}_p\left((t)\right)\left(t^{1/p^n}\right)}.$$

Galois extensions of K and $K^\flat$ correspond precisely by swapping out t for p as above. Furthermore, this correspondence preserves degree, so that $G_K \cong G_{K^\flat}$. This gives us a powerful tool allowing us to transport Galois theoretic data between K and $K^\flat$. This equivalence may be less surprising considering the relationship between the integral subrings of K and $K^\flat$. In particular, we have the following isomorphism.

$$\mathbb{Z}_p\left[p^{1/p^\infty}\right]/(p) \cong \mathbb{F}_p\left[p^{1/p^\infty}\right]/(t).$$

2.2 Topological Rings and Fields

Establishing the isomorphism above relies on using the topologies of the fields K and $K^\flat$, so we begin by establishing the necessary background on algebraic objects equipped with topologies.

Definition 2.1. A *topological ring* is a ring R endowed with a topology on which the maps $(r, s) \mapsto r + s$, $(r, s) \mapsto rs$ and $r \mapsto -r$ are all continuous. A *topological field* is a field K which is a topological ring and furthermore the map $x \mapsto x^{-1}$ is continuous on $K^\times$ with the subspace topology.

Remark 2.2

For a topological ring R, and any $r \in R$, the map $x \mapsto x + r$ is a homeomorphism of R, and the map $x \mapsto xr$ is continuous.

The most important cases for us will be *adic* topologies and topologies induced by a nonarchimedean absolute value.

Definition 2.3. A topological ring R is called *adic* if there exists an ideal I of R such that $\{I^n; n \geq 0\}$ is a neigborhood basis for 0. This implies that $\{r + I^n : r \in R, n \geq 0\}$ forms a basis for the topology of R. I is called an *ideal of definition* for R.

Example 2.4

The ring of integers $\mathbb{Z}$ with the p-adic topology is an adic ring with ideal of definition (p). The ring of formal power series $k[[t]]$ with the t-adic topology is an adic ring with ideal of definition (t).

Definition 2.5. A nonarchimedean field is a field K endowed with an absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying the following properties.

(a) For $x \in K$, $|x| = 0$ if and only if $x = 0$.

(b) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

(c) $|1| = 1$.

(d) $|xy| = |x| \cdot |y|$ for all $x, y \in K$.

A *normed* K*-algebra* is a K-algebra R together with a norm $|| \cdot || : R \to \mathbb{R}_{\geq 0}$ extending that of K and satisfying (a) and (b) above, as well as the following properties.

(c') $||1|| \leq 1$.

(d') $||xy|| \leq ||x|| \cdot ||y||$ for all $x, y \in R$..

(e') $||\lambda x|| = |\lambda| \cdot ||x||$ for all $\lambda \in K$, and $x \in R$.

If R satisfies (c) and (d) instead of (c') and (d'), then R is called a *multiplicative normed* K*-algebra.*

We make K into a topological field by letting the sets $\{|x| \leq \gamma\}$ for $\gamma \in \mathbb{R}_{>0}$ form a neighborhood basis for 0, and make R into a topological ring the same way. These topological rings and fields are called *nonarchimedean.*

Example 2.6

The field $\mathbb{Q}$ with the topology induced by the p-adic absolute value is a nonarchimedean topological field. It contains $\mathbb{Z}$ as an open subset whose topology coincides with the p-adic topology it caries as an adic ring. Notice that $\mathbb{Q}$ with this topology is not adic, because p generates the unit ideal. We will later define a ring which is not necessarily adic but whose topology can be generated by an open adic subring as *Huber.*

Example 2.7

Let K be a nonarchimedean field and let $K^\circ = \{x : |x| \leq 1\}$ be its valuation ring. Then K° is a nonarchimedean open subring of K (although not a K-algebra). For any $x \in K \setminus K^\circ$, we have $K = K^\circ[x]$. Indeed, fix any $y \in K$. Since $|x| > 1$ we have $|x^{-1}| < 1$ and so some N we have $|yx^{-N}| \leq 1$. Thus

$$y = yx^{-N} \cdot x^N \in K^\circ[x].$$

Example 2.8

Let K be a nonarchimedean field. We can form the polynomial ring $K[T]$ and endow it with the *Gauss norm* $||a_nT^n + \cdots + +a_1T + a_0|| = \max\{|a_i|\}$. This makes $K[T]$ into a multiplicatively normed K-algebra.

Definition 2.9. A sequence of elements $(x_1, x_2, \cdots)$ in a topological ring R is said to *converge to* $x \in R$ if for all neighborhoods U of 0, there is some $N >> 0$ such that for all $n > N$ we have $x_n - x \in U$. If (x_n) converges to x, we write $\lim_{n\to\infty} x_n = x$.

A sequence of elements $(x_1, x_2, \cdots)$ in a topological ring R is called *Cauchy* if for all neighborhoods U of 0 there exists some $N >> 0$ such that for all $n, m > N$, $x_n - x_m \in U$.

A topological ring is *complete* if every Cauchy sequence coverges to a value $x \in R$.

Under modest assumptions, one can formally adjoin Cauchy sequences to a topological ring in order to make it complete.

Definition/Theorem 2.10 (Completions of Topological Rings [5] III.6.5)

Let R be a topological ring with a neighborhood basis of 0 consisting of open subgroups. Then there is a complete topological ring $\hat{R}$ together with a continuous homomorphism $i : R \to \hat{R}$ which is initial among continuous homomorphisms from R to complete topological rings. $\hat{R}$ is called the *completion* of R. The formation of the completion is functorial among topological rings with this bases of 0 consisting of open subgroups.

Corollary 2.11

Both adic rings and nonarchimedean rings satisfy the conditions of Theorem 2.10, and therefore have completions which are unique up to unique isomorphism. The same is true for topological rings which contain an adic ring as an open subring (these we will later define as Huber rings, see Definition 3.7 below).

Example 2.12

If R is adic with ideal of definition I, then $\hat{R} \cong \varprojlim_n R/I^n$. If R is noetherian then the natural map $R \to \hat{R}$ is flat.

Example 2.13

Let $\mathbb{Z}$ have the p-adic topology. Then $\hat{\mathbb{Z}} \cong \mathbb{Z}_p$ is the ring of p-adic integers.

The completion of $\mathbb{Q}$ with respect to the p-adic norm is the field $\mathbb{Q}_p$ of p-adic numbers. Notice that $\mathbb{Z}_p$ is naturally an open subring of $\mathbb{Q}_p$, and and is in fact the valuation ring of $\mathbb{Q}_p$, consisting of elements $x \in \mathbb{Q}_p$ with $|x| \leq 1$ in the p-adic norm. Therefore $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$.

Definition 2.14. Let R be a topological ring. An element $x \in R$ is called *topologically nilpotent* if

$$\lim_{n\to\infty} x^n = 0.$$

That is, if for all open neighborhoods U of 0, there is some $N >> 0$ such that for all $n > N$, $x^n \in U$.

A subset B of R is *bounded* if for every neighborhood U of 0, there exists a neighborhood V of 0 such that $V \cdot B \subseteq U$.

An element $x \in R$ is called *power-bounded* if the set $\{x^n : n \geq 1\}$ is bounded.

We denote the set of power-bounded elements by R°, and the set of topologically nilpotent elements by $R^{\circ\circ}$.

The following example justifies our terminology.

Example 2.15 ([37] Example 5.29)

Let R be a nonarchimedean ring. An element x is power-bounded if and only if $|x| \leq 1$ and topologically nilpotent if and only if $|x| < 1$. A subset B of R is bounded if and only if there is some $\lambda \in \mathbb{R}_{>0}$ such that $|b| \leq \lambda$ for all $b \in B$.

Proposition 2.16 ([37] Proposition 5.30)

Let R be a nonarchimedean ring. The set R° of power-bounded elements is an integrally closed subring of R, and the set $R^{\circ\circ}$ of topologically nilpotent elements is a radical ideal of R°.

Example 2.17

The p-adic integers $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ form the subring of power-bounded elements, and the ideal of topologically nilpotent elements is precisely the maximal ideal $p\mathbb{Z}_p$ of $\mathbb{Z}_p$.

More generally if K is a nonarchimedean field, then K° is its valuation ring with maximal ideal $K^{\circ\circ}$, and the quotient $K^\circ/K^{\circ\circ}$ is the residue field of K.

Definition 2.18. Let K be a complete nonarchimedean field. A complete normed K-algebra is called a *Banach K-algebra*.

Example 2.19

Let K be a complete nonarchimedean field. The completion of $K[T]$ with the topology induced by the Gauss norm is the ring $K\langle T\rangle$ of convergent power series over K. It consists of formal power series $\sum a_n T^n$ where $\lim_{n\to\infty} a_n = 0$. The Gauss norm extends as $||\sum a_n T^n|| = \sup\{|a_n|\}$ making $K\langle T\rangle$ into a multiplicative Banach K-algebra.

More generally, given a complete nonarchimedean ring R, we can define the ring of convergent power series $R\langle T_1, \cdots, T_n\rangle$ by completing the polynomial ring $R[T_1, \cdots, T_n]$ equipped with the topology induced by the Gauss norm.

Banach algebras need not be commutative.

Example 2.20

Let K be a complete nonarchimedean field, and endow the matrix algebra $M_n(K)$ with the *Gauss norm* $||(a_{ij})|| = \max\{||a_{ij}||\}$. This makes $M_n(K)$ into a Banach K-algebra.

More generally, given a Banach algebra R, the Gauss norm makes $M_n(R)$ into a Banach algebra as well.

There are a number of useful technical results that we record here for later reference. The first is that an integer is always power-bounded.

Lemma 2.21

In any nonarchimedean group, any integer has absolute value less than or equal to one.

Proof. $||n|| = ||1 + \cdots + 1|| \leq \max\{||1||, \cdots, ||1||\} = 1$.

Another important fact about nonarchimedean rings is that "all triangles are isosceles".

Lemma 2.22

Let R be a nonarchimedean ring, and $a, b \in R$. If $||a|| \neq ||b||$ then $||a + b|| = \max\{||a||, ||b||\}$.

Proof. Without loss of generality we let $||a|| > ||b||$. If $||a + b|| < \max\{||a||, ||b||\} = ||a||$, then

$$||a|| = ||a + b - b|| \leq \max\{||a + b||, ||b||\} < ||a||,$$

a contradiction.

The nonarchimedean property can make it much easier to check if a limit converges. In particular, we only need to check sequential elements.

Lemma 2.23

A sequence $(a_1, a_2, \cdots)$ of elements of a nonarchimedean ring R are Cauchy if and only if for all $\varepsilon > 0$, there is some N such that for all $m > N$, we have $||a_{m+1} - a_m|| < \varepsilon$.

Proof. Fix $n \geq m > N$. Then

$$\begin{aligned} ||a_n - a_m|| &= ||(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{m+1} - a_m)|| \\ &\leq \max\{||a_n - a_{n-1}||, \cdots, ||a_{m+1} - a_m||\} \\ &< \varepsilon. \end{aligned}$$

The nonarchimedean property also makes it easier for infinite sums to converge.

Lemma 2.24

Let R be a complete nonarchimedean ring, and $a_n \in R$. The infinite sum $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose (a_n) converges to 0. We must show the partial sums $s_m = \sum_{n=0}^{m} a_n$ converge. For every $\varepsilon > 0$ there is some N such that for any $m > N$ we have $|a_m| < \varepsilon$. In particular, for $m \geq r > N$

$$||s_m - s_r|| = ||a_m + a_{m-1} + \cdots + a_r|| \leq \max\{||a_m||, ||a_{m-1}||, \cdots, ||a_r||\} < \varepsilon,$$

so that the partial sums are Cauchy. Since R is complete they converge. The other direction is immediate.

Remark 2.25

Notice that this implies that the ring $K\langle T\rangle$ defined in Example 2.19 consists precisely of formal power series which converge when evaluated on K°, justifying the nomenclature.

We get the following immediate consequence.

Lemma 2.26

Let R be a complete nonarchimedean ring, and $f \in R$ topologically nilpotent. Then $1 - f$ is a unit in R.

Proof. The geometric series $\frac{1}{1-f} = \sum_{n=0}^{\infty} f^n$ converges to an inverse of $1 - f$ by Lemma 2.24.

For homomorphisms between Banach algebras, continuity is readily checked.

Definition 2.27. Let $f : R \to S$ be a homomorphism of Banach algebras. We say f is bounded if there is some $\rho \in \mathbb{R}$ such that for all $x \in R$, $||f(x)|| \leq \rho||x||$.

Proposition 2.28 ([4] Corollary 2.1.8.3)

A homomorphism of Banach algebras is continuous if and only if it is bounded.

2.3 A Convergence Result

The following is a general convergence result for certain nonarchimedean Banach algebras. It will prove useful several times throughout the paper.

Proposition 2.29

Let K be a complete nonarchimedean field, either of characteristic p or 0. Suppose that p is topologically nilpotent and that we can choose a consistent sequence of pth power roots of p, that is $(p, p^{1/p}, \cdots)$. Let R be a Banach K-algebra. Let $(a_0, a_1, a_2, \cdots)$ and $(b_0, b_1, b_2, \cdots)$ be sequences of elements in R such that for all i, $a_{i+1}^p = a_i$, and $b_{i+1}^p = b_i$. Then $\lim_{n\to\infty} (a_n + b_n)^{p^n}$ converges in R.

Remark 2.30

K having consistent sequences of pth power roots of p is actually quite close to K being perfectoid, and any perfectoid field will have this property. See Definition 2.37.

The characteristic p case is immediate, as each term in the limit is $a_0 + b_0$. Therefore we assume that the characteristic is 0. The proof will involve a series of reductions. Here is the first.

Claim 2.31

It suffices to show the result for sequences with $a_0, b_0 \in R^\circ$.

Proof. Let $\gamma = p^N$ for n large enough so that $||\gamma a_0||, ||\gamma b_0|| < 1$. Using the sequence of pth power roots for p, we can form a sequence of pth power roots for γ. For each m, define $a'_m = \gamma^{1/p^m} a_m$ and $b'_m = \gamma^{1/p^m} b_m$. Then $a'^p_{m+1} = a'_m$ and $b'^p_{m+1} = b'_m$. Since $||a'_0||, ||b'_0|| \leq 1$, they are in R°, so we may assume that $\lim_{n\to\infty} (a'_n + b'_n)^{p^n}$ converges, and therefore so does

$$\begin{aligned}\gamma^{-1} \lim_{n\to\infty} (a'_n + b'_n)^{p^n} &= \lim_{n\to\infty} \left(\gamma^{-1/p^n} a'_n + \gamma^{-1/p^n} b'_n\right)^{p^n} \\ &= \lim_{n\to\infty} \left(\gamma^{-1/p^n}\gamma^{1/p^n} a_n + \gamma^{-1/p^n}\gamma^{-1/p^n} b_n\right)^{p^n} \\ &= \lim_{n\to\infty} (a_n + b_n)^{p^n}.\end{aligned}$$

Here is the second reduction.

Claim 2.32

It suffices to show that if $a, a', b, b' \in R^\circ$, with $a'^p = a$ and $b'^p = b$ we have

$$\left|\left|(a+b)^{p^m} - (a'+b')^{p^{m+1}}\right|\right| \leq ||p^m||.$$

Proof. Fix any real number $\varepsilon > 0$. Since p is topologically nilpotent there is some $N >> 0$ such that for all $m > N$ we have $||p^m|| < \varepsilon$. The difference between the mth and $(m+1)$st terms in the sequence is

$$(a_m + b_m)^{p^m} - (a_{m+1} + b_{m+1})^{p^{m+1}}.$$

Letting $a = a_m$, $a' = a_{m+1}$, $b = b_m$, and $b' = b_{m+1}$, we may assume that this difference has absolute value $< \varepsilon$. Thus by Lemma 2.23 we are done.

A final reduction follows.

Claim 2.33

It suffices to show that for $x, y \in R^\circ$,

$$\left|\left|(x+py)^{p^m} - x^{p^m}\right|\right| \leq ||p^m||.$$

Proof. Notice that $(a_0 + b_0)^{p^{m+1}} = ((a_0 + b_0)^p)^{p^m} = (a + b + p \cdot y)^{p^m}$ for some $y \in R^\circ$. Letting $x = a + b$ we have

$$\left|\left|(a+b)^{p^m} - (a_0+b_0)^{p^{m+1}}\right|\right| = \left|\left|x^{p^m} - (x+py)^{p^m}\right|\right| \leq ||p^m||.$$

Before completing the final step we need the following combinatorial result.

Lemma 2.34

For $n \in \mathbb{Z}$, we denote by $v_p(n) := \max\{k : p^k | n\}$. Then

$$v_p\left(\binom{p^k}{i}\right) = k - v_p(i).$$

Proof. We follow a proof of Bruno Joyal in [18].

Let $q = p^k$. We hope to compute $v_p(q!) - v_p(i!) - v_p((q-i)!)$.

Notice first that for any n, we have

$$v_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor .$$

In particular, $v_p(q!) = \frac{p^k-1}{p-1}$. Also for any $x \in R$ we have

$$\lfloor -x \rfloor + \lfloor x \rfloor = \begin{cases} 0 & : x \in \mathbb{Z} \\ -1 & : x \notin \mathbb{Z} \end{cases}$$

Therefore,

$$\begin{aligned} v_p((q-i)!) + v_p(i!) &= \sum_{j=1}^{n} \left\lfloor \frac{p^n - i}{p^j} \right\rfloor + \left\lfloor \frac{i}{p^j} \right\rfloor \\ &= \sum_{j=1}^{n} \left(p^{n-j} + \left\lfloor \frac{-i}{p^j} \right\rfloor + \left\lfloor \frac{i}{p^j} \right\rfloor \right) \\ &= \frac{p^k - 1}{p-1} - (k - v_p(i)). \end{aligned}$$

Subtracting this from $v_p(q!)$ gives the result.

Now we can complete the proof of Proposition 2.29 with the following lemma.

Lemma 2.35
Let $x, y \in R^\circ$. Then

$$\left|\left|(x+py)^{p^m} - x^{p^m}\right|\right| \leq ||p^m||.$$

Proof. First notice that for $a \in \mathbb{Z}$, if $p^m | a$ in $\mathbb{Z}$, then $||a|| \leq ||p^m||$. Indeed, if $p^m \cdot r = a$, then $||p^m|| \cdot ||r|| = ||a||$. Since r is an integer, we know by Lemma 2.21 that $||r|| \leq 1$, so that $||a|| \leq ||p^m||$. Now consider,

$$\begin{aligned} \left|\left|(x-py)^{p^m} - x^{p^m}\right|\right| &= \left|\left| \left(\sum_{i=0}^{p^m} \binom{p^m}{i} x^{p^m - i} p^i y^i \right) - x^{p^m} \right|\right| \\ &= \left|\left| \sum_{i=1}^{p^m} \binom{p^m}{i} x^{p^m - i} p^i y^i \right|\right| \\ &\leq \max_{i=1,\cdots,p^m} \left\{ \left|\left| \binom{p^m}{i} p^i \right|\right| \right\}, \end{aligned}$$

using in the last step that $||x||, ||y|| \leq 1$. Therefore it suffices to show that $p^m \left| \binom{p^m}{i} \cdot p^i \right.$ But this follows from Lemma 2.34. Indeed, for any r, if $p^r \leq i < p^{r+1}$, then Lemma 2.34 shows that $p^{m-r} \left| \binom{p^m}{i} \right.$ Certainly $p^r | p^i$, so that p^m divides their product.

2.4 Perfectoid Fields

We now can define some of the central objects of our study.

Definition 2.36. Let K be a nonarchimedean valued field. The *residue characteristic* of K is the characteristic of the residue field $K^\circ / K^{\circ\circ}$.

Definition 2.37. A *perfectoid field* is a nondiscretely valued complete nonarchimedean field K of residue characteristic p on which the Frobenius morphism $x \mapsto x^p$ on $K^\circ/(p)$ is surjective.

Remark 2.38
The condition that the valuation is nondiscrete eliminates unramified extensions of $\mathbb{Q}_p$.

Definition 2.39. Every perfectoid field K has a (not necessarily unique) topologically nilpotent unit ϖ, that is, a nonzero element with $|\varpi| < 1$ called a *pseudouniformizer.*

Example 2.40

1. The main examples in characteristic 0 are the p-adic completions of $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$ and $\mathbb{Q}_p\left(\mu_{p^\infty}\right)$. Also of interest is $\mathbb{C}_p$, the completed algebraic closure of $\mathbb{Q}_p$. In each case, p serves as a pseudouniformizer.

2. A complete nonarchimedean field of characteristic $p > 0$ is perfectoid if and only if it is perfect. A fundamental example is the completion of $k\left(\left(t^{1/p^\infty}\right)\right)$. Indeed, if K is any perfectoid field of characteristic p, and residue field k, then K contains $k\left(\left(t^{1/p^\infty}\right)\right)$, where t is any element of K with $0 < |t| < 1$. Notice that t serves as a pseudouniformizer, and that K is endowed with the t-adic topology.

Definition 2.41. Let K be a perfectoid field. The *tilt* of K is defined as follows. First, as a multiplicative monoid we define

$$K^\flat := \varprojlim_{x \mapsto x^p} K.$$

The addition law of $K^\flat$ follows the rule $(a_n) + (b_n) = (c_n)$, where

$$c_n = \lim_{m \to \infty} \left(a_{n+m} + b_{n+m}\right)^{p^m}.$$

The limit converges due to Proposition 2.29. Notice that the formal elements of the tilt correspond to consistent sequences of pth power roots of elements of K.

Lemma 2.42 ([27] Proposition 3.8)

(i) The reduction map $K^\circ \to K^\circ/p$ induces an isomorphism of topological multiplicative monoids

$$K^{\circ\flat} := \varprojlim_{x \mapsto x^p} K^\circ \to \varprojlim_{x \mapsto x^p} K^\circ/p,$$

whose inverse is $(a_n \bmod p) \mapsto (b_n)$ where

$$b_n = \lim_{m \to \infty} a_{m+n}^{p^n}.$$

The limit does not depend on the lift of a_n. This gives $K^{\circ\flat}$ a canonical ring structure.

(ii) Give $K^{\circ\flat}$ the induced ring structure, $K^\flat$ is the field of fractions.

(iii) Projection onto the first coordinate defines a map of topological monoids $K^\flat \to K$, which we call the Teichmüller map *and denote by $x \mapsto x^\sharp$. The absolute value $|x|_{K^\flat} := |x^\sharp|_K$ induces a topology on $K^\flat$ making it into a perfectoid field of characteristic p, with integral subring $K^{\circ\flat}$, i.e., $K^{\circ\flat} \cong K^{\flat\circ}$.*

(iv) The Teichmüller map $\sharp$ induces an isomorphism:

$$K^{\flat\circ}/K^{\flat\circ\circ} \cong K^\circ/K^{\circ\circ}.$$

Furthermore, for any nonzero $\varpi \in K^{\flat\circ}$, we have:

$$K^{\flat\circ}/\varpi \cong K^\circ/\varpi^\sharp.$$

(v) If K is of characteristic p, then $K^\flat \cong K$.

Remark 2.43

Lemma 2.42 allows nonisomorphic fields to have isomorphic tilts. Indeed, take any perfectoid field K of characteristic 0. Then K and $K^\flat$ both tilt to $K^\flat$. In fact, nonisomorphic fields of characteristic 0 can tilt to the same field of characteristic p, see Example 2.44 below.

Example 2.44

1. Let $K = \widehat{\mathbb{Q}_p\left(p^{1/p^\infty}\right)}$. Then we have the element $t = (p, p^{1/p}, p^{1/p^2}, \ldots) \in K^\flat$, and $|t| = |p| < 1$. Our discussion in Example 2.40 shows that $\mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right) \subseteq K^\flat$. To see that they are the same, notice that they have the same integral subrings. Indeed,

$$K^\circ/p \cong \mathbb{Z}_p\left[p^{1/p^\infty}\right]/(p) \cong \mathbb{F}_p\left[t^{1/p^\infty}\right]/(t).$$

Applying $\lim_{x\mapsto x^p}$ to both sides, and applying part (iii) of Lemma 2.42 we see that

$$K^{\flat\circ} \cong \left(\mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right)\right)^\circ.$$

Taking fields of fractions shows

$$K^\flat \cong \mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right).$$

2. Let $K = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, then a similar argument shows that $K^\flat \cong \mathbb{F}_p\left(\left(t^{1/p^\infty}\right)\right)$. Indeed, viewing $K^\flat$ as the fraction field of $\varprojlim K^\circ/p$, we can take $t = (1-\zeta_p, 1-\zeta_{p^2}, \ldots)$, where ζ_{p^i} are primitive p^ith roots of unity satisfying $\zeta_{p^{i+1}}^p = \zeta_{p^i}$

Remark 2.45

One can construct a type of moduli space of fields tilting to a given perfectoid field. It is a projective curve whose closed points correspond to untilts, and can be realized as a scheme or an adic space (see Section 3). This curve is called the *Fargues-Fontaine curve*, or also the *fundamental curve of p-adic Hodge theory.* For a more detailed discussion, see, for example [8].

The following theorem of Scholze generalizes the isomorphism of Fontaine and Wintenberger (Theorem 1.1).

Theorem 2.46 ([27] Theorem 3.7)

Let K be a perfectoid field.

(i) Let L be a finite extension of K. Then L with the natural topology as a finite K vector space is a perfectoid field.

(ii) Let $K^\flat$ be the tilt of K. Then the tilting functor $L \to L^\flat$ induces an equivalence of categories between the category of finite algebraic extensions of K and finite algebraic extensions of $K^\flat$. This equivalence preserves degrees, and therefore induces a topological isomorphism of absolute Galois groups $G_K \cong G_{K^\flat}$.

Remark 2.47

This result was independently obtained by Kedlaya-Liu, [21] Theorem 3.5.9.

Remark 2.48

It is useful to describe the quasi-inverse explicitly. Since we are moving from characteristic p to characteristic 0, one should expect the ring of Witt vectors to arise. Let us briefly review the definition of this ring.

Given a perfect ring R of characteristic p, we assign to it a ring of characteristic 0, $W(R)$, which is complete in the p-adic topology, along with a map of multiplicative monoids $R \to W(R)$, denoted $x \mapsto [x]$, which is initial among maps of multiplicative monoids $R \to S$, where S is p-adically complete ring of characteristic 0 and $R \to S \to S/p$ is a ring homomorphism. $W(R)$ is called the ring of *Witt vectors*, and its formation is functorial in R. The elements of $W(R)$ can be written uniquely as formal power series $[x_0] + [x_1]p + [x_2]p^2 + \cdots$ with $x_i \in R$.

Notice that $K^{\flat\circ}$, is a perfect ring, so it has a ring of Witt vectors $W(K^{\flat\circ})$. Because $x \mapsto x^\sharp$ is a map of multiplicative monoids from $K^{\flat\circ} \to K^\circ$ which becomes a ring map after reducing modulo p, the universal property gives us a ring homomorphism $\theta : W(K^{\flat\circ}) \to K^\circ$. By, [38] Lemma 2.2.1, after inverting p, θ induces a surjective homomorphism, $\theta : W(K^{\flat\circ})\,[1/p] \to K$, making K into a $W(K^{\flat\circ})$-algebra.

We use this to describe the inverse to the functor $L \to L^\flat$. Let $M/K^\flat$ be a finite extension. Then M° is perfect and $W(M^\circ)$ is a finite $W(K^{\circ\flat})$-algebra. We define

$$M^\sharp := W(M^\circ) \otimes_{W(K^{\flat\circ})} K.$$

Then $M^\sharp$ is a perfectoid field, and the map $M \to M^\sharp$ sending $x \mapsto x^\sharp := [x] \otimes 1$ is a multiplicative map. The map $M \to M^{\sharp\flat}$ given by $x \mapsto \big(x^\sharp, (x^{1/p})^\sharp, \ldots\big)$ is an isomorphism.

Remark 2.49
Lemma 2.2.1 in [38] shows that every characteristic 0 field whose tilt is $K^\flat$ must be a quotient of $W(K^{\flat\circ})\,[1/p]$ by some maximal ideal.

2.5 Perfectoid Algebras

If perfectoid fields are going to correspond to 'points' of our spaces, then our affine patches (in this case the term will be affinoid) should correspond to certain classes of rings. To start, we fix a perfectoid field K and pseudouniformizer $\varpi \in K$. We will also need a pseudouniformizer $\varpi^\flat \in K^\flat$, and we may assume that $(\varpi^\flat)^\sharp = \varpi$. It will turn out that none of the following depends on the choices of ϖ and $\varpi^\flat$, but it is useful to have one fixed at the outset. See, for example, [27] Section 3.

Definition 2.50. A Banach K-algebra R is called a *perfectoid* K-algebra if the subset $R^\circ \subset R$ of power-bounded elements is open and bounded, and the Frobenius morphism $\Phi : R^\circ/\varpi \to R^\circ/\varpi$ given by $x \mapsto x^p$ is surjective. Morphisms between perfectoid algebras are continuous homomorphisms of K-algebras.

Remark 2.51
Scholze in [27] initially only defined perfectoid algebras over perfectoid fields, but the same definition makes sense for general rings, defining a *perfectoid ring.* In fact, there are examples of perfectoid rings which do not arise as algebras over a perfectoid field. See, for example, [19] Exercise 2.4.10. We will focus on perfectoid K-algebras in this paper.

Remark 2.52 ([19] Section 2)
If R is a perfectoid K-algebra, then $R = R^\circ[1/\varpi]$.

Remark 2.53 ([19] Corollary 2.9.3)
A perfectoid K-algebra is noetherian if and only if it is a finite direct sum of perfectoid fields.

Lemma 2.54
If R is a perfectoid K-algebra, then R is reduced.

Proof. If $x \in R$ is nilpotent, λx is nilpotent for all $\lambda \in K$, and therefore topologically nilpotent and thus power-bounded. In particular, $||\lambda x|| \leq 1$ for all $\lambda \in K$. This can only happen if $x = 0$.

As with fields, in positive characteristic, being perfectoid is the same as being perfect.

Proposition 2.55 ([27] Proposition 5.9)

If K has characteristic p, and R a Banach K-algebra such that the set of power-bounded elements is open and bounded, then R is perfectoid if and only if it is perfect.

We can tilt perfectoid algebras the same way we tilt perfectoid fields.

Definition 2.56. Given R perfectoid over K, we form $R^\flat := \varprojlim_{x \mapsto x^p} R$ as a multiplicative monoid, with addition defined by the rule, $(a_n) + (b_n) = (c_n)$ where

$$(c_n) = \lim_{m \to \infty} (a_{n+m} + b_{n+m})^{p^m}.$$

The limit converges due to Proposition 2.29. As with fields, there is a continuous map of multiplicative monoids $\sharp : R^\flat \to R$ called the *Teichmüller map*, given by projection onto the first coordinate, which we denote by $x \mapsto x^\sharp$. Defining an absolute value on $R^\flat$ by $|x|_{R^\flat} = |x^\sharp|_R$ makes $R^\flat$ into a perfectoid $K^\flat$-algebra. The map $x \mapsto x^\sharp$ descends to an isomorphism of rings

$$R^{\flat\circ}/(\varpi^\flat) \xrightarrow{\sim} R^\circ/(\varpi).$$

The following extends Theorem 2.46.

Theorem 2.57 ([27] Theorem 5.2)

The functor $R \mapsto R^\flat$ is an equivalence from the category of perfectoid K-algebras to the category of perfectoid $K^\flat$-algebras. A quasi-inverse is given by

$$S \mapsto W(S^\circ) \otimes_{W(K^{\flat\circ})} K.$$

Example 2.58

In classical algebraic geometry, the polynomial ring serves as the ring of regular functions for affine space. Analogously, in rigid analytic geometry, there is the Tate algebra $K\langle T_1, \dots, T_n\rangle$ of power series over a complete nonarchimedean field K which converge on the unit polydisk. This serves as the ring of regular functions for the unit polydisk over K. If K is a perfectoid space, we see that the Tate algebra is not a perfectoid K algebra, because the indeterminates don't have pth power roots. The natural extension is called the *perfectoid Tate algebra*, denoted,

$$T_{n,K}^{\text{perf}} := K\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle.$$

It is constructed as follows. We first consider,

$$K^\circ\left[T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right] = \bigcup_{m \geq 0} K^\circ\left[T_1^{1/p^m}, \cdots, T_n^{1/p^m}\right],$$

and then take the completion with respect to the ϖ-adic topology to form $\left(T_{n,K}^{\text{perf}}\right)^\circ$. Finally we invert ϖ. The Gauss norm makes $T_{n,K}^{\text{perf}}$ into a multiplicatively normed Banach K-algebra. We study this ring extensively be in Sections 5 through 7.

It is worth noting that the same construction allows us to build $R\left\langle T_1^{1/p^\infty} \cdots, T_n^{1/p^\infty}\right\rangle$ for any perfectoid algebra R.

The construction of the perfectoid Tate algebra is compatible with the tilting operation.

Proposition 2.59 ([27] Proposition 5.20)

$T_{n,K}^{\text{perf}}$ is a perfectoid K algebra whose subring of power-bounded elements is $\left(T_{n,K}^{\text{perf}}\right)^\circ$. Its tilt is,

$$T_{n,K^\flat}^{\text{perf}} = K^\flat\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle,$$

where by abuse of notation $T_i^\sharp = T_i$.

Remark 2.60

This generalizes as well. If R is a perfectoid K-algebra, then so is $R\left\langle T_1^{1/p^\infty}, \cdot, T_n^{1/p^\infty}\right\rangle$ and its tilt is $R^\flat\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle$.

If R is a perfectoid K-algebra, there is a tilting equivalence for certain classes of algebras over R and $R^\flat$. Indeed, the natural generalization of algebraic field extensions over K and $K^\flat$ are finite étale algebras over R and $R^\flat$. It turns out we get a direct generalization of Theorem 2.46 and Theorem 2.57.

Theorem 2.61 ([27] Theorem 7.12)

Let R be a perfectoid K-algebra with tilt $R^\flat$,

(i) The functor $S \mapsto S^\flat$ defines an equivalence of categories between perfectoid R-algebras and perfectoid $R^\flat$-algebras.

(ii) A finite étale R-algebra S is perfectoid. The functor $S \mapsto S^\flat$ defines an equivalence of categories $R_{\text{fét}} \to R^\flat_{\text{fét}}$.

Since this occurs as rings, if we can 'glue' the tilting equivalences together in some reasonable way, we should get a geometric theory allowing us to tilt a space of characteristic 0 to characteristic p. Theorem 2.61 seems to imply that this would produce a new space with an isomorphic étale site. Unfortunately, as the following example illustrates, this procedure seems analytic in nature and would probably require a notion of limits which schemes do not allow.

Example 2.62

Suppose K is algebraically closed. The affine line over K should parametrize elements of K. Suppose we want to tilt the affine line $\mathbb{A}^1_K$. The result should be the affine line $\mathbb{A}^1_{K^\flat}$. By the tilting procedure, this should be 'equal to' the inverse limit $\varprojlim_{T\mapsto T^p} \mathbb{A}^1_K$ where T is the coordinate on $\mathbb{A}^1$. Nevertheless, the explicit map between $\mathbb{A}^1_{K^\flat}$ and $\varprojlim_{T\mapsto T^p} \mathbb{A}^1_K$ would require taking a p-adic limit. Therefore to actually formalize this isomorphism we need an analytic framework which respects the p-adic topology. Rigid spaces are not the right framework for nonnoetherian rings, and so Scholze settled on Huber's generalization of these, called adic spaces.

3 Adic Spaces

To build the prime spectrum of a ring R we consider the prime ideals of R as the underlying topological space and R as the ring of regular functions. Unfortunately the coarse Zariski topology makes it impossible to make analytic arguments go through without passing to some other category. If we are working over $\mathbb{C}$ then the we can use Serre's analytification functor and GAGA [31] in order to study our variety with its (much finer) complex analytic topology. If we are working over nonarchimedean fields over then we can view the variety as a rigid analytic space, where the topological space is locally $\operatorname{Spm}(R)$, the set of maximal ideals, and then we 'fill in' all the missing points by constructing a Grothendieck topology of admissible coverings (see [10]). Rigid spaces admit the following pathology, one can construct open immersions which are bijective but not isomorphisms, or even admissible coverings in the Grothendieck topology.

Example 3.1

Let $X = \operatorname{Spm} K\langle T\rangle$ be the rigid-analytic closed unit disk. Let Y be the disjoint union of the open unit disk U and the circle $S = \operatorname{Spm} K\langle T, T^{-1}\rangle$. Then $Y \to X$ is an open immersion which is bijective on the level of points, but it is not even a covering of X. See [10].

This problem is fixed by Huber in [15], where he builds the underlying topological space by suitably topologizing a set of valuations on the ring R. This fixes our pathological example because we get extra points on X which are 'between' S and U, so that $Y \to X$ is no longer surjective, see Section 3.4 below for the details.

A more pressing issue for us is that rigid analytic geometry can only speak about certain classes of rings, all of which are noetherian. Perfectoid algebras are rarely noetherian (see Remark 2.53), so if we want to use rigid analytic methods to study perfectoid algebras we need a different framework. The advantage of Huber's approach is it allows us to study a far larger class of rings which are not affinoid in the sense of rigid geometry. Among these are all perfectoid rings and algebras.

3.1 Valuation Spectra

If X is an 'analytic' space of some sort, we should be able to take limits of points on X. An important tool in doing this is to be able to define subsets of X by inequalities. That is, for any $f \in \Gamma(X, \mathscr{O}_X)$, the subset $\{x : |f(x)| \leq 1\}$ should make sense. In particular, any point $x \in X$ should give rise to a *valuation* on $\Gamma(X, \mathscr{O}_X)$. First let us recall what a valuation is.

Definition 3.2. Let R be a commutative ring. A valuation on R is given by a map $|\cdot| : R \to \Gamma \cup \{0\}$ where Γ is some totally ordered group, satisfying the following properties.

(i) $|a+b| \leq \max\{|a|, |b|\}$ for all $a, b \in R$.

(ii) $|ab| = |a| \cdot |b|$ for all $a, b \in R$.

(iii) $|0| = 0$ and $|1| = 1$.

The *support* of $|\cdot|$ is $\operatorname{supp}(|\cdot|) := |\cdot|^{-1}(0)$. The *value group* of $|\cdot|$, denoted $\Gamma_{|\cdot|}$, is the group closure of the monoid $|R^\times|$ in Γ.

Notice that $\operatorname{supp}(|\cdot|)$ is a prime ideal so that the quotient $R/\operatorname{supp}(|\cdot|)$ is a domain. Let L be the fraction field. Then the valuation $|\cdot|$ factors as $R \to L \to \Gamma \cup \{0\}$ for a unique valuation on L. Let $A(|\cdot|) = \{x \in L : |x| \leq 1\}$, be the associated valuation ring.

Definition 3.3 ([37] Proposition/Definition 1.27). Two valuations $|\cdot|$ and $|\cdot|'$ are called equivalent if the following equivalent conditions are satisfied.

(i) There is an isomorphism of totally ordered groups $\alpha : \Gamma_{|\cdot|} \to \Gamma_{|\cdot|'}$ and $|\cdot|' = \alpha \circ |\cdot|$.

(ii) $\operatorname{supp}(|\cdot|) = \operatorname{supp}(|\cdot|')$, and $A(|\cdot|) = A(|\cdot|')$.

(iii) For all $a, b \in R$, we have $|a| \leq |b|$ if and only if $|a|' \leq |b|'$.

Remark 3.4

For the rest of this paper we will conflate the notions of valuations and equivalence classes of valuations. For example, in the next definition we define the valuation spectrum simply as the set of valuations on a ring, when what we really mean is the set of equivalence classes of valuations. This is a standard abuse of notation to avoid overly cumbersome wording.

Definition 3.5. Let R be a ring. The *valuation spectrum of* R, denoted $\operatorname{Spv} R$, is the set of all valuations on R. The topology on $X = \operatorname{Spv} R$ is generated by the open sets

$$X\left(\frac{T}{g}\right) := \{|\cdot| \in X : |t| \leq |g| \neq 0 \text{ for all } t \in T\},$$

for all finite subsets $T \subseteq R$ and all $g \in R$.

If $\phi : R \to S$ is a homomorphism of rings, pulling back valuations induces a continuous map $\operatorname{Spv}(\phi) : \operatorname{Spv} S \to \operatorname{Spv} R$ which maps $|\cdot| \mapsto |\cdot| \circ \phi$. This makes Spv into a contravariant functor from rings to topological spaces.

The support map $\operatorname{supp} : \operatorname{Spv} R \to \operatorname{Spec} R$ is continuous. Indeed, if $f \in R$, then the preimage of the associated distinguished open $D(f)$ is $X\left(\frac{f}{f}\right)$. Futhermore, this map is surjective. Indeed, for $\mathfrak{p} \in \operatorname{Spec} R$ let $|\cdot|_{\mathfrak{p}}$, be the valuation which evaluates to 0 on $\mathfrak{p}$, and 1 everywhere else. Then $\operatorname{supp}(|\cdot|_{\mathfrak{p}}) = \mathfrak{p}$. Thus we can view $\operatorname{Spv} R$ as fibered over $\operatorname{Spec} R$. Even better, this fibration localizes.

Proposition 3.6 ([37] Remark 4.4)

Let R be a ring.

(i) Let $S \subset R$ be a multiplicative subset, and $\phi : R \to S^{-1}R$ the canonical localization map. Then $\operatorname{Spv}(\phi)$ is a homeomorphism of $\operatorname{Spv}(S^{-1}R)$ onto the open subspace

$$\{v \in \operatorname{Spv} R : \operatorname{supp}(v) \cap S = \emptyset\}.$$

(ii) Let $\mathfrak{a} \subset R$ be an ideal, and $\pi : R \to R/\mathfrak{a}$ the quotient map. Then $\operatorname{Spv}(\pi)$ is a homeomorphism of $\operatorname{Spv}(R/\mathfrak{a})$ onto the closed subspace $\{v \in \operatorname{Spv} R : \operatorname{supp}(v) \supseteq \mathfrak{a}\}$.

In particular, if R' is a localization or quotient of R, then

$$\begin{tikzcd} \operatorname{Spv} R' \arrow[r] \arrow[d] & \operatorname{Spv} R \arrow[d] \\ \operatorname{Spec} R' \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

is Cartesian.

The space of all valuations on a ring is much too large for our purposes, and it does not detect the topology of a ring. An (affinoid) adic space will be a subspace of the space of all valuations, constructed on certain topological rings.

3.2 Huber Rings

Recall from Example 2.13 that although $\mathbb{Q}_p$ is not adic, it is quite close. In particular, it is the field of fractions of an adic ring that it contains as an open subring, namely $\mathbb{Z}_p$. Perfectoid rings arise similarly, with the topology on R induced by an adic topology on the subring of power-bounded elements R°. This motivates the following definition.

Definition 3.7. A *Huber ring* is a topological ring R containing an adic open subring R_0, whose topology is induced by a finitely generated ideal of definition I. R_0 is called a *ring of definition*, (R_0, I) is called a *pair of definition*. We remark that the data of R_0 and I are not packaged with the definition, instead only their existence is asserted.

A Huber ring is *Tate* if it contains a topologically nilpotent unit, called a *pseudouniformizer*.

More generally, a Huber ring is *analytic* if the unit ideal can be generated by topologically nilpotent elements.

If the subset R° of power-bounded elements is bounded, then R is called *uniform*.

Example 3.8

1. Any ring R with the discrete topology is a Huber ring with $R_0 = R$ and $I = 0$.
2. Any nonarchimedean field K with nontrivial metric is Huber. $K^\circ = \{x : |x| \leq 1\}$ serves as a ring of definition and (ϖ) the ideal of definition for any pseudouniformizer $\varpi \in K^\times$. In particular, perfectoid fields are Huber, and even Tate. It is clear from the definition that perfectoid fields are uniform.
3. For a complete nonarchimedean field K, the Tate algebra $K\langle X_1, \cdots, X_n\rangle$ is a Huber ring, with ring of definition $K^\circ\langle X_1, \cdots, X_n\rangle$ and ideal of definition (ϖ). The existence of ϖ implies that the Tate algebra is Tate.
4. Suppose K is perfectoid. Then any perfectoid K-algebra R is Huber with ring of definition R° and ideal of definition (ϖ). Therefore all perfectoid K-algebras are Tate. It is clear from the definition of a perfectoid algebra that R is uniform. In general, not all perfectoid rings are Tate, although they are Huber, and even analytic (see [19] Section 2).

A subtlety in the definition of an adic space is its reliance on fixing an open integrally closed subring.

Definition 3.9. A pair of rings (R, R^+) is called a *Huber pair* if R is a Huber ring and $R^+ \subset R^\circ$ is an open, integrally closed subring of R, consisting of power-bounded elements. A morphism of Huber pairs, $\phi : (R, R^+) \to (S, S^+)$, is a ring map $\phi : R \to S$ such that $\phi(R^+) \subseteq S^+$.

Remark 3.10

Notice here that attached to a Huber ring R we now have 3 open subrings, R°, the ring of power-bounded elements, R_0 a ring of definition, and now R^+. These rings will often coincide. Indeed, for a uniform Huber ring, R^+ will always be a ring of definition (see, for example, [19] 2.1.1). Certainly R° is the largest open integrally closed subring of contained in R°, and in fact it is is the union of all possible R^+.

Definition 3.11. A Huber pair (R, R^+) is called *perfectoid* (resp. *Tate*, *analytic*, *uniform*, *complete*, etc.) if R is perfectoid (resp. Tate, analytic, uniform, complete, etc.)

A morphism of Huber pairs $\phi : (R, R^+) \to (S, S^+)$ is *continuous* (resp. finite étale) if $\phi : R \to S$ is.

3.3 The Adic Spectrum of a Huber Pair

Notation 3.12

Since we are working with Banach rings R endowed with an absolute value, we will henceforth use the notation $|\cdot|$ for the absolute value on R, and the notation x for the other valuations on R when regarded as points on $\operatorname{Spv} R$. Notice that $|\cdot|$ will have value group $\mathbb{R}_{>0}$ and we will denote by Γ_x the value group of a valuation x. For any $f \in R$, we will denote the image of f under x by $|f(x)| \in \Gamma_x$.

Definition 3.13. A valuation on a topological ring R is called *continuous* if the sets $\{f : |f(x)| < \lambda\}$ are open in R for every $\lambda \in \Gamma_x$, or equivalently if the topology defined by x is coarser than that of R. We denote by $\operatorname{Cont}(R)$ the subspace of $\operatorname{Spv}(R)$ constisting of continuous valuations.

Definition 3.14. Let (R, R^+) be Huber pair. We define the *adic spectrum* of (R, R^+) to be the subspace

$$\operatorname{Spa}(R, R^+) := \left\{x \in \operatorname{Cont}(R) : |f(x)| \leq 1 \text{ for all } f \in R^+\right\}$$

endowed with the subspace topology. Let $X = \operatorname{Spa}(R, R^+)$. A *rational subset* of X is one of the form:

$$X\left(\frac{T}{g}\right) = \{x \in X : |t(x)| \leq |g(x)| \neq 0 \text{ for all } t \in T\},$$

where $T \subseteq R$ is a finite set generating the unit ideal, and $g \in R$. Since $\operatorname{Spa}(R, R^+)$ is a subspace of $\operatorname{Spv} R$, these sets are open in X and in fact form a basis for the topology of X.

Let $\phi : (R, R^+) \to (S, S^+)$ be a continuous homomorphism of Huber pairs. This induces a continuous map $\operatorname{Spa}(\phi) : \operatorname{Spa}(S, S^+) \to \operatorname{Spa}(R, R^+)$ given by pulling back valuations along ϕ. This makes Spa into a contravariant functor from Huber pairs with continuous homomorphisms to topological spaces.

$X = \operatorname{Spa}(R, R^+)$ is *Tate* (resp. *analytic*) if the pair (R, R^+) is.

Example 3.15

Let K be a nonarchimedean field, with valuation ring K°. Then the space $\operatorname{Spa}(K, K^\circ)$ contains only one point, corresponding to the absolute value on K. Indeed, if x were some other point, we would have $|f(x)| \leq 1$ for all $f \in K^\circ$. Let $\mathscr{O}_x = \{f \in K : |f(x)| \leq 1\}$ be the valuation ring. Then $K^\circ \subseteq \mathscr{O}_x$. By Example 2.7 we have $\mathscr{O}_x = K^\circ$ or $\mathscr{O}_x = K$. In the first case we have x equivalent to $|\cdot|$. In the second, x is the trivial valuation sending each unit in K to 1, which is not continuous.

Remark 3.16

1. Since $|g(x)| \leq |g(x)|$ for any x, we see that $X\left(\frac{f_1, \cdots, f_n}{g}\right) = X\left(\frac{f_1, \cdots, f_n, g}{g}\right)$, so that we can always assume that f_n is equal to g.
2. If g is a unit, then $X\left(\frac{f_1, \cdots, f_n}{g}\right)$ is always rational, as (including g), the f_i generate the unit ideal. Therefore sets of the form
$$X\left(\frac{f}{1}\right) = \{x \in X : |f(x)| \leq 1\}$$
are always rational.

Lemma 3.17

The intersection of two rational subsets is again rational.

Proof. Let $T_1 = \{f_1, \cdots, f_n\}$ and $T_2 = \{g_1, \cdots, g_m\}$ be two finite sets generating the unit ideal. Let $U = X\left(\frac{T_1}{s}\right)$ and $V = X\left(\frac{T_2}{t}\right)$ be the corresponding rational subsets of $\operatorname{Spa}(R, R^+)$. By our remark, we may assume $f_n = s$ and $g_m = t$. Let

$$W = X\left(\frac{T_1 \cdot T_2}{st}\right).$$

Then we claim that $U \cap V = W$. Indeed, to prove the left hand side includes in the right notice that if $x \in U \cap V$, then $|f_i(x)| \leq |s(x)| \neq 0$ and $|g_j(x)| \leq |t(x)| \neq 0$. Multiplicativity implies $|f_i(x)g_j(x)| \leq |s(x)t(x)| \neq 0$ so $x \in W$. On the other hand, if $x \in W$ then $|f_i(x)g_m(x)| = |f_i(x)t(x)| \leq |s(x)t(x)| \neq 0$, so that $|f_i(x)| \leq |s(x)| \neq 0$, and so $x \in U$. A similar argument shows $x \in V$ completing the proof.

The adic spectrum $\mathrm{Spa}(R, R^+)$ has a lot of properties similar to those of $\mathrm{Spec}\, A$ for a ring A.

Definition 3.18 ([27] Definition/Proposition 2.9). A topological space X is called *spectral* if it satisfies the following equivalent properties.

(i) There is some ring A with X homeomorphic to $\mathrm{Spec}\, A$.

(ii) The space X is quasicompact, has a basis of quasicompact open subsets stable under finite intersections, and every irreducible closed subset has a unique generic point.

Remark 3.19
On a spectral space we can define the *Krull dimension* to be the maximal chain of irreducible closed subsets.

Theorem 3.20 ([37] Theorem 7.35)
Let (R, R^+) be an Huber pair. Then $\mathrm{Spa}(R, R^+)$ is a spectral space whose rational subsets form a basis of quasi-compact opens, stable under finite intersection. In particular, $\mathrm{Spa}(R, R^+)$ is quasicompact.

All perfectoid algebras are complete, and rigid geometry also deals in complete rings, but we have made no such assumptions for adic spaces. It turns out, that we are safe in restricting our attention to complete Huber pairs. Indeed, every continuous valuation on R extends uniquely to one on $\hat{R}$. In fact, we can do better.

Proposition 3.21 ([16] Proposition 3.9)
Let (R, R^+) be a Huber pair with completion $(\hat{R}, \hat{R}^+)$. Then the inclusion $R \hookrightarrow \hat{R}$ induces a homeomorphism $\mathrm{Spa}(\hat{R}, \hat{R}^+) \to \mathrm{Spa}(R, R^+)$ identifying rational subsets.

3.4 The Adic Unit Disk

In this section we define and study the adic unit disk, comparing it to the rigid unit disk from rigid geometry. We will show that the adic unit disk avoids the pathology highlighted in Example 3.1. We will also use this example to illustrate the role the integrally closed subring R^+ plays in defining the adic spectrum $\mathrm{Spa}(R, R^+)$.

Let K be a nonarchimedean field. Recall from Example 3.1 that the *rigid unit disk*, is defined to be the maximal spectrum of convergent power series over K, $\mathbb{D}^1 = \mathrm{Spm}\, K\langle X\rangle$. Recall also that we have a bijective open immersion $S \sqcup U \to \mathbb{D}^1$, where S is the circle $\{x \in K : |x| = 1\}$, and U is the open disk $\{x \in K : |x| < 1\}$. This map is not an isomorphism or even a covering in the Grothendieck topology! What if instead we considered the space $\mathrm{Cont}(K\langle X\rangle)$?

We can embed $\mathbb{D}^1 \hookrightarrow \mathrm{Cont}(K\langle X\rangle)$ as follows. Given some maximal ideal $\mathfrak{m} \in \mathrm{Spm}(K\langle X\rangle)$, we get a field $K\langle X\rangle/\mathfrak{m}$ which is a finite extension of K, and therefore has a unique absolute value extending that of K. Pulling this absolute value back to $K\langle X\rangle$ gives us an element of $\mathrm{Cont}(K\langle X\rangle)$ whose support is precisely $\mathfrak{m}$, showing that this mapping is injective.

Of course, $\mathrm{Cont}(K\langle X\rangle)$ has many more points. One such point, x^-, can be defined as follows. Let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ be an ordered group given by the relation $a < \gamma < 1$ for all $a \in \mathbb{R}_{>0}$ with $a < 1$. That is, γ is a number infinitesimally smaller than 1. Then we can define x^- as a valuation according to the rule

$$\sum_{n=0}^{\infty} a_n X^n \mapsto \sup_{n \geq 0} |a_n| \gamma^n.$$

This is a point on which the function X evaluates to γ which is infinitesimally smaller than 1, so morally it should be in the unit disk. We can view this point as strictly between the rigid open disk and the rigid unit circle, and is essentially the 'missing point' which made Example 3.1 look pathological.

Unfortunately, it seems that we have picked up too many extra points. We can define another valuation $x^+ \in \mathrm{Cont}(K\langle X\rangle)$ in the same way, except now with γ infinitesimally larger than 1. In particular, $|X(x^+)| > 1$.

Morally speaking, there should not be any point in the unit disk on which the function X evaluates to something greater than 1. Here, the integrally closed open subring comes to the rescue.

We define the adic unit disk to be $\mathbb{D}^{1,\mathrm{ad}} = \mathrm{Spa}(K\langle X\rangle, K^\circ\langle X\rangle)$. Since $X \in K^\circ\langle X\rangle$ any $x \in \mathbb{D}^{1,\mathrm{ad}}$ must obey $|X(x)| \leq 1$. Therefore $x^- \in \mathbb{D}^{1,\mathrm{ad}}$ but $x^+ \notin \mathbb{D}^{1,\mathrm{ad}}$. We see that the subring R^+ we specify allows us to sharpen our edges, by specifying what sort of infinitesimal wiggling is allowed. Suppose we let

$$R^+ = \left\{ \sum_{n=0}^{\infty} a_n X^n \in K^\circ\langle X\rangle : |a_n| < 1 \text{ for all } n \geq 1 \right\},$$

and define $\mathbb{D}^{1,\mathrm{ad}+} := \mathrm{Spa}(K\langle X\rangle, R^+)$. Because $R^+ \subseteq K^\circ\langle X\rangle$ we have $\mathbb{D}^{1,\mathrm{ad}} \subseteq \mathbb{D}^{1,\mathrm{ad}+}$, and it turns out that we miss only one point: $\mathbb{D}^{1,\mathrm{ad}+} \setminus \mathbb{D}^{1,\mathrm{ad}} = \{x^+\}$. The interested reader could check that if we embed $\mathbb{D}^{1,\mathrm{ad}}$ into an open disk of a larger radius, then it will be an open subset, with closure $\mathbb{D}^{1,\mathrm{ad}+}$.

To get a sense of the points on adic spaces, we include a classification of points on the adic unit disk, which is examined quite carefully in [27] Section 2 and [38] Section 7.7.

Example 3.22 (Classification of Points on the Adic Unit Disk)
We assume for the following example that K is also algebraically closed. We classify the 5 types of points appearing on the adic unit disk $\mathbb{D}^{1,\mathrm{ad}} = \mathrm{Spa}(K\langle X\rangle, K^\circ\langle X\rangle)$.

1. Type 1 points are the classical points coming from the rigid unit disk. They correspond precisely to the elements $x \in K^\circ$, and the associated valuations come from evaluation: $f \mapsto |f(x)|$. More precisely
$$x : \left(f = \sum_{n=0}^{\infty} a_n X^n\right) \mapsto \left|\sum a_n x^n\right|.$$

2. Points of types 2 and 3 are called *Gauss points*, and are remeniscent of generic points of irreducible closed subschemes. They correspond to closed disks $D(\alpha, r)$ in K°. If $\alpha \in K^\circ$ and $r \leq 1$ is a real number, we define the disk of radius r centered at α to be $D(\alpha, r) := \{\beta \in K^\circ : |\alpha - \beta| < r\}$. The associated valuation is
$$f \mapsto \sum_{\beta \in D} |f(\beta)|.$$
If we expand f as a series about α, so $f(X) = \sum_{n=0}^{\infty} a_n (X - \alpha)^n$, the valuation is $f \mapsto \sup |a_n| r^n$. If $r \in |K^\times|$, then the point is type 2, else it is type 3. If $r = 1$, then point is called *the Gauss point of the disk.*

3. Type 4 points correspond to descending sequences of disks. If $D_1 \supset D_2 \supset \cdots$, we can define a valuation
$$f \mapsto \inf_i \sup_{x \in D_i} |f(x)|.$$
If the intersection of these disks is a point, then this will be equivalent to the type 1 valuation associated to that point (and will therefore be a classical point). If the intersection is a disk, then this will be equivalent to the type 2 or 3 valuation associated to that disk. Otherwise it is a new type of point which we call type 4. Type 4 points do not exist precisely when K is *spherically complete* (in fact, this may be taken as the definition of spherical completeness).

4. Type 5 points correspond to infinitesimal thickenings or thinnings of type 2 points, and are points like x^- defined above. Indeed, fixing any $\alpha \in K^\circ$ and $r \leq 1$, and a sign + or −, we let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, be the ordered group with γ infinitesimally larger or smaller than r, depending on the chosen sign. We have a continuous valuation:
$$f = \sum a_n (X - \alpha)^n \mapsto \sup |a_n| \gamma^n.$$

If we let x be a type 2 Gauss point, and x^- and x^+ the corresponding type 5 points, we have

$$|f(x^-)| < |f(x)| < |f(x^+)|$$

albeit infinitesimally. If x were a type 3 point, then $x^- = x = x^+$. Also notice that for $r = 1$, $x^+ \notin \mathbb{D}^{1,\mathrm{ad}}$.

All these points are closed except for type 2 points, which are open. The closure of a type 2 point x contains x^+.

3.5 The Structure Presheaf $\mathscr{O}_X$

To construct a scheme, after defining the topological space $X = \operatorname{Spec} R$, we define a structure sheaf $\mathscr{O}_X$ which tells us the regular functions on a given open set. It suffices to define $\mathscr{O}_X$ on the basis of open sets X_f, where f does not vanish, and on these we define $\mathscr{O}_X(X_f) = R[1/f]$. This has the pleasant consequence that $X_f \cong \operatorname{Spec} R[1/f]$. Similar considerations hold for adic spectra. We start by defining the analog of localization.

Definition 3.23 (Rational Localization). Let R be a Huber ring, $T = \{f_1, \dots, f_n\} \subseteq R$ a finite set which generates the unit ideal, and $g \in R$. There is a unique nonarchimedean topology on the localization R_g making it into a topological ring such that the set $\{f_i/g\}$ is power-bounded, and such that the continuous homomorphism $R \to R_g$ is initial among continuous homomorphisms $\phi : R \to S$ where $\phi(g)$ is invertible and the set $\{\phi(f_i)\phi(g)^{-1}\}$ is power-bounded in S. We denote R_g with this topology by $R\left(\frac{T}{g}\right)$.

$R\left(\frac{T}{g}\right)$ is constructed as follows. Let (R_0, I) be a pair of definition and let

$$A := R_0\left[\frac{f_1}{g}, \cdots, \frac{f_n}{g}\right] \subseteq R_g.$$

We topologize R_g by letting $(A, I \cdot A)$ be a pair of definition. In particular, $I^n \cdot A$ is a neighborhood basis for 0 in $R\left(\frac{T}{g}\right)$. It is clear from the definition that we have constructed a Huber ring.

Let $R^+ \subseteq R$ be a ring of integral elements, so that (R, R^+) is a Huber pair. Then we define $R^+\left(\frac{T}{g}\right)$ to be the integral closure of $R^+\left[\frac{f_1}{g}, \cdots, \frac{f_n}{g}\right]$ in R_g. This makes

$$\left(R\left(\frac{T}{g}\right), R^+\left(\frac{T}{g}\right)\right),$$

into a Huber pair. The *rational localization* of R (with respect to T and g) is the completion of this Huber pair, denoted

$$\left(R\left\langle\frac{T}{g}\right\rangle, R^+\left\langle\frac{T}{g}\right\rangle\right).$$

Notice that because of the universal property of $R\left(\frac{T}{g}\right)$, the map $R \to R\left\langle\frac{T}{g}\right\rangle$ is initial among continuous homomorphisms $\phi : R \to S$ where S is complete, $\phi(g)$ is invertible, and the set $\{\phi(f_i)\phi(g)^{-1}\}$ is power-bounded in S. Indeed, we will show below in Proposition 3.27 that this universal property is satisfied.

Remark/Open Problem 3.24
Notice that because we are taking completions of rings that are not necessarily noetherian, we cannot assert that in general a rational localization of a Huber ring is flat. In fact, the question as to whether rational localizations of perfectoid algebras are flat is open.

Remark 3.25

In [19], Definition 1.2.1, Kedlaya defines the rational localization $R\left\langle \frac{f_1,\cdots,f_n}{g}\right\rangle$ to be the quotient of the algebra $R\langle T_1,\cdots T_n\rangle$ by the closure of the ideal $(gT_1 - f_1,\cdots,gT_n - f_n)$. With analytic Huber rings, this definition is equivalent.

If all rational localizations of a ring have a certain property then that ring is said to *stably* have that property. See for example the following definition.

Definition 3.26. Recall from Definition 3.8 the definition of a Huber ring being uniform. A Huber ring R is *stably* uniform if all its rational localizations are uniform.

Let $X = \mathrm{Spa}(R, R^+)$. It is straightforward to check that $\mathrm{Spa}\left(R\left\langle \frac{T}{g}\right\rangle, R^+\left\langle \frac{T}{g}\right\rangle\right) \to X$ factors through $X\left(\frac{T}{g}\right) \hookrightarrow X$. In fact, we can do better.

Proposition 3.27 ([15] Proposition 1.3)

The universal property of rational localization described in Definition 3.23 is satisfied by $R\left\langle \frac{T}{g}\right\rangle$*, and has the following geometric strengthening. For every complete Huber pair* (S, S^+)*, with a continuous morphism* $(R, R^+) \to (S, S^+)$ *so that the induced map* $\mathrm{Spa}(S, S^+) \to X$ *factors over* $X\left(\frac{T}{g}\right)$*, there is a unique continuous morphism of Huber pairs*

$$\left(R\left\langle \frac{T}{g}\right\rangle, R^+\left\langle \frac{T}{g}\right\rangle\right) \to (S, S^+),$$

so that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Spa}\left(R\left\langle \frac{T}{g}\right\rangle, R^+\left\langle \frac{T}{g}\right\rangle\right) \arrow[dr] & & \\
& X\left(\frac{T}{g}\right) \arrow[r, hook] & X \\
\mathrm{Spa}(S, S^+) \arrow[uu, dashed] \arrow[ur] & &
\end{tikzcd}$$

As a consequence the pair

$$\left(R\left\langle \frac{T}{g}\right\rangle, R^+\left\langle \frac{T}{g}\right\rangle\right)$$

depends only on the open set $U = X\left(\frac{T}{g}\right)$.

Definition 3.28. Define presheaves $\mathscr{O}_X$ and $\mathscr{O}_X^+$ on rational subsets by the rule

$$(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) = \left(R\left\langle \frac{f_1,\cdots,f_n}{g}\right\rangle, R^+\left\langle \frac{f_1,\cdots,f_n}{g}\right\rangle\right),$$

and for general open $W \subset X$ as

$$\mathscr{O}_X(W) = \varprojlim_{U\subset W \text{rational}} \mathscr{O}_X(U),$$

and

$$\mathscr{O}_X^+(W) = \varprojlim_{U\subset W \text{rational}} \mathscr{O}_X^+(U).$$

This is well defined due to Proposition 3.27, and makes $\mathscr{O}_X$ into a presheaf of complete Huber rings and $(\mathscr{O}_X, \mathscr{O}_X^+)$ into a presheaf of complete Huber pairs.

It is worth mentioning the following equivalent characterization of $\mathscr{O}_X^+$.

Lemma 3.29

The presheaf of integral elements obeys the following rule for all open $U \subseteq X$.

$$\mathscr{O}_X^+(U) = \{f \in \mathscr{O}_X(U) : |f(x)| \leq 1 \text{ for all } x \in U\}.$$

Proof. [37] Definition 8.13 and Proposition 8.15.

With this perspective in mind, the following subpresheaf of $\mathscr{O}_X^+$ will be of some interest.

Definition 3.30. The *presheaf of topologically nilpotent elements*, denoted $\mathscr{O}_X^{++}$, is defined via the following rule. For each open $U \subseteq X$

$$\mathscr{O}_X^{++}(U) = \{f \in \mathscr{O}_X(U) : |f(x)| < 1 \text{ for all } x \in U\}.$$

Denote by $\tilde{\mathscr{O}}_X$ the quotient $\mathscr{O}_X^+/\mathscr{O}_X^{++}$.

Remark 3.31

If $X = \mathrm{Spa}(R, R^+)$, then $(\mathscr{O}_X(X), \mathscr{O}_X^+(X)) = (\hat{R}, \hat{R}^+)$. Therefore, if R is complete, taking global sections of these presheaves returns the original Huber pair.

The next proposition shows that rational subets of X can be given the natural structure of an adic spectrum.

Proposition 3.32 ([37] Proposition 8.2)

The natural map

$$\mathrm{Spa}\left(R\left\langle\frac{T}{g}\right\rangle, R^+\left\langle\frac{T}{g}\right\rangle\right) \to X,$$

is a homeomorphism onto the open set $X\left(\frac{T}{g}\right)$. As a consequence, for each rational subset U there is a natural homeomorphism $U \cong \mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$, so that rational subspaces of adic spectra are naturally adic spectra.

Definition 3.33 (Stalks of the Structure Presheaf). For $x \in X$, we can define the stalk of $\mathscr{O}_X$ in the usual way,

$$\mathscr{O}_{X,x} := \varinjlim_{x \in U \text{ open}} \mathscr{O}_X(U) = \varinjlim_{x \in U \text{ rational}} \mathscr{O}_X(U).$$

$\mathscr{O}_{X,x}$ is a local ring (see [37] Proposition 8.6). Here we take the colimit in the category of rings, so $\mathscr{O}_{X,x}$ does not have an induced topology. That being said, for every rational subset $U \subset X$ containing x, the valuation defined by x extends uniquely to a valuation $x_U : \mathscr{O}_X(U) \to \Gamma_x \cup \{0\}$. Passing to the colimit we obtain a valuation $v_x : \mathscr{O}_{X,x} \to \Gamma_x \cap \{0\}$, whose support is the maximal ideal of $\mathscr{O}_{X,x}$.

If we denote by $k(x)$ the residue field of $\mathscr{O}_{X,x}$, the valuation v_x induces a nonarchimedean absolute value on $k(x)$ with valuation ring $k(x)^+$. Thus we get a map

$$(R, R^+) \to (k(x), k(x)^+) \subseteq (\widehat{k(x)}, \widehat{k(x)}^+)$$

with dense image. The converse is also true.

Proposition 3.34 ([27] Proposition 2.27)

The points of $\mathrm{Spa}(R, R^+)$ *are in bijection with maps $(R, R^+) \to (k, k^+)$ with dense image, where k is a complete nonarchimedean field and $k^+ \subseteq k^\circ$ is a valuation ring.*

3.6 The Categories of Pre-Adic and Adic Spaces

Schemes are a subcategory of locally ringed spaces. Similarly for adic spaces we start off with a larger category, and give conditions on an object to be a pre-adic or adic space.

Definition 3.35 ([37] Section 8). Let $\mathscr{C}$ be the category of tuples $X = (X, \mathscr{O}_X, (v_x)_{x \in X})$ where,

(i) X is a topological space,

(ii) $\mathscr{O}_X$ is a presheaf of complete topological rings such that the stalk $\mathscr{O}_{X,x}$ is a local ring, and

(iii) v_x is (an equivalence class of) a valuation on the stalk $\mathscr{O}_{X,x}$ such that the support of v_x is the maximal ideal of $\mathscr{O}_{X,x}$.

A morphism $f : X \to Y$ is a pair $(f, f^{\#})$ where $f : X \to Y$ is a continuous map, $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of presheaves of topological rings, so that evaluated on any open set the induced ring map is continuous. In addition, for all $x \in X$, the induced ring homomorphism $f_x^{\#}$ satisfies:

$$\begin{tikzcd}
\mathscr{O}_{Y,f(x)} \arrow[r, "f_x^{\#}"] \arrow[d, "v_{f(x)}"] & \mathscr{O}_{X,x} \arrow[d, "v_x"] \\
\Gamma_{v_{f(x)}} \cup \{0\} \arrow[r, hook] & \Gamma_{v_x} \cup \{0\}.
\end{tikzcd}$$

Notice that in particular, this implies that $\Gamma_{v_{f(x)}} \subseteq \Gamma_{v_x}$ and that $f_x^{\#}$ must be a local homomorphism.

Definition 3.36. If X is an object in $\mathscr{C}$, and $U \subseteq X$ is an open subset of the underlying topological space, then $(U, \mathscr{O}_X|_U, (v_x)_{x \in U})$ is also an object in $\mathscr{C}$. A morphism $j : Y \to X$ in $\mathscr{C}$ is called an *open immersion* if j is a homeomorphism of Y onto an open subspace U inducing an isomorphism

$$(Y, \mathscr{O}_Y, (v_y)_{y \in Y}) \xrightarrow{\sim} (U, \mathscr{O}_X|_U, (v_x)_{x \in U}).$$

Definition 3.37. Our discussion shows that $\mathrm{Spa}(R, R^+)$ is an object of $\mathscr{C}$. An *affinoid pre-adic space* is an object of $\mathscr{C}$ which is isomorphic to $\mathrm{Spa}(R, R^+)$ for a Huber pair (R, R^+).

An object X of $\mathscr{C}$ is called a *pre-adic space* if there is an open covering $\{U_i \to X\}$ such that each U_i is affinoid. The open affinoids form a basis for the underlying topology of X.

A pre-adic space X is an *adic space* if the presheaf $\mathscr{O}_X$ is a sheaf.

An adic space is *Tate* (resp. *analytic*) if it can be covered by affinoid adic spaces which are Tate (resp. analytic).

In general, even if R is Tate, there is no guarantee that $\mathscr{O}_X$ is a sheaf. There are certain assumptions we can make on R to guarantee that the structure presheaf is a sheaf.

Definition 3.38. Let R be a Huber ring. If for all integrally closed open subrings $R^+ \subseteq R$, the pre-adic space $X = \mathrm{Spa}(R, R^+)$ is an adic space (that is $\mathscr{O}_X$ is a sheaf), then R is called *sheafy.*

Theorem 3.39 ([37] Theorem 8.27 for (i), [6] for (ii))
Let R be a Huber ring. In the following cases R is sheafy.

(i) R is Tate and strongly noetherian, *that is for all n, the ring of convergent power series $R\langle T_1, \cdots, T_n\rangle$ is a noetherian ring. This includes the Tate algebra and in fact all affinoid algebras from rigid analytic geometry. In particular the adic unit disk is an adic space.*

(ii) R is stably uniform.

Remark 3.40

We will see in Section 4 that perfectoid algebras are stably uniform, so that in particular they are sheafy.

Remark 3.41

The category of adic spaces in general does not contain all fibered products (see, for example, [37] Section 8.6). Nevertheless, we will see in Section 4 that perfectoid spaces do contain all fibered products.

In algebraic geometry, the Spec functor is an equivalence of categories between rings and affine schemes. Proposition 3.21 seems to suggest the same result is not true for adic spaces. Nevetheless, with appropriate restrictions to our category of Huber pairs, we do get an analogous result.

Proposition 3.42 ([37] Proposition 8.18)

Let (R, R^+) and (S, S^+) be Huber pairs. If S is complete, the map

$$\operatorname{Hom}((R, R^+), (S, S^+)) \to \operatorname{Hom}(\operatorname{Spa}(S, S^+), \operatorname{Spa}(R, R^+)), \qquad \phi \mapsto \phi^*$$

is bijective.

In particular, the adic spectrum functor is an equivalence between complete sheafy Huber pairs to affinoid adic spaces. Furthermore, we have the following characterization of morphisms to affinoid adic spaces (compare to [14] Exercise II.2.4).

Proposition 3.43 ([27] Proposition 2.19)

Let X be an adic space, (R, R^+) a complete sheafy huber pair, and $Y = \operatorname{Spa}(R, R^+)$. Then the natural map

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}\left((R, R^+), (\mathscr{O}_X(X), \mathscr{O}_X^+(X))\right)$$

is bijective.

3.7 Examples of Adic Spaces

Let K be a nonarchimedean field, a fix a pseudouniformizer ϖ. We've already seen the adic unit disk $\mathbb{D}^{1,\mathrm{ad}}$ in Section 3.4. Let's enumerate a few more examples.

Example 3.44 (The Adic Affine Line)

Let $\mathbb{D}^{1,\mathrm{ad}}$ be the closed adic unit disk. The map $T \mapsto \varpi T$ embeds $\mathbb{D}^{1,\mathrm{ad}}$ into a closed disk which we can think of as having radius $|\varpi|^{-1} > 1$. We define $\mathbb{A}^{1,\mathrm{ad}} := \varinjlim \mathbb{D}^{1,\mathrm{ad}}$ along this embedding. Thus $\mathbb{A}^{1,\mathrm{ad}}$ is the ascending union of closed disks of unbounded radius. Since this cover by disks has no finite subcover, $\mathbb{A}^{1,\mathrm{ad}}$ is not quasicompact, so that by Proposition 3.20 it cannot be affinoid.

Example 3.45 (The Adic Unit Polydisk)

We generalize the adic unit disk (of Section 3.4) to higher dimensions. The *adic unit n-polydisk* is the adic space associated to the Tate algebra,

$$\mathbb{D}^{n,\mathrm{ad}} = \operatorname{Spa}\left(K\left\langle T_1, \cdots, T_n\right\rangle, K^\circ\left\langle T_1, \cdots, T_n\right\rangle\right).$$

Example 3.46 (Adic Projective Line)

We can construct $\mathbb{P}^{1,\mathrm{ad}}$ by gluing two copies of $\mathbb{D}^{1,\mathrm{ad}}$ together along the map $T \mapsto T^{-1}$, on the adic circle $\{|T| = 1\}$, which is a rational open subset of the closed disk. Notice that $\mathbb{P}^{1,\mathrm{ad}}$ contains $\mathbb{A}^{1,\mathrm{ad}}$ as an open subspace, whose complement is a single point.

Example 3.47 (Varieties and Rigid Analytic Spaces)

In [15], Huber showed that the category of rigid analytic spaces embeds into the category of adic spaces. Affinoid locally, the space $\operatorname{Spm} R$ (where R is a Banach algebra) can be viewed as the adic space $\operatorname{Spa}(R, R^{\circ})$. Furthermore, this construction is compatible with rational localization so that any rigid space can be viewed as an adic space. Under this construction, the rigid disk, rigid affine line, and rigid projective line map to respective adic counterparts.

In fact, if X is any variety over a nonarchimedean field, one can construct an associated rigid space X^{an}, and this analytification functor satisfies a GAGA principle, and in particular is fully faithful (see for example [10] Chapter 5). Thus, composing with Huber's adic analyitification allows us to view any variety over a nonarchimedean field as an adic space. In particular, the adic analytifications of the scheme theoretic affine and projective lines produce the adic affine and projective lines.

3.8 Sheaves on Adic Spaces

Definition 3.48. Let (R, R^+) be a Huber pair and $X = \operatorname{Spa}(R, R^+)$. For an R-module M, we define $\tilde{M}$ to be the presheaf on X defined on the basis of rational sets $U = \operatorname{Spa}(S, S^+)$ by the rule $\tilde{M}(U) = M \otimes_R S$.

Definition 3.49 (Vector Bundles on Adic Spaces). Let $\mathbf{FPMod}_R$ be the category of finite projective R-modules. A vector bundle on an adic space X is a sheaf $\mathscr{E}$ of $\mathscr{O}_X$-modules which is locally of the form $\tilde{M}$ for M finite projective. That is, there is a covering $\{U_i \to X\}$ with $U_i = \operatorname{Spa}(R_i, R_i^+)$ affinoid and $\mathscr{E}|_U \cong \tilde{M}_i$ for M_i a finite projective R_i module. Equivalently, a vector bundle is a locally free $\mathscr{O}_X$-module. We denote the category of vector bundles on X by $\mathbf{Vec}_X$, where morphisms are those of $\mathscr{O}_X$-modules.

Remark 3.50

This is the definition appearing in [19] Section 1.4, and is analogous to the situation for schemes, where vector bundles on affine schemes correspond to finite projective modules on their coordinate rings, and more generally vector bundles on schemes correspond to locally free sheaves. See for example [14] Exercise II.5.18.

Theorem 3.51 ([19] Theorem 1.3.4)

If R is sheafy, $X = \operatorname{Spa}(R, R^+)$, and M a finite projective R module, then $H^i(X, \tilde{M}) = 0$ for all $i > 0$.

Theorem 3.52 ([19] Theorem 1.4.2)

If R is sheafy, $X = \operatorname{Spa}(R, R^+)$, then the functor $\mathbf{FPMod}_R \to \mathbf{Vec}_X$ is an equivalence of categories. In particular, every vector bundle is acyclic.

Remark 3.53

Theorem 3.52 implies that the pullback functor $\mathbf{Vec}_{\operatorname{Spec}(R)} \to \mathbf{Vec}_X$ is an equivalence of categories, and so understanding vector bundles on adic spaces can be locally reduced to questions in commutative algebra.

Definition 3.54. An R-module M is *pseudocoherent* if it admits a projective resolution (possibly infinite) by finite projective R-modules. We let $\mathbf{PCoh}_R$ be the category of pseudocoherent R-modules that are complete for the natural topology.

Although rational localizations are not known to be flat in general, if the underlying ring is sheafy we get close.

Theorem 3.55 ([19] Theorem 1.4.13)

If R is sheafy and $(R, R^+) \to (S, S^+)$ is a rational localization, then base extension defines an exact functor $\mathbf{PCoh}_R \to \mathbf{PCoh}_S$.

Definition 3.56. A *pseudocoherent sheaf* on an adic space X is a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules which is locally of the form $\tilde{M}$ for M a complete pseudocoherent module. By Theorem 3.55, if $X = \operatorname{Spa}(R, R^+)$ is an affinoid adic space, the functor $\mathbf{PCoh}_R \to \mathbf{PCoh}_X$, $M \mapsto \tilde{M}$ is exact.

Theorem 3.57 ([19] Theorem 1.4.15)

If R is sheafy, then for any $M \in \boldsymbol{PCoh}_R$, the sheaf $\tilde{M}$ is acyclic.

Open Problem 3.58

Over affine schemes, all coherent sheaves are acyclic, and Serre showed in [32] that this property characterizes affine schemes. Theorem 3.57 says that over affinoid adic spaces, pseudocoherent sheaves are acyclic, but in fact no such cohomological characterization of affinoid adic spaces exists. In fact, Liu in [24] constructed examples of non-affinoid rigid spaces with no coherent cohomology. It is still unknown if such counterexamples exist for perfectoid spaces.

4 Perfectoid Spaces

We now have the requisite theory to construct perfectoid spaces. We fix one and for all the following data. Begin with a perfectoid field K with integral subring K° whose unique maximal ideal is the ideal $K^{\circ\circ}$ of topologically nilpotent elements. Let $K^\flat$ be the tilt of K, and $(\varpi^\flat)$ a pesudouniformizer of $K^\flat$ so that $\varpi = \left(\varpi^\flat\right)^\sharp$ is a pseudouniformizer of K. In this section we study adic spaces built out of adic spectra of Huber pairs (R, R^+) where R is a perfectoid K-algebra, and extend the tilting results of Section 2.5 to this geometric context.

4.1 Affinoid Perfectoid Spaces

We begin by extending the tilting equivalence (Theorem 2.57) to perfectoid Huber pairs.

Definition 4.1. A Huber pair (R, R^+) is called an *affinoid perfectoid K-algebra* if R is a perfectoid K-algebra. The tilt of (R, R^+) is $(R^\flat, R^{\flat +})$, where $R^{\flat+} = \varprojlim_{x \mapsto x^p} R^+$ with addition structure inherited as a subset of $R^\flat$ (or equivalently, following the same rule as in Definition 2.56).

An *affinoid perfectoid space over K* is an affinoid (pre-)adic space $X \cong \operatorname{Spa}(R, R^+)$ where (R, R^+) is an affinoid perfectoid K-algebra.

Remark 4.2

Notice that since R^+ is integrally closed and open in R, we must have that $\varpi \in R^+$. Indeed, since ϖ is topologically nilpotent, $\varpi^N \in R^+$ for some N, but then ϖ is a solution to the monic polynomial $T^N - \varpi^N$.

Lemma 4.3 ([27] Lemma 6.2)

Let (R, R^+) be an affinoid perfectoid K-algebra. Then $(R^\flat, R^{\flat+})$ is an affinoid perfectoid $K^\flat$-algebra. Furthermore, the functor $(R, R^+) \mapsto (R^\flat, R^{\flat+})$ is an equivalence of categories between affinoid perfectoid K-algebras and affinoid perfectoid $K^\flat$-algebras. Furthermore, the Teichmüller map $x \mapsto x^\sharp$ induces an isomorphism of rings $R^{\flat+}/\varpi^\flat \cong R^+/\varpi$.

Importantly, the notion of being perfectoid is preserved under rational localization.

Lemma 4.4 ([27] Corollary 6.8)

Let (R, R^+) be an affinoid perfectoid K-algebra and let $X = \operatorname{Spa}(R, R^+)$ be the associated affinoid perfectoid space. Then for all rational $U \subseteq X$, the pair $\left(\mathscr{O}_X(U), \mathscr{O}_X^+(U)\right)$ is an affinoid perfectoid K-algebra.

This has the following important corollary.

Corollary 4.5

Affinoid perfectoid K-algebras are sheafy, and hence affinoid perfectoid spaces are adic spaces.

Proof. The discussion in Example 3.8 showed that perfectoid K-algebras are uniform, so that Lemma 4.4 implies they are stably uniform. Therefore by Theorem 3.39, they are sheafy giving the result.

It turns out that the algebraic tilting functor preserves quite a bit of the geometry of the associated adic spaces.

Theorem 4.6 ([27] Theorem 6.3)

Let (R, R^+) be a affinoid perfectoid K-algebra, and $X = \operatorname{Spa}(R, R^+)$, with associated sheaves $\mathscr{O}_X$ and $\mathscr{O}_X^+$. Let $(R^\flat, R^{\flat+})$ be the tilt, viewed as an affinoid perfectoid $K^\flat$-algebra. Let $X^\flat = \operatorname{Spa}(R^\flat, R^{\flat+})$, with associated sheaves $\mathscr{O}_{X^\flat}$ and $\mathscr{O}_{X^\flat}^+$.

(i) Pulling back along the Teichmüller defines a homeomorphism $\phi : X \to X^\flat$. Explicitly, ϕ maps a valuation $x \in X$ to the valuation $|f(\phi(x))| = |f^\sharp(x)|$.

(ii) The homeomorphism ϕ identifies rational subsets. Explicitly, for any rational subset U, the pair $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is again a perfectoid K-algebra. The map $(R^\flat, R^{\flat+}) \to (\mathscr{O}_X(U)^\flat, \mathscr{O}_X^+(U)^\flat)$ induces a map $U^\flat \to X^\flat$ which can be identified with the inclusion $\phi(U) \hookrightarrow X^\flat$. Under this identification $\phi(U) = U^\flat$ is a rational subset of X. In particular, the tilt of $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$.

(iii) For any open $U \subseteq X$, U is rational if and only if $\phi(U) \subseteq X^\flat$ is.

(iv) The cohomology groups $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for all $i > 0$.

(v) The cohomology groups $\mathrm{H}^i(X, \mathscr{O}_X^+)$ are $K^{\circ\circ}$-torsion for all $i > 0$.

Notation 4.7

As an abuse of notation, for any subset $M \subseteq X$, we denote the image $\phi(M)$ in $X^\flat$ by $M^\flat$. Lemma 4.3 and Theorem 4.6 say that if M is a rational open subset, there is no confusion.

4.2 Globalization of the Tilting Functor

Definition 4.8. An adic space X over K (i.e. with a map to $\mathrm{Spa}(K, K^\circ)$) is a *perfectoid space* if it has an open cover by affinoid perfectoid spaces. Morphisms between perfectoid spaces are morphisms of adic spaces.

Open Problem 4.9

It is currently unknown if a perfectoid space in mixed characteristic which is affinoid as an adic space is affinoid as a perfectoid space. Explicitly, if (R, R^+) is a complete sheafy Huber pair and $X = \mathrm{Spa}(R, R^+)$ is a perfectoid space (that is, it is covered by adic spectra of affinoid perfectoid algebras), does this imply that (R, R^+) was a perfectoid pair to begin with? The positive characteristic case is settled in the affirmative by [6] Corollary 10 or [21] Proposition 3.1.16.

With the machinery of adic spaces tilting glues.

Definition 4.10. Let X be a perfectoid space over K. We call a perfectoid space $X^\flat$ over $K^\flat$ the *tilt of* X if for any affinoid perfectoid K-algebra (R, R^+) there are isomorphisms

$$\mathrm{Hom}(\mathrm{Spa}(R, R^+), X) \cong \mathrm{Hom}(\mathrm{Spa}(R^\flat, R^{\flat+}), X^\flat).$$

which are functorial in (R, R^+).

With this definition in hand, we can globalize Theorem 4.6.

Theorem 4.11 ([27] Proposition 6.17)

(i) Any perfectoid space X over K admits a tilt $X^\flat$ which is unique up to unique isomorphism. The functor $X \mapsto X^\flat$ is an equivalence of categories between the category of perfectoid spaces over K and the category of perfectoid spaces over $K^\flat$.

(ii) The underlying topological spaces X and $X^\flat$ are naturally identified and homeomorphic. We denote by $M^\flat$ the image of a subset $M \subseteq X$ under this identification.

(iii) A perfectoid space X is affinoid if and only if $X^\flat$ is.

(iv) For any affinoid perfectoid subspace $U \subset X$, the pair $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is an affinoid perfectoid K-algebra with tilt $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$.

Although the category of adic spaces does not contain fibered products, the category of perfectoid spaces does.

Theorem 4.12 ([27] Proposition 6.18)

If $X \to Y \leftarrow Z$ are morphisms of perfectoid spaces over K, then the fibered product $X \times_Y Z$ exists in the category of adic spaces over K, and is a perfectoid space.

Remark 4.13

Although we do not include the full proof, it is worth describing what the fibered product is on the level of affinoid perfectoid spaces. Let $X = \mathrm{Spa}(A, A^+), Y = \mathrm{Spa}(B, B^+)$, and $Z = \mathrm{Spa}(C, C^+)$, together with morphisms as in Theorem 4.12. We can identify $X \times_Y Z$ with $\mathrm{Spa}(D, D^+)$, where $D = A \widehat{\otimes}_B C$ is the completed tensor product of A and C over B (that is, the completion of the ordinary tensor product with its inherited topological structure), and D^+ is the integral closure of the image of $A^+ \otimes_{B^+} C^+$ in D.

4.3 The Étale Site

We can also globalize Theorem 2.61.

Definition 4.14. (i) A morphism $(R, R^+) \to (S, S^+)$ of affinoid perfectoid K-algebras is called *finite étale* if S is a finite étale R-algebra, and S^+ is the integral closure of the image of R^+ in S.

(ii) A morphism $f : X \to Y$ of perfectoid spaces over K is called *finite étale* if there is a cover of Y by affinoids V such that the preimage $U = f^{-1}(V)$ is affinoid, and the associated morphism $(\mathscr{O}_Y(V), \mathscr{O}_Y^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is finite étale.

(iii) A morphism $f : X \to Y$ of perfectoid spaces over K is called *étale* if for any $x \in X$, there are open neighborhoods U and V of x and $f(x)$ respectively together with a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f|_U"'] & W \arrow[d, "p"] \\ & V \end{tikzcd}$$

where j is an open immersion and p is finite étale.

As one would hope, the notion of being finite étale is preserved under base change.

Lemma 4.15 ([27] Lemma 7.3)

Let $X \to Y$ be a finite étale morphism of perfectoid spaces and $Z \to Y$ any morphism of perfectoid spaces. Then the induced map $X \times_Y Z \to Z$ is finite étale.

As a consequence being étale can be checked locally.

Proposition 4.16 ([27] Proposition 7.6)

If $f : X \to Y$ is a finite étale morphism of perfectoid spaces, then for any affinoid perfectoid $V \subset Y$, its preimage U is affinoid perfectoid, and the ring map

$$(\mathscr{O}_Y(V), \mathscr{O}_Y^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$$

is finite étale.

Remark 4.17

Scholze's proof of this Proposition 4.16 required being in positive characteristic and used the notions of completed perfections. Nevertheless, by Theorem 2.61 and the fact that tilting preserves rational subsets, we may first tilt and a proof in positive characteristic suffices. This is a common method of proof for theorems about perfectoid objects, and one we will use in the following sections.

We can now define the étale site of a perfectoid space.

Definition 4.18. Let X be a perfectoid space. The (small) étale site of X, denoted $X_{\text{ét}}$ consists of perfectoid spaces which are étale over X. The coverings are given by topological coverings, i.e., jointly surjective morphisms.

Our preparations show that all the conditions on a site are satisfied, and a morphism $f : X \to Y$ of perfectoid spaces induces a morphism of sites $X_{\text{ét}} \to Y_{\text{ét}}$. Theorem 2.61 along with the results of this section immediately imply the following theorem.

Theorem 4.19 ([27] Theorem 7.12)
Let X be a perfectoid space over K with tilt $X^\flat$ over $K^\flat$. The tilting functor induces an equivalence of sites $X_{\text{ét}} \cong X^\flat_{\text{ét}}$ which is functorial in X.

4.4 Examples of Perfectoid Spaces and their Tilts

Let's exhibit a few naturally arising perfectoid spaces. These examples will be our central objects of study for the remainder of this work. Compare to the examples of adic spaces in Section 3.7.

Example 4.20 (The Perfectoid Unit Disk)
Much like the definition of the to the adic unit disk (Section 3.4), we define the *perfectoid unit disk* to be the adic space associated to the perfectoid Tate algebra in one variable,

$$\mathbb{D}_K^{1,\text{perf}} := \operatorname{Spa}\left(K\left\langle T^{1/p^\infty}\right\rangle, K^\circ\left\langle T^{1/p^\infty}\right\rangle\right).$$

We can similarly define the *n-dimensional perfectoid unit polydisk* as the adic space associated the perfectoid Tate algebra in n variables,

$$\mathbb{D}_K^{n,\text{perf}} := \operatorname{Spa}\left(K\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle, K^\circ\left\langle T_1^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle\right).$$

Example 4.21 (The Perfectoid Unit Circle)
We define the *perfectoid unit circle* to be

$$\mathbb{S}_K^{1,\text{perf}} = \operatorname{Spa}\left(K\left\langle T^{\pm 1/p^\infty}\right\rangle, K^\circ\left\langle T^{\pm 1/p^\infty}\right\rangle\right).$$

Notice that the circle can be identified with the rational open subset $\mathbb{D}^{1,\text{perf}}\left(\frac{1}{T}\right)$ in the perfectoid disk, and consists of points on the disk on which the function T evaluates to something with absolute value 1, precisely what our idea of the circle is.

Similarly, we can define the *perfectoid unit sphere* by

$$\mathbb{S}_K^{n,\text{perf}} = \operatorname{Spa}\left(K\left\langle T_1^{\pm 1/p^\infty}, \cdots, T_n^{\pm 1/p^\infty}\right\rangle, K^\circ\left\langle T_1^{\pm 1/p^\infty}, \cdots, T_n^{\pm 1/p^\infty}\right\rangle\right).$$

It is open in $\mathbb{D}_K^{n,\text{perf}}$, as the intersection of all the rational opens of the form $\mathbb{D}_K^{n,\text{perf}}\left(\frac{1}{T_i}\right)$, and consists of points on which the all the coordinate functions T_i evaluate to something with absolute value 1.

Example 4.22 (The Projectivoid Line)
Analogously to the construction of the Riemann sphere or of $\mathbb{P}^1$ in rigid analytic geometry or algebraic geometry, we can build the perfectoid analog of he projecitve line by gluing two copies of the perfectoid unit disk along the perfectoid unit circle.

Explicitly, the inclusion $K\left\langle T^{1/p^\infty}\right\rangle \to K\left\langle T^{\pm 1/p^\infty}\right\rangle$ corresponds to the open immersion $\mathbb{S}^{1,\text{perf}} \hookrightarrow \mathbb{D}^{1,\text{perf}}$. The map $K\left\langle T^{-1/p^\infty}\right\rangle \to K\left\langle T^{\pm 1/p^\infty}\right\rangle$ also corresponds to an open immersion of the circle into the disk, where now the disk has coordinate T^{-1}. Identifying the circles on each of these disks and gluing produces *the projectivoid line*, denoted $\mathbb{P}_K^{1,\text{perf}}$.

Example 4.23 (Projectivoid Space)

As with the projectivoid line, we could define projectivoid n-space, denoted $\mathbb{P}^{n,\mathrm{perf}}$, by gluing together $n+1$ perfectoid unit n-polydisks along their associated perfectoid sphere exteriors as in Example 4.22. This is a useful perspective as is provides projectivoid space with a cover by perfectoid unit polydisks. In [28] Section 7, Scholze showed that we could also define projectivoid space in the following way. Let $\mathbb{P}^n_K$ be projective space over K, which can be viewed as an adic space as in Example 3.47, that is, first viewed as a rigid space using the rigid analytification functor, and then as an adic space as in [15]. Let $\varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be the morphism given in projective coordinates by $(T_0 : \cdots : T_n) \mapsto (T_0^p : \cdots : T_n^p)$. Then

$$\mathbb{P}^{n,\mathrm{perf}}_K \sim \varprojlim_{\varphi} \mathbb{P}^n_K.$$

Note that here "$\sim \varprojlim$" does not denote the categorical inverse limit (since these are not in general unique in the category of adic spaces). Nevertheless, it should be thought of affinoid locally as corresponding to the completed directed limit, and if such a limit exists as a perfectoid space, it is unique among all perfectoid spaces and satisfies the usual universal property (among perfectoid spaces). See [29] Definition 2.4.1 and subsequent discussion.

The constructions of these examples are compatible with tilting.

Proposition 4.24

Let K be a perfectoid field with tilt $K^\flat$. Then:

$$\left(\mathbb{D}^{n,\mathrm{perf}}_K\right)^\flat \cong \mathbb{D}^{n,\mathrm{perf}}_{K^\flat}$$

$$\left(\mathbb{S}^{n,\mathrm{perf}}_K\right)^\flat \cong \mathbb{S}^{n,\mathrm{perf}}_{K^\flat}$$

$$\left(\mathbb{P}^{n,\mathrm{perf}}_K\right)^\flat \cong \mathbb{P}^{n,\mathrm{perf}}_{K^\flat}$$

Proof. The cases for the polydisk and the sphere follow from Proposition 2.59 together with Theorem 4.6. The case for projectivoid space follows from these two and Theorem 4.11 since tilts are constructed locally and are compatible with rational localizations.

5 The Perfectoid Tate Algebra

Let K be a perfectoid field with pseudouniformizer ϖ. We defined the perfectoid Tate algebra $T_{n,K}^{\mathrm{perf}}$ in Example 2.58. In this section we will study the algebraic structures and module theory of this ring in order to understand the structure of vector bundles on the associated perfectoid space $\mathbb{D}^{n,\mathrm{perf}}$ defined in Example 4.20.

Let us present an alternative but equivalent definition of the perfectoid Tate algebra.

$$T_{n,K}^{\mathrm{perf}} = K\left\langle X_1^{1/p^\infty}, \cdots, X_n^{1/p^\infty}\right\rangle = \widehat{\bigcup_{n\geq 0} K\left\langle X_1^{1/p^n}, \cdots, X_n^{1/p^n}\right\rangle}.$$

This way, we express the perfectoid Tate algebra as a completed union of Tate algebras (whose module theory is well understood, mostly due to Lütkebohmert, see [25]). The completion is taken with respect to topology induced by the absolute value coming at each finite level (that is with respect to the Gauss norm inherited from K).

In this way we see that the perfectoid Tate algebra consists of formal power series over K which converge on the unit disk. Letting $X = (X_1, \cdots, X_n)$ be an n-tuple, we can write down the elements of this ring.

$$T_{n,K}^{\mathrm{perf}} = \left\{ \sum_{\alpha\in\left(\mathbb{Z}[1/p]_{\geq 0}\right)^n} a_\alpha X^\alpha : \text{ for all } \lambda \in \mathbb{R}_{>0} \text{ only finitely many } |a_\alpha| \geq \lambda \right\}.$$

This ring inherits the Gauss norm, $||\sum_\alpha a_\alpha X^\alpha|| = \sup\{|a_\alpha|\}$.

$\left(T_{n,K}^{\mathrm{perf}}\right)^\circ$ is the subring $\{||f|| \leq 1\}$ of power-bounded elements of $T_{n,K}^{\mathrm{perf}}$, and consists of power series with coefficients in K°. The ideal $\left(T_{n,K}^{\mathrm{perf}}\right)^{\circ\circ}$ of topologically nilpotent elements consists of power series with coefficients in $K^{\circ\circ}$. The quotient is

$$\tilde{T}_{n,K}^{\mathrm{perf}} := \left(T_{n,K}^{\mathrm{perf}}\right)^\circ / \left(T_{n,K}^{\mathrm{perf}}\right)^{\circ\circ} = k\left[X_1^{1/p^\infty}, \cdots, X_n^{1/p^\infty}\right]$$

where $k = K^\circ/K^{\circ\circ}$ is the residue field. Notice that every element in the quotient ring is a polynomial, because a power series $\left(T_{n,K}^{\mathrm{perf}}\right)^\circ$ can only have finitely many coefficients of norm 1.

When there will be no confusion, we omit K from the notation.

Remark 5.1

We have established that the perfectoid Tate algebra is the completed union of Tate algebras, and that its ring of power-bounded elements is the completed union of rings of power-bounded elements of Tate algebras. Therefore using the notation of Example 3.45 and Example 4.23 we have

$$\mathbb{D}^{n,\mathrm{perf}} \sim \varprojlim_{\varphi} \mathbb{D}^{n,\mathrm{ad}},$$

where φ is the pth power map on coordinates.

Remark 5.2 (A Note on Convergence)

Suppose $f = \sum f_\alpha X^\alpha \in T_n^{\mathrm{perf}}$. Morally speaking, saying that the sum converges on the unit disk should mean that evaluating at any $x \in (K^\circ)^n$ should give an element of K. Since are sums are not taken over $\mathbb{Z}_{\geq 0}^n$, but rather $(\mathbb{Z}[1/p]_{\geq 0})^n$, we must be more careful in defining what convergence means. Let us begin by studying $f(1, 1, \cdots, 1) = \sum f_\alpha$. This should converge, so to make sense of this we define the partial sums

$$s_m = \sum_{\alpha\in\left(\frac{\mathbb{Z}}{p^m}\right)^n : 0\leq\alpha_i\leq m} f_\alpha.$$

These partial sums are approaching the infinite sum, and if the sequence (s_m) converges, we define the infinite sum to be the limit. Let us check that the convergence of the power series f implies convergence of $\sum f_\alpha$ in this sense. Fixing some $\varepsilon > 0$, there are only finitely many f_α with $|f_\alpha| \geq \varepsilon$. Therefore, there is some large N such that for each such f_α we have $\alpha = (\alpha_1, \cdots, \alpha_n) \in \left(\frac{\mathbb{Z}}{p^N}\right)^n$ $0 < \alpha_i < N$. Therefore, fixing $m \geq r > N$, the differences $s_m - s_r$ have none of the coefficients f_α with absolute value larger than ε, so that by the nonarchimedean property $|s_m - s_r| < \varepsilon$. This shows that the sum converges to an element $f(1) \in K$.

We remark now that if $|g_\alpha| \leq 1$, the same argument would show that $\sum f_\alpha g_\alpha$ also converges. In particular $f(x) \in K$ for all $x \in (K^\circ)$ (letting $g_\alpha = \Pi x_i^{\alpha_i}$).

We record a useful normalization trick for further use down the line.

Lemma 5.3 (Normalization)

Let $f \in T_n^{\mathrm{perf}}$ be nonzero. There is some $\lambda \in K$ such that $||\lambda f|| = 1$.

Proof. Since only finitely many coefficients in f have absolute value above $||f|| - \varepsilon$, the supremum of that absolute values of the coefficients is achieved by some f_α. Taking $\lambda = f_\alpha^{-1}$ completes the proof.

5.1 The Group of Units

As a first step towards understanding the perfectoid Tate algebra, we compute its group of units.

Proposition 5.4

Let $f \in T_n^{\mathrm{perf}}$ with $||f|| = 1$. The following are equivalent:

- *(i) f is a unit in $\left(T_n^{\mathrm{perf}}\right)$.*
- *(ii) f is a unit in $\left(T_n^{\mathrm{perf}}\right)^\circ$.*
- *(iii) The image of $\overline{f}$ of f in $\tilde{T}_n^{\mathrm{perf}}$ is a nonzero constant $\lambda \in k^\times$.*
- *(iv) $|f(0)| = 1$ and $||f - f(0)|| < 1$.*

Proof. (i) $\iff$ (ii). An inverse to f must have absolute value 1, and therefore would also lie in $\left(T_n^{\mathrm{perf}}\right)^\circ$.

(ii) $\implies$ (iii). The map $\left(T_n^{\mathrm{perf}}\right)^\circ \to \tilde{T}_n^{\mathrm{perf}}$ must send units to units, and the group of units of $\tilde{T}_n^{\mathrm{perf}}$ is precisely the nonzero constant polynomials. Indeed, the inverse to any element of $\tilde{T}_n^{\mathrm{perf}}$ would also have to be a polynomial (in X^{1/p^m} for some m), implying that they both must be constants.

(iii) $\iff$ (iv). This is immediate.

(iv) $\implies$ (i). If $|f(0)| = 1$ then $f(0) \in K^\times \subseteq (T_n^{\mathrm{perf}})^\times$. Therefore $1 - \frac{f}{f(0)} \in T_n^{\mathrm{perf}}$ and

$$\left|\left|1 - \frac{f}{f(0)}\right|\right| = ||f(0)|| \cdot \left|\left|1 - \frac{f}{f(0)}\right|\right| = ||f(0) - f|| < 1.$$

Therefore $1 - \frac{f}{f(0)}$ is topologically nilpotent, so that by Lemma 2.26, $\frac{f}{f(0)}$ is a unit. Since $f(0)$ is too, we can conclude that f is a unit.

Corollary 5.5

$f = \sum f_\alpha X^\alpha \in \left(T_n^{\mathrm{perf}}\right)^\circ$ is a unit if and only if $|f_0| = 1$ and $|f_\alpha| < 1$ for all $\alpha \neq 0$.

Corollary 5.6

$f = \sum f_\alpha X^\alpha \in T_n^{\mathrm{perf}}$ is a unit if and only if $|f_\alpha| < |f_0|$ for all $\alpha \neq 0$.

Proof. Using our normalization trick, we know $||\lambda f|| = 1$ for some $\lambda \in K^\times$. The f is a unit if and only if λf, if and only if $|\lambda f_\alpha| < 1 = |\lambda f_0|$ for all $\alpha \neq 0$. Cancelling shows this holds if and only if $|f_\alpha| < |f_0|$ for all $\alpha \neq 0$.

5.2 Krull Dimension in Characteristic p

If the characteristic of K is p, we can compute the Krull dimension of T_n^{perf}, using the notion of perfections.

Proposition 5.7 ([12])

Let R be a domain of characteristic p. There is a perfect ring R^{perf} of characteristic p and a morphism $R \to R^{\text{perf}}$ which is initial among morphisms of R to perfect rings of characteristic p. This map is injective, so we can identify R with a subset of R^{perf}. Under this identification, for any $f \in R^{\text{perf}}$ there is some m such that $f^{p^m} \in R$.

Proposition 5.8

Let $R \to R^{\text{perf}}$ be the perfection. Then the Krull dimensions of R and R^{perf} agree.

Proof. First let

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_r$$

be an ascending chain of prime ideals in R^{perf}. Intersection with R gives a chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_r$. Suppose $\mathfrak{q}_i = \mathfrak{q}_{i+1}$ for some i. Then for any $f \in \mathfrak{p}_{i+1}$ we have

$$f^{p^m} \in \mathfrak{p}_{i+1} \cap R = \mathfrak{q}_{i+1} = \mathfrak{q}_i \subseteq \mathfrak{p}_i.$$

Since $\mathfrak{p}_i$ is prime then $f \in \mathfrak{p}_i$ contradicting that $\mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1}$. This shows that $\operatorname{Kdim} R^{\text{perf}} \leq \operatorname{Kdim} R$. For the converse we first we establish the following fact.

Claim 5.9

Let $\mathfrak{q} \subseteq R$ be prime. Then $\sqrt{\mathfrak{q}R^{\text{perf}}}$ is prime.

Proof. Let $fg \in \sqrt{\mathfrak{q}R^{\text{perf}}}$. Thus for some m we have $(fg)^m = a_1s_1 + \cdots + a_ts_t$ for $s_i \in \mathfrak{q}$. Since we are in characteristic p we have

$$(a_1s_1 + \cdots + a_ts_t)^{p^N} = a_1^{p^N}s_1^{p^N} + \cdots + a_t^{p^N}s_t^{p^N}.$$

Choosing N large enough so that each $a_i^{p^N}, f^{mp^N}, g^{mp^N} \in R$, we have that $(fg)^{mp^N} = f^{mp^N}g^{mp^N} \in \mathfrak{q}$. So then f^{mp^N} or g^{mp^N} are in $\mathfrak{q} \subseteq \mathfrak{q}R^{\text{perf}}$. Thus either f or g are in $\sqrt{\mathfrak{q}R^{\text{perf}}}$.

Now let $\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_t$ be a chain of prime ideals in R. Then $\sqrt{\mathfrak{q}_1R^{\text{perf}}} \subset \cdots \subset \sqrt{\mathfrak{q}_tR^{\text{perf}}}$ is a chain of primes in R^{perf}. Suppose $\sqrt{\mathfrak{q}_iR^{\text{perf}}} = \sqrt{\mathfrak{q}_{i+1}R^{\text{perf}}}$ for some i. Then for any $g \in \mathfrak{q}_{i+1} \subseteq \sqrt{\mathfrak{q}_{i+1}R^{\text{perf}}} = \sqrt{\mathfrak{q}_iR^{\text{perf}}}$ we have $g^m = a_1s_1 + \cdots + a_ks_k$ with $s_i \in \mathfrak{q}_i$. Using the same trick as in the proof of Claim 5.9, we can see that $g^{mp^N} \in \mathfrak{q}_i$ so that g must be. Since g was arbitrary, this contradicts that $\mathfrak{q}_i \subsetneq \mathfrak{q}_{i+1}$, so that $\operatorname{Kdim} R^{\text{perf}} \geq \operatorname{Kdim} R$ completing the proof.

Definition/Proposition 5.10

For a ring R with an I-adic topology, one can form the *completed perfection* $\widehat{R^{\text{perf}}}$ by taking the IR^{perf}-adic completion of the perfection R^{perf} of R. If we have a Tate Huber ring R with pseudouniformizer ϖ, one can obtain the *completed perfection* by taking

$$\widehat{R^{\text{perf}}} := \widehat{R^{\circ\text{perf}}}[1/\varpi].$$

The completed perfection of a Huber ring is the initial continuous homomorphism $R \to \widehat{R^{\text{perf}}}$ to a complete perfect ring. If R is a Tate Huber ring, then completed perfection preserves Krull dimesion.

Proof. The universal property follows from the sequential combination of the universal properties of perfection and completion. Further, we just saw that perfection preserves Krull dimension. Completion preserves Krull dimension because ϖ is contained in the Jacobson radical (since $1 + \varpi$ is a unit by Proposition 5.4).

Recall from Example 2.19 that the Tate algebra $T_n := K\langle X_1, \cdots, X_n\rangle$ is defined to be the completion of $K[X_1, \cdots, X_n]$ with respect to the Gauss norm. It is well known that over any complete nonarchimedean field, $\operatorname{Kdim} T_n = n$ (see, for example, [4] Section 6.1).

Proposition 5.11

In characteristic p, the the completed perection of the Tate algebra is the perfectoid Tate algebra. That is, $\widehat{(T_n)^{\text{perf}}} = T_n^{\text{perf}}$.

Proof. We check that T_n^{perf} satisfies the universal property of completed perfections. It is perfect, because it is perfectoid. Let $\phi : T_n \to R$, be a contiuous homomorphism to a complete perfect ring R which maps $X_i \mapsto a_i$. This factors uniquely through the perfection $(T_n)^{\text{perf}} \to R$ by mapping $X_i^{1/p^r} \mapsto a_i^{1/p^r}$, where the pth power roots of each a_i exist and are unique because R is perfect. This in turn passes uniquely to the completion T_n^{perf} because R is complete.

Corollary 5.12

In characteristic p, $\operatorname{Kdim} T_n^{\text{perf}} = n$.

Open Problem 5.13

We hope to use the tilting equivalence to deduce this result in characteristic 0. Unfortunately, it is not known in general that tilting preserves Krull dimension. Since the adic spectrum of the tilt is homeomorphic to the adic spectrum of the original ring, we would have the result if the $\operatorname{Kdim} R = \operatorname{Kdim} \operatorname{Spa} R$. Huber in [17] proves this result for certain classes of Huber rings, but his proofs relied heavily of these rings being noetherian. We conjecture that this should also be true for these perfectoid algebras.

5.3 Weierstrass Division

For the Tate algebra $K\langle X_1, \cdots, X_n\rangle$ there is a notion of a power series f being regular in X_n of degree d. Morally, this means that f is close to being a monic polynomial of degree d in the variable X_n. Weierstrass division ([4] Section 5.2.1 Theorem 2) then says we can do Euclidean division when dividing by a regular element f, and and as a corollary one can show that $f = ug$ where u is a unit and g is a polynomial in X_n of degree d. This is generally called the Weierstrass preparation trick.

Furthermore, for any power series f, there is an automorphism Φ of the Tate algebra so that $\Phi(f)$ is regular in X_n of some degree. Morally speaking, this shows that every convergent power series, after perhaps an automorphism of the ring, is close to being a polynomial in X_n. This machinery allows many arguments in rigid geometry to be reduced to the cases of polynomial rings. See [4] Chapter 5 for a good summary of this process in the rigid analytic context.

Unfortunately for the perfectoid Tate algebra the analogous result is not quite as strong. In particular, for power series regular in X_n of degree d, we can show that $f = ug$ for u a unit and g having degree d in X_n. Because pth power roots of X_n all exist, this does not imply that g is a polynomial in X_n (see Example 5.19).

Over the next two sections we explore how far these techniques carry over to the perfectoid Tate algebra. The work was begun by Das in [7], where he proved the perfectoid analog of Weiestrass division. We record his proof here, and then continue with our contribution to this question, namely, the existence and uniqueness of the Weierstrass preperation of a regular element, and the construction of an automorphism of T_n^{perf} taking a given element to a regular element.

Definition 5.14. Fix element $f \in T_n^{\text{perf}}$, and write it as a power series in X_n.

$$f = \sum_{\alpha \in \mathbb{Z}[1/p]} f_\alpha X_n^\alpha,$$

for $f_\alpha \in T_{n-1}^{\text{perf}} = K\langle X_1^{1/p^\infty}, \cdots, X_{n-1}^{1/p^\infty}\rangle$. We say that f is *regular in* X_n of degree d if

(i) $||f|| = ||f_d|| > ||f_\alpha||$ for all $\alpha > d$.

(ii) $f_d \in T_{n-1}^{\text{perf}}$ is a unit.

If $||f|| = 1$ there is an equivalent characterization of regularity.

Lemma 5.15

Suppose $f \in T_n^{\text{perf}}$ with $||f|| = 1$. Then f is regular in X_n of degree d if and only if

$$\overline{f} = \lambda X^d + g \in \tilde{T}_n^{\text{perf}}, \tag{1}$$

where $\lambda \in k^\times$ and $\deg_{X_n} g < d$.

Proof. If $f = \sum f_\alpha X_n^\alpha$ is regular in X_n of degree d and $||f|| = 1$, then $1 = ||f_d|| > ||f_\alpha||$ for all $\alpha > d$. Thus $\overline{f_d} = \lambda \in k^\times$, and $\overline{f_\alpha} = 0$ for all $\alpha > d$ showing that $\overline{f}$ satisfies Equation 1

Conversely, if $\overline{f}$ satisfies Equation 1. Then $|f_\alpha| < 1 = |f_d|$ for all $\alpha > d$. Proposition 5.4 applied to T_{n-1}^{perf} shows that since $\overline{f_d} = \lambda \in k^\times$, f_d must be a unit. Thus f is regular in X_n of degree d.

Theorem 5.16 (Weierstrass Division, [7] Proposition 4.4.2)

Let $f \in T_n^{\text{perf}}$ be regular in X_n of degree d. For any $g \in T_n^{\text{perf}}$, there exist unique $q, r \in T_n^{\text{perf}}$ with $\deg_{X_n} r < d$ such that

$$g = qf + r.$$

Furthermore, $||g|| = \max\{||q|| \cdot ||f||, ||r||\}$.

Proof. We first reduce to the case that $||f|| = 1$. Suppose we can divide by regular elements of degree 1. Replace g by λg for $\lambda \in K^\times$ with $||\lambda f|| = 1$. Then $\lambda g = q(\lambda f) + r$, so that $g = qf + \lambda^{-1} r$ and $\deg_{X_n} \lambda^{-1} r = \deg_{X_n} r < d$. The rest is easily checked. Therefore we may assume $||f|| = 1$.

We may also assume $||g|| > 0$, since if $||g|| = 0$ then $g = 0$ and $r = s = 0$.

We begin by proving the statement about absolute values. Suppose that $g = qf + r$ is a Weierstrass quotient. Since $||f|| = 1$, we have $||g|| \leq \max\{||q||, ||r||\}$. Assume that $||g|| < \max\{||q||, ||r||\}$. By Lemma 2.22, $||q|| = ||r|| > 0$. Normalize so that $||\lambda q|| = ||\lambda r|| = 1$. Then $\lambda g = \lambda q f + \lambda r$, and pass to $\tilde{T}_n^{\text{perf}}$. Then

$$0 = \overline{\lambda q f} + \overline{\lambda r}.$$

We have that $\overline{\lambda q}$ is a nonzero polynomial and $\overline{f}$ has degree d in X_n. Thus $\overline{\lambda q f}$ is a polynomial of degree $\geq d$ in X_n. But we also know $\overline{\lambda r}$ is a polynomial of degree $< d$ in X_n, a contradiction. So $||g|| = \max\{||q||, ||r||\}$.

We now prove the division result in 2 steps. For the first, we assume that $f = \sum f_\alpha X_n^\alpha \in T_{n-1}^{\text{perf}}[X_n^{1/p^n}]$ is a Weierstrass polynomial. That is, it is a degree d polynomial in X_n which is also regular in degree d. For each α, there is some constant $N(\alpha)$ so that both

$$X_n^\alpha, f \in T_{n-1}^{\text{perf}} \left[X_n^{1/p^{N(\alpha)}} \right]$$

are polynomials in X_n. Furthermore, the leading coefficient of f is a unit, so we can do polynomial long division.

$$X_n^\alpha = q_\alpha f + r_\alpha,$$

where $\deg_{X_n} r_\alpha < d$. Furthermore, $\max\{||q_\alpha||, ||r_\alpha||\} = ||X_n^\alpha|| = 1$.

Write g as $\sum g_\alpha X_n^\alpha$. The discussion in Remark 5.2 shows that

$$q = \sum g_\alpha q_\alpha$$

and

$$r = \sum g_\alpha r_\alpha$$

both converge to elements in T_n^{perf}. Since $\deg_{X_n} g_\alpha = 0$ for all α, we have that

$$\deg_{X_n} r \leq \max\{\deg_{X_n}(g_\alpha r_\alpha)\} \leq \max\{\deg_{X_n} r_\alpha\} \leq d.$$

Now we can directly check that,

$$\begin{aligned} qf + r &= \sum (g_\alpha q_\alpha f + g_\alpha r_\alpha) \\ &= \sum g_\alpha (q_\alpha f + r_\alpha) \\ &= \sum g_\alpha X_n^\alpha \\ &= g. \end{aligned}$$

For the general case, we have $f = \sum f_\alpha X_n^\alpha$. We let $f_0 = \sum_{|f_\alpha|=1} f_\alpha X_n^\alpha$. Then f_0 is a Weierstrass polynomial of degree d. Let $D = f_0 - f$. We have $||D|| < ||f_0|| = ||f|| = 1$. Because f_0 is a Weierstrass polynomial, we can divide g by f_0. That is, there are unique q_0, r_0 with $\deg_{X_n} r_0 < d$ so that

$$g = q_0 f_0 + r_0 = q_0 f + q_0 D + r_0.$$

We let $g_1 = q_0 D$. Then by our first statement about absolute values we have $||g_1|| \leq ||g|| \cdot ||D||$. We divide g_1 by f_0 next:

$$g_1 = q_1 f_0 + r_1 = q_1 f + q_1 D + r_1.$$

Now letting $g_2 = q_1 D$, we have $||g_1|| \leq ||g_1|| \cdot ||D|| \leq ||g|| \cdot ||D||^2$. We continue in this fashion, letting each $g_{i+1} = g_i \cdot D$ where $g_i = q_i f_0 + r_i$. In each case, $||q_i||, ||r_i|| \leq ||g_i|| < ||g|| \cdot ||D||^i$. In particular, because $||D|| < 1$, we have $\lim_{i\to\infty} q_i = 0$ and $\lim_{i\to\infty} r_i = 0$. Therefore by Lemma 2.24 we can form the infinite sums

$$q = \sum_{i=0}^{\infty} q_i$$

and

$$r = \sum_{i=0}^{\infty} r_i.$$

Then we can check (setting $g_0 = g$) that

$$\begin{aligned} qf + r &= \sum_{i=0}^{\infty} (q_i(f_0 - D) + r_i) \\ &= \sum_{i=0}^{\infty} ((q_i f_0 + r_i) - q_i D) \\ &= \sum_{i=0}^{\infty} g_i - g_{i+1} \\ &= g \end{aligned}$$

To check uniqueness we suppose that $g = qf + r = q'f + r'$ with $\deg r, \deg r' < d$. Then $(q-q')f + (r-r') = 0$. Since $||f|| = 1$ this implies that $||q - q'|| = ||r - r'||$. If both are equal to 0 we are done, otherwise normalize by λ. Then we have in $\tilde{T}_n^{\text{perf}}$

$$\overline{\lambda(q-q')f} + \overline{\lambda(r-r')} = 0.$$

Since $\overline{(\lambda(q-q'))} \neq 0$ we have that $\deg_{X_n}\left(\overline{\lambda(q-q')f}\right) \geq d$. But $\deg_{X_n}(r - r') < d$, a contradiction.

Corollary 5.17 (Weierstrass Preparation)

If f is regular of degree d, then $f = u \cdot g$ for g a monic and of degree d in X_n, and u a unit.

PROOF. Without loss of generality we may assume that $||f|| = 1$. Since f is regular of degree d we can write $X_n^d = qf + r$ and because $\deg_{X_n} r < d$ and $||r|| \leq ||X^d||$ we have that $X_n^d - r$ is regular of degree d. Thus we can divide f by it.

$$\begin{aligned} f &= (X_n^d - r)p + s \\ &= ((fq + r) - r)\, p + s \\ &= f \cdot qp + s. \end{aligned}$$

Since $\deg_{X_n} s < d$, then this is the unique Weierstrass division of f by f, so that in particular $qp = 1$ and $s = 0$. Therefore, letting $g = X_n^d - r$ and $u = p$ gives us our preparation.

Unfortunately in the perfectoid Tate algebra having finite degree is not the same as being a polynomial. Therefore it is not true in general that g is a polynomial in X_n. In Example 5.19 we will construct an example of some f which is regular in X_n of degree d, but such that there is no factorization $f = ug$ with u a unit and g a polynomial in X_n^{1/p^m} for any m. First we need a uniqueness result, that any two Weierstrass preparations differ by a constant multiple.

Proposition 5.18 (Uniqueness of Preparation)

Suppose f is regular in X_n of degree d and $f = ug = vh$ are two factorizations with u, v units, and g, h of degree d in X_n. Then $u^{-1}v \in K^\times$.

PROOF. Normalize so that $||u|| = ||g|| = ||v|| = ||h|| = 1$. Passing to $\tilde{T}_n^{\mathrm{perf}}$ we see that $\overline{vh} = \lambda X_n^d + l$ for $\deg_{X_n} l < d$, since f is regular in degree d. Therefore the leading coefficient of h is a unit, and passing it to v we may assume that h is monic. Write $h = \sum h_\alpha X_n^\alpha$. Notice that $g = u^{-1}vh$. We can write $u^{-1}v = \sum q_\alpha X_n^\alpha$, and by Corollary 5.5 we have $||q_0|| = 1$ and $||q_\alpha|| < 1$ for all $\alpha \neq 0$.

Assume some $||q_\alpha|| = \varepsilon > 0$ for $\alpha > 0$. Then only finitely many q_α have absolute value greater than or equal to ε. Let τ be the maximum absolute value achieved by one of these coefficients (not including q_0), and choose γ the largest index such that $|q_\gamma| = \tau$. For every $\alpha > \gamma$ we have $||q_\alpha|| < ||q_\gamma||$. We examine the coefficient of $X_n^{d+\gamma}$ in $u^{-1}vh$. In particular, since it must be 0, we know that

$$\sum_{\alpha+\beta=d+\gamma} q_\alpha h_\beta = 0.$$

Since $h_d = 1$, we have $q_\gamma = -\sum_{\alpha>\gamma} q_\alpha h_\beta$. Since each $|h_\beta| \leq 1$ we have $||q_\gamma|| > ||q_\alpha h_\beta||$ for each $\alpha > \gamma$. But by the nonarchimedean property, then $||-\sum_{\alpha>\gamma} q_\alpha h_\beta|| < ||q_\gamma||$, a contradiction. So $q_\gamma = 0$, completing the proof.

We can now produce a counterexample to [7] Corollary 4.4.4.

Example 5.19

Let $f = \sum_{n=0}^{\infty} \varpi^n X^{1/p^n} \in T_1^{\mathrm{perf}}$. Then f is regular of degree 1, and $f = 1 \cdot f$ is a Weierstrass preparation of f. Any other preparation, $f = ug$ must have $u \in K^\times$, so that g cannot be a polynomial.

5.4 Generating Regular Elements

Although not every regular element in T_n^{perf} is a unit away from being a polynomial, it is a unit away from having finite degree. Next we show that for every $f \in T_n^{\mathrm{perf}}$, there is an automorphism of T_n^{perf} taking f to a regular element. Therefore every element in T_n^{perf} is an automorphism and a unit away from having finite degree.

Theorem 5.20

For every nonzero $f \in T_n^{\mathrm{perf}}$, there is some K-linear automorphism Φ of T_n^{perf} depending on f, such that $\Phi(f)$ is regular in X_n of some degree.

Notice that without loss of generality, we may assume $||f|| = 1$. If not we can normalize by $\lambda \in K^\times$ so $||\lambda f|| = 1$. Then $\Phi(\lambda f) = \lambda\Phi(f)$ is regular in X_n so that $\Phi(f)$ is regular in X_n of the same degree.

Taking this into account, we fix an element $f \in T_n^{\mathrm{perf}}$ with $||f|| = 1$. The proof will take several steps. First we define the map and show that it is well defined. Then we prove Theorem 5.20 in characteristic p. Finally we use the tilting equivalence to deduce the characteristic 0 case.

Let's begin by defining Φ. We let $d = \deg \overline{f} \in \tilde{T}_n^{\mathrm{perf}}$. Write $\overline{f} = \sum f_\alpha X^\alpha$. Consider all $\alpha \neq \beta$ such that $f_\alpha, f_\beta \neq 0$. Without loss of generality let $\alpha < \beta$ lexicographically, and let r be the first index such that $\alpha_r < \beta_r$. Then there is some m such that $\beta_r - \alpha_r \geq 1/p^m$. Since $\overline{f}$ has only finitely many nonzero coefficients, we can let M be larger than all m found in this way. We inductively define constants $\lambda_n, \cdots, \lambda_1 \in \mathbb{Z}$ as follows. Let $\lambda_n = 1$. For all $1 \leq j < n$ we define

$$\lambda_{n-j} = 1 + p^M d \sum_{i=0}^{j-1} \lambda_{n-i}.$$

Then we define $\Phi = \Phi_{d,M} : T_n^{\mathrm{perf}} \to T_n^{\mathrm{perf}}$ on the generators X_i^{1/p^m} of T_n^{perf} as follows, and extend K-linearly.

$$\Phi\left(X_n^{1/p^m}\right) = X_n^{1/p^m},$$

and for $i < n$

$$\Phi\left(X_i^{1/p^m}\right) = \lim_{r\to\infty}\left(X_i^{1/p^{m+r}} + X_n^{\lambda_i/p^{m+r}}\right)^{p^r}.$$

The limit converges due to Proposition 2.29.

Lemma 5.21

Φ is a well defined continuous ring homomorphism.

Proof. To see that Φ is well defined, it suffices to show that $\Phi\left(X_i^{1/p^{m+1}}\right)^p = \Phi\left(X_i^{1/p^m}\right)$. For $i = n$, this is immediate. Otherwise,

$$\begin{aligned}\Phi\left(X_i^{1/p^{m+1}}\right)^p &= \lim_{r\to\infty}\left(X_i^{1/p^{m+1+r}} + X_n^{\lambda_i/p^{m+1+r}}\right)^{p^{r+1}} \\ &= \Phi\left(X_i^{1/p^m}\right),\end{aligned}$$

where the last step follows because $r \to \infty$ if and only if $r + 1 \to \infty$.

Continuity follows from Proposition 2.28, since Φ is bounded. Indeed, for all i we have

$$\left|\left|\Phi\left(X_i^{1/p^r}\right)\right|\right| = 1 = \left|\left|X_i^{1/p^r}\right|\right|.$$

Since $||g|| = \max\{|a_\alpha|\}$ among all coefficients a_α, and because Φ is K linear, the strong triangle inequality shows that $||\Phi(g)|| \leq ||g||$, so that Φ is bounded by $\rho = 1$, and is therefore continuous.

If the characteristic of K is p, the map Φ simplifies considerably. In this case we still have

$$\Phi\left(X_n^{1/p^m}\right) = X_n^{1/p^m},$$

but for $i < n$

$$\Phi\left(X_i^{1/p^m}\right) = X_i^{1/p^m} + X_n^{\lambda_i/p^m},$$

Proposition 5.22

If the characteristic of K is p, then Φ is an isomorphism.

PROOF. We first check surjectivity. We only need to check that each generator of T_n^{perf} over K is in the image of Φ. Certainly X_n^{1/p^m} is in the image for each m. For $i < n$,

$$\begin{aligned} X_i^{1/p^m} &= X_i^{1/p^m} + X_n^{\lambda_i/p^m} - X_n^{\lambda_i/p^m} \\ &= \Phi\left(X_i^{1/p^m}\right) - \Phi\left(X_n^{1/p^m}\right)^{\lambda_i} \\ &= \Phi\left(X_i^{1/p^m} - X_n^{\lambda_i/p^m}\right). \end{aligned}$$

Before proving injectivity we establish the following fact.

Claim 5.23

If $\alpha \neq \beta \in \mathbb{Z}[1/p]_{\geq 0}$, then $\Phi\left(X_i^\alpha\right)$ and $\Phi\left(X_i^\beta\right)$ are linearly independent as elements of T_n^{perf} viewed as a module over $T_{n-1}^{\text{perf}} = K\langle X_1^{1/p^\infty}, \cdots, \widehat{X_i^{1/p^\infty}}, \cdots, X_n^{1/p^\infty}\rangle$.

PROOF. If $i = n$ this is clear. Otherwise, view $\alpha = a/p^k$ and $\beta = b/p^l$ with a, b coprime to p and without loss of generality suppose $k \geq l$. Suppose that there are $g, h \in T_{n-1}^{\text{perf}}$ (embedded into T_n^{perf} away from the ith coordinate).

$$g\Phi\left(X_i^\alpha\right) + h\Phi\left(X_i^\beta\right) = 0.$$

Then raising both sides to the p^k power, we have

$$g^{p^k}\Phi\left(X_i\right)^a + h^{p^k}\Phi\left(X_i\right)^{bp^{k-l}} = g^{p^k}\left(X_i + X_n^{\lambda_i}\right)^a + h^{p^k}\left(X_i + X_n^{\lambda_i}\right)^{bp^k - l} = 0.$$

Notice that $a \neq bp^{k-l}$. Indeed, if $k = l$ then $a \neq b$ because $\alpha \neq \beta$. Otherwise they can't be equal because a is coprime to p. Unless $g = 0$, we have

$$\deg_{X_i}\left(g^{p^k}\left(X_i + X_n^{\lambda_i}\right)^a\right) = a,$$

and similarly, unless $h = 0$

$$\deg_{X_i}\left(h^{p^k}\left(X_i + X_n^{\lambda_i}\right)^{bp^k - l}\right) = bp^{k-l}.$$

Since their difference is 0, we must have $g = h = 0$.

We can now complete the proof of injectivity using induction on n. For $n = 0, 1$, Φ is the identity map so we are done. In general, fix g and write it as $\sum g_\alpha X_i^\alpha$ for some $i \neq n$. Then $\Phi(g) = \sum \Phi(g_\alpha) X_i^\alpha$. If this is 0, Claim 5.23 shows that $\Phi(g_\alpha) = 0$ for all α. But $g_\alpha \in T_{n-1}^{\text{perf}}$, and Φ restricted to this ring is an isomorphism by the inductive hypothesis, and therefore injective. Thus $g_\alpha = 0$ for all α, and so $g = 0$.

Now that we have established that Φ is an automorphism of the perfectoid Tate algebra (at least in characteristic p), let's show that the image of f under Φ is regular of degree d.

Proposition 5.24

Let K have characteristic p. Let $\gamma = (\gamma_1, \cdots, \gamma_n)$ be the lexicographically maximal m-tuple of elements of $\mathbb{Z}[1/p]_{\geq 0}$ such that $|f_\gamma| = ||f|| = 1$. Then $\Phi(f)$ is regular in X_n of degree $\delta = \lambda_1\gamma_1 + \cdots + \lambda_n\gamma_n$.

PROOF. Write $\overline{f} = \sum \overline{f_\alpha} X^\alpha$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be tuples such that $\overline{f_\alpha}, \overline{f_\beta} \neq 0$. By the definition of d, we have $d \geq \alpha_i, \beta_i$ for all i. We first check that $\alpha < \beta$ lexicographically if and only if $\sum \lambda_i \alpha_i < \sum \lambda_i \beta_i$. Indeed, since $\alpha < \beta$ there is some r with $0 \leq r < n$ so that

$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_{r-1} = \beta_{r-1}$ and $\alpha_r < \beta_r$. We know that $\beta_r - \alpha_r < 1/p^M$, so that

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i \alpha_i &\leq \sum_{i=1}^{r-1} \lambda_i \beta_i + \lambda_r(\beta_r - 1/p^M) + \sum_{i=r+1}^{n} \lambda_i d \\ &= \left(\sum_{i=1}^{r} \lambda_i \beta_i \right) - 1 \\ &< \sum_{i=1}^{n} \lambda_i \beta_i. \end{aligned}$$

Recall that $\overline{f} = \sum \overline{f_\alpha} X^\alpha = \sum \overline{f_\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Write $\alpha_i = c_i/p^{k_i}$ for c_i prime to p. Then we have

$$\begin{aligned} \overline{\Phi}\left(\overline{f}\right) &= \sum \overline{f_\alpha} \left(X_1^{1/p^{k_1}} + X_n^{\lambda_1/p^{k_1}} \right)^{c_1} \cdots \left(X_{n-1}^{1/p^{k_{n-1}}} + X_n^{\lambda_{n-1}/p^{k_{n-1}}} \right)^{c_{n-1}} X_n^{\alpha_n} \\ &= \sum \left((\text{lower order terms}) + \overline{f_\alpha} X_n^{\sum_{i=1}^{n} \lambda_i \alpha_i} \right). \end{aligned}$$

Our computation above shows that the maximal degree of X_n corresponds precisely to the maximal lexicographic γ with $\overline{f_\gamma} \neq 0$, or equivalently, with $|f_\gamma| = 1$, so that $\overline{\Phi}\left(\overline{f}\right)$ is monic in X_n of degree δ, and so $\Phi(f)$ is regular in X_n of the same degree.

Now that we have the desired result in characteristic p, we hope to extend it to the general case.

Lemma 5.25

Let K be an arbitrary perfectoid field, and $K^\flat$ its characteristic p tilt. Let Φ_K and $\Phi_{K^\flat}$ be the maps constructed above. Then $(\Phi_K)^\flat = \Phi_{K^\flat}$.

Proof. Let $T_{n,K}^{\text{perf}} = K\langle X_1^{1/p^\infty}, \cdots, X_n^{1/p^\infty} \rangle$ and $T_{n,K^\flat}^{\text{perf}} = K^\flat \langle Y_1^{1/p^\infty}, \cdots, Y_n^{1/p^\infty} \rangle$. Viewing the perfectoid Tate algebra over $K^\flat$ as the tilt of the perfectoid Tate algebra over K, we can view $Y_i^{1/p^m} = \left(X_i^{1/p^m}, X_i^{1/p^{m+1}}, \cdots \right)$. Then we have

$$\begin{aligned} \Phi_{K^\flat}\left(Y_n^{1/p^m} \right) &= Y_n^{1/p^m} \\ &= \left(X_n^{1/p^m}, X_n^{1/p^{m+1}}, \cdots \right) \\ &= \left(\Phi_K\left(X_n^{1/p^m} \right), \Phi_K\left(X_n^{1/p^{m+1}} \right), \cdots \right) \\ &= \Phi_K^\flat\left(Y_n^{1/p^m} \right). \end{aligned}$$

For $i < n$, we consider the rth coordinate of $\Phi_{K^\flat}\left(Y_i^{1/p^m} \right)$.

$$\begin{aligned} \left(\Phi_{K^\flat}\left(Y_i^{1/p^m} \right) \right)_r &= \left(\left(X_i^{1/p^m}, X_i^{1/p^{m+1}}, \cdots \right) + \left(X_n^{\lambda_i/p^m}, X_n^{\lambda_i/p^{m+1}}, \cdots \right) \right)_r \\ &= \lim_{t \to \infty} \left(X_i^{1/p^{m+r+t}} + X_n^{\lambda_i/p^{m+r+t}} \right)^{p^t} \\ &= \Phi_K\left(X_i^{1/p^{m+r}} \right) \\ &= \left(\Phi_K^\flat\left(Y_i^{1/p^m} \right) \right)_r, \end{aligned}$$

completing the proof.

Corollary 5.26

In any characteristic, Φ_K is an isomorphism.

Proof. By Theorem 2.57, tilting defines an equivalence of categories $K_{\text{perf}} \to K^{\flat}_{\text{perf}}$. Since $\Phi^{\flat}_K = \Phi_{K^{\flat}}$ is an isomorphism, Φ_K must be as well.

Finally, we show that Proposition 5.24 extends to characteristic 0. We use the tilting equivalence here. The important point is that $\tilde{T}^{\text{perf}}_{n,K}$ and $\tilde{T}^{\text{perf}}_{n,K^{\flat}}$ can be identified via the Teichmüller map, and under this identification the reductions of Φ_K and $\Phi_{K^{\flat}}$ agree. Futhermore, Lemma 5.15 allows regularity to be checked after reducing modulo topologically nilpotent elements.

Proposition 5.27

In any characteristic, $\Phi(f)$ is regular in X_n of degree δ.

Proof. Identify $\tilde{T}^{\text{perf}}_{n,K}$ and $\tilde{T}^{\text{perf}}_{n,K^{\flat}}$ along the Teichmüller map, and choose some $f^{\flat} \in K^{\flat\circ}$ whose image in $\tilde{T}^{\text{perf}}_{n,K}$ is $\overline{f}$. Since the invariants d, M used in defining Φ only relied on the image of f in the quotient ring, we see that $\Phi_{K^{\flat}}(f^{\flat})$ is regular in X_n of degree δ. Under our identification, $\overline{\Phi}_K = \overline{\Phi}_{K^{\flat}}$, so that,

$$\overline{\Phi}_K\left(\overline{f}\right) = \overline{\Phi}_{K^{\flat}}\left(\overline{f^{\flat}}\right) = \lambda X_n^{\delta} + g,$$

where $\deg_{X_n} g < \delta$. In particular, $\Phi(f)$ is regular of degree δ.

6 Vector Bundles on the Perfectoid Unit Disk

There is a well known correspondence between finite projective modules over a ring, and finite dimensional (algebraic) vector bundles over the associated affine scheme, and more generally, between vector bundles over a locally ringed space and locally free sheaves on that space (see, for example, [14] Exercise 2.5.18). In [30], Serre asked whether there could be finite projective modules which are not free over the polynomial ring $A = k[x_1, \cdots, x_n]$ for a field k. This became known as Serre's conjecture, and can be interpreted geometrically as asking whether there are any nontrivial vector bundles over affine space $\mathbb{A}^n = \operatorname{Spec} A$. In 1976, Quillen [26] and independently Andrei Suslin proved Serre's conjecture, which is now known as the Quillen-Suslin theorem. Using these methods, Lütkebohmert in [25] was able to extend the result to the Tate algebra $K\langle X_1, \cdots, X_n\rangle$ of convergent power series over a complete nonarchimedean field.

In this section we establish a perfectoid analog of the Quillen-Suslin theorem. Specifically, we prove that all finite projective modules on the perfectoid Tate algebra are isomorphic to free modules. This implies that the perfectoid unit disk has no nontrivial finite vector bundles. Along the way we will show that both the subring of integral elements $\left(T_n^{\mathrm{perf}}\right)^\circ$, and the residue ring $\tilde{T}_n^{\mathrm{perf}}$ also have no nontrivial finite projective modules. Although these results are not necessary to establish the result for the perfectoid Tate algebra, they will be important in asserting the acyclicity of certain sheaves in Section 7.

In his 1986 paper [36], Leonid Vaseršteĭn gave a greatly simplified version of the Quillen-Suslin theorem which later appeared in Serge Lang's *Algebra*, [23]. It is this proof that we loosely follow in Section 6.3 so we begin by summarizing their methods.

6.1 Finite Free Resolutions and Unimodular Extension

There are two main ingredients to proving Serre's conjecture. The first step is showing that every finite projective module over the polynomial ring is stably free.

Definition 6.1. Let R be a commutative ring. An R-module M is said to be *stably free* if $M \oplus F$ is free for some finite free module F.

To do this, we use finite free resolutions.

Definition 6.2. A *finite free resolution* of a module M is an exact sequence (possibly infinite)

$$\cdots \to F^m \to F^{m-1} \to \cdots \to F^0 \to M \to 0$$

where each F^i is a finite free module. If the exact sequence has finite length, then M is said to have a *finite free resolution of finite length*

Proposition 6.3
A projective R-module P is stably free if and only if it has a finite free resolution of finite length.

Proof. If $P \oplus F \cong R^n$ for some n, then $0 \to F \to R^n \to P \to 0$ is a finite free resolution of length 1. Conversely, let $0 \to F^m \to \cdots \to F^0 \to M \to 0$ be a finite free resolution, we proceed by induction on m. If $m = 0$, P is free and so stably free. For $m \geq 1$, we let $M_1 = \ker(F^0 \to M)$. This gives rise to an exact sequence $0 \to M_1 \to F^0 \to P \to 0$. Since P is projective, $F^0 \cong P \oplus M_1$. But M_1 has a finite free resolution of length $m-1$ and is therefore stably free. Thus for some finite free F, $F^0 \oplus F \cong P \oplus M_1 \oplus F$ is free with $M_1 \oplus F$ free, and so P is stably free.

Serre proved finite projective modules over a polynomial ring are stably free in [33]. Indeed, using associated primes and the following lemma, one can reduce where M is a prime ideal of R.

Lemma 6.4 ([23] Theorem XXI.2.7)
Let $0 \to P \to M \to N \to 0$ be an exact sequence of R-modules. If any two of these modules have finite free resolutions of finite length, then so does the third.

Theorem 6.5 ([33] Proposition 10)

If k is a field, every finite $k[x_1, \cdots, x_n]$-module admits a finite free resolution of finite length. In particular, finite projective modules are stably free.

To complete the proof, we must show that every stably free module over this ring is in fact free. To do this, we introduce the concept of unimodular extension.

Definition 6.6. Let R be a commutative ring. A vector $v = (r_1, \cdots, r_n) \in R^n$ is called *unimodular* if the elements r_i generate the unit ideal.

A unimodular vector $v \in R^n$ is said to have the *unimodular extension property* if its transpose v^t is the first column of a matrix $M \in \mathrm{GL}_n(R)$. Equivalently, if there is some $M \in \mathrm{GL}_n(R)$ such that $Me_1 = u$ for the standard basis element $e_1 = (1, 0, \cdots, 0)$.

R is said to have the *unimodular extension property* if every unimodular vector has the unimodular extension property. Equivalently, if for all n the group $\mathrm{GL}_n(R)$ acts transitively on the set of unimodular vectors of length n.

This turns out to be the exact condition necessary for stably free module to be free.

Theorem 6.7

Let R have the unimodular extension property. If a finitely generated module E is stably free, then it is free.

Proof. First suppose that $R \oplus E = R^n$. Let $\pi : R \oplus E \to R$ be projection onto the first coordinate. Let $u_1 \in \pi^{-1}(1)$, and write $u_1 = (r_1, \cdots, r_n)$. Then $\pi(u_1) = r_1\pi(e_1) + \cdots + r_n\pi(e_n) = 1$. In particular, the $r_1, \cdots, r_n$ generate the unit ideal, and so u_1 is unimodular. Since R has the unimodular extension property, we can find an invertible matrix $M : R^n \to R^n$ such that $Me_1 = u_1$. For $j > 1$ define vectors $u_j = Me_j$. We have the following diagram.

$$\begin{tikzcd}
R^n \arrow[r, "M"] \arrow[dr, "\tilde{\pi}"'] & R^n \cong R \oplus E \arrow[d, "\pi"] \\
 & R
\end{tikzcd}$$

Define $c_j = \tilde{\pi}(u_j) \in R$, and with them define a new basis for R^n by $b_1 = u_1$ and $b_j = u_j - c_ju_1$ for $j > 1$. Then $\pi(b_j) = 0$ for all $j > 1$. We now have the following diagram of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d] & R \oplus E \arrow[r, "\pi"] \arrow[d, "b_i \mapsto e_i"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R^{n-1} \arrow[r] & R^n \arrow[r, "\pi_1"] & R \arrow[r] & 0.
\end{tikzcd}$$

Since the two righthand vertical maps are isomorphisms, so is the vertical map on the left, and so E is free.

For the general case we proceed by descending induction. If $R^m \oplus E$ is free, then our discussion shows that $R^{m-1} \oplus E$ is also free, and continuing in this way we conclude that E is free.

To complete the proof of the Quillen-Suslin theorem for the polynomial ring over a field, all that remains is to show that the polynomial ring has the unimodular extension property, which was the result that Quillen and Suslin independently arrived at. See [23] Section XXI for a complete proof of the theorem.

6.2 Coherent Rings

To extend the result from the polynomial ring to the Tate algebra, Lütkebohmert uses certain tools which are absent from the case from the perfectoid Tate algebra. To illustrate this we summarize the proof given in [20]. To prove that a finite projective module M has a finite free resolution of finite length, we can use the noetherian condition and Lemma 6.4 to reduce to proving that the associated primes of M have finite free resolutions of finite lengths, and the noetherian condition again to state that these primes are finitely generated ideals. Then Weierstrass preparation ([4] Section 5.2.2) implies that these ideals may be assumed to generated by polynomials and thus we can reduce to the polynomial case. Similarly, to prove that the Tate algebra has the unimodular extension property we may use Weierstrass preparation to assume that our unimodular vector is made up of polynomials (up to perhaps a unit) and again reduce to the polynomial case. In Example 5.19 we showed that the Weierstrass preparation theorem for the perfectoid Tate algebra, Theorem 5.17, is not strong enough to reduce us to the case of polynomials. Also the perfectoid Tate algebra is far from being noetherian (unless $n = 0$, see Remark 2.53). Nevertheless, the residue ring $\tilde{T}_n^{\mathrm{perf}}$ is something pretty close.

Definition 6.8. Let R be a commutative ring. A finitely generated R-module M is called *coherent* if every finitely generated submodule of M is finitely presented.

We call R a *coherent ring* if it is coherent as a module over itself. Equivalently, if every finitely generated ideal of R is finitely presented.

Coherent rings are studied extensively by Glaz in [11], and have a lot of the pleasant properties of noetherian rings. We will use the following facts.

Proposition 6.9 ([11] Theorem 2.2.1)
Let R be a commutative ring and $0 \to P \to N \to M \to 0$ a short exact sequence of R-modules. If any two modules in the sequence are coherent, so is the third.

Corollary 6.10 ([11] Corollary 2.2.2)
Let $\phi : M \to N$ be a homomorphism of coherent R-modules. Then $\ker \phi, \operatorname{im} \phi$, and $\operatorname{coker} \phi$ are coherent modules. In particular, these are all finitely generated.

There is also the following characterization of coherent rings.

Theorem 6.11 ([11] Theorem 2.3.2)
A commutative ring R is coherent if and only every finitely presented R module is coherent.

Corollary 6.12
Let M be a coherent module over a coherent ring R. Every finitely generated submodule of M is coherent.

From this we can deduce the following.

Lemma 6.13
Let M be a coherent module over a coherent ring R. Then there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M,$$

with subquotients $M_i/M_{i-1} \cong R/J_i$ for finitely generated ideals J_i.

Proof. Let $x_1, \cdots, x_r$ be a minimal generating set for M. We induct on r. $M_1 = Rx_1$ is a finitely generated submodule of M, and is therefore coherent. We have the presentation

$$0 \to J_1 \to R \to M_1 \to 0.$$

Since M_1 and R are coherent, so is J_1, so that in particular it is finitely generated. (This also serves as a base case.)

Now we apply induction to $M' = M/M_1$ and let $\pi : M \to M'$ be the projection. By induction there is a finite filtration

$$0 = M'_1 \subseteq M'_2 \subseteq \cdots \subseteq M'_n = M',$$

and $M'_i/M'_{i-1} \cong R/J_i$ for J_i finitely generated in R. Letting $M_i = \pi^{-1}(M'_i)$ produces a filtration of M with the same subquotients.

6.3 Finite Projective Modules on the Residue Ring

In this section we prove that all finite projective modules are free over the residue ring

$$\tilde{T}_n^{\text{perf}} = k\left[X_1^{1/p^\infty}, \cdots, X_n^{1/p^\infty}\right] = \bigcup_m k\left[X_1^{1/p^m}, \cdots, X_n^{1/p^m}\right],$$

where, as before, $k = K^\circ/K^{\circ\circ}$ is the residue field of K. As a first step, we prove that $\tilde{T}_n^{\text{perf}}$ is coherent. To simplify notation we will often denote the tuple $(X_1, \cdots, X_n)$ by X.

Proposition 6.14

Let $I = (f_1, \cdots, f_r) \subseteq \tilde{T}_n^{\text{perf}}$ be a finitely generated ideal. Then I admits a finite free resolution of finite length. In particular, I is finitely presented so that $\tilde{T}_n^{\text{perf}}$ is a coherent ring.

Proof. Each $f_i \in k\left[X^{1/p^{N_i}}\right]$ for some N_i. Let $N \geq \max N_i$ Then for all $m > N$ we have $f_i \in k\left[X^{1/p^m}\right]$. Let I_m be the ideal generated by the f_i in this ring. Since it is a finite module over a polynomial ring, Theorem 6.5 implies that I_m has a finite free resolution

$$0 \to \left(k\left[X^{1/p^m}\right]\right)^{n_l} \to \cdots \to \left(k\left[X^{1/p^m}\right]\right)^{n_1} \to \left(k\left[X^{1/p^m}\right]\right)^{n_0} \to (f_1, \cdots, f_r) = I_m \to 0.$$

Taking the union as $m \to \infty$ and noting that filtered colimits are exact produces a finite free resolution of I of finite length.

Corollary 6.15

Let M be a coherent $\tilde{T}_n^{\text{perf}}$-module. Then M has a finite free resolution of finite length.

Proof. We induct on the number of generators of M. If there is only 1, then we have

$$0 \to I \to \tilde{T}_n^{\text{perf}} \to M \to 0.$$

By Proposition 6.9 I is coherent and hence finitely generated. Therefore I has a finite free resolution of finite length by Proposition 6.14, so that by Lemma 6.4, M has a finite free resolution of finite length.

For the general case we notice that by Lemma 6.13, we have a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M,$$

where r is a minimal number of generators for M. Consider the exact sequence

$$0 \to M_1 \to M_r \to M_r/M_1 \to 0.$$

Proposition 6.9 applied to this short exact sequence implies that M/M_1 is coherent, and therefore has a finite free resolution of finite length by the induction hypothesis (having $r-1$ generators). M_1 is coherent by Proposition 6.9 applied to $0 \to J_1 \to R \to M_1 \to 0$, and therefore has a finite free resolution of finite length by the induction hypothesis noting it has 1 generator. Therefore M_r has a finite free resolution of finite length by Lemma 6.4.

This together with Proposition 6.3 implies that all finite projective $\tilde{T}_n^{\mathrm{perf}}$-modules are stably free. To show that they are in fact free, we prove that $\tilde{T}_n^{\mathrm{perf}}$ has the unimodular extension property.

Proposition 6.16

$\tilde{T}_n^{\mathrm{perf}}$ has the unimodular extension property.

Proof. Let $u = (f_1, \cdots, f_m)$ be a unimodular vector in $\left(\tilde{T}_n^{\mathrm{perf}}\right)^m$. Then there are some g_i such that $\sum f_i g_i = 1$. Each $f_i, g_i \in k\left[X^{1/p^N}\right]$ for some large N. Therefore $(f_1, \cdots, f_m)$ is a unimodular vector in $\left(k\left[X^{1/p^N}\right]\right)^m$ which is a polynomial ring and therefore has the unimodular extension property. Therefore there is some matrix

$$M \in \mathrm{GL}_m\left(k\left[X^{1/p^N}\right]\right) \subseteq \mathrm{GL}_m\left(\tilde{T}_n^{\mathrm{perf}}\right)$$

with $Me_1 = u$.

We now completely understand finite projective modules on $\tilde{T}_n^{\mathrm{perf}}$.

Corollary 6.17

Every finite projective $\tilde{T}_n^{\mathrm{perf}}$-module is free.

Proof. A finite projective module is stably free by Corollary 6.15 and Proposition 6.3, and is therefore free by Theorem 6.7 and Proposition 6.16.

6.4 Finite Projective Modules on the Ring of Integral Elements

We extend Corollary 6.17 to the subring of power-bounded elements of the perfectoid Tate algebra, $\left(T_n^{\mathrm{perf}}\right)^\circ$, using Nakayama's lemma. We first fix some notation.

Notation 6.18

For a commutative ring R and an ideal I contained in the Jacobson radical of R we let $R_0 = R/I$. For an R module M we will denote by M_0 the R_0 module M/IM, and for a homomorphism ϕ of R-modules we denote by ϕ_0 its reduction mod I. If $m \in M$, then we denote by $\overline{m}$ its image in M_0.

Lemma 6.19

Let R be a commutative ring, and I an ideal contained in the Jacobson radical of R. If M and N are two projective R-modules such that there exists an isomorphism $\phi : M_0 \xrightarrow{\sim} N_0$, then ϕ lifts to an isomorphism $\psi : M \xrightarrow{\sim} N$.

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, dashed, "\psi"] \arrow[d, two heads, "\pi"] & N \arrow[d, two heads, "\rho"] \\
M_0 \arrow[r, "\phi"] & N_0
\end{tikzcd}$$

Indeed, a lift ψ exists because M is projective. Notice that $\psi_0 = \phi$. Indeed,

$$\begin{aligned}
\psi_0(\overline{m}) &= \overline{\psi(m)} \\
&= \rho\psi(m) \\
&= \phi\pi(m) \\
&= \phi(\overline{m}).
\end{aligned}$$

Since ϕ surjects, so does ψ by Nakayama's lemma. Since N is projective, ψ has a section σ which is necessarily injective. We claim that $\sigma_0 = \phi^{-1}$. Indeed, we can check this after applying ϕ.

$$\begin{aligned}\phi\sigma_0(\overline{n}) &= \phi\pi\sigma(n) \\ &= \rho\psi\sigma(n) \\ &= \rho(n) \\ &= \overline{n}.\end{aligned}$$

Therefore σ_0 surjects, so that σ surjects by Nakayama's lemma. Thus σ is an isomorphism.

Corollary 6.20

With the same setup as Lemma 6.19, we let P be a projective R-module. If P_0 is a free R_0 module, then P is free.

Proof. Suppose $P_0 \cong R_0^m$. We also have $(R^m)_0 \cong R_0^m$ so that by the lemma, $P \cong R^m$.

The result now follows.

Corollary 6.21

Let P be a finite projective $\left(T_n^{\text{perf}}\right)^{\circ}$ module. Then P is free.

Proof. Notice that P_0 is a finite projective $\tilde{T}_n^{\text{perf}}$ module. Then P_0 is free by Corollary 6.17. Therefore by Corollary 6.20 it suffices to show that the kernel of the reduction map is contained in the Jacobson radical. Let f be in the kernel. Then $||f|| < 1$ so that for all g, $||fg|| < 1$ and so $\overline{fg} = 0$. Therefore $\overline{1 - fg} = 1 \in k^{\times}$. By Proposition 5.4 we know that $1 - fg$ is a unit. Since g was arbitrary, f is in the Jacobson radical.

Remark 6.22

The author remarks that $\left(T_n^{\text{perf}}\right)^{\circ}$ also exhibits the unimodular extension property, but this fact is not necessary in the proof and so is not included.

6.5 The Quillen-Suslin Theorem for the Perfectoid Tate Algebra

We now prove the main theorem of this section.

Theorem 6.23

Finite projective modules on the perfectoid Tate algebra are free.

Keeping Theorem 3.52 in mind, we have the following geometric restatement.

Theorem 6.24

All finite vector bundles on the perfectoid unit disk are free.

We do not use the setup of the previous sections. Instead, recall that

$$T_n^{\text{perf}} = \widehat{\bigcup_i K\left\langle T_1^{1/p^i}, \cdots, T_n^{1/p^i}\right\rangle},$$

so that the perfectoid Tate algebra is the completed union of Tate algebras. Lütkebohmert showed that finite projective modules on Tate algebras are free ([25] Satz 1), so we have reduced to proving the following proposition.

Proposition 6.25

Suppose $A = \widehat{\cup A_i}$ where A_i are complete Banach algebras. Then any finite projective module on A is isomorphic to the base extension of a finite projective module on one of the factors.

We devote the rest of this section to the proof of Proposition 6.25, drawing inspiration from proof of Lemma 5.6.8 in [22].

Fix a finite projective module P over A, and take a presentation $\pi : A^n \to P$, as well as a section of this projection σ. The composition $\sigma \circ \pi$ can be thought of as a projector matrix $U \in M_n(A)$, which is idempotent (i.e., $U^2 = U$).

$$\begin{tikzcd}
A^n \arrow[rr, "U"] \arrow[rrrr, bend left, "id"] \arrow[dr, "\pi"] & & A^n \arrow[rr, "U"] \arrow[dr, "\pi"] & & A^n \\
& P \arrow[ur, "\sigma"] \arrow[rr, equal] & & P \arrow[ur, "\sigma"] &
\end{tikzcd}$$

Conversely, the image of an idempotent matrix is always projective, with the section just given by the natural inclusion $\operatorname{im} U \subseteq A^n$. In this way, we get a (non unique) correspondence between finite projective modules and idempotent matrices over A. To approximate a projective module over A by one over some A_i, we will try to approximate the associated idempotent matrix over A by one over A_i for some i.

We view $M_n(A)$ as a complete noncommutative nonarchimedean Banach A-algebra with the supremum norm $||(a_{ij})|| = \max\{|a_{ij}|\}$. This is a submultiplicative norm, that is, $||X \cdot Y|| \leq ||X|| \cdot ||Y||$, where we cannot insist on equality because matrix multiplication involves addition. Since A is the completion of the union of the A_i we have that $M_n(A)$ is the completed union of the $M_n(A_i)$. In particular, fixing any $U \in M_n(A)$ and any $\varepsilon > 0$, there is some i and $V \in M_n(A_i)$ such that $||V - U|| < \varepsilon$.

Fix a matrix $U \in M_n(A)$ whose image is P. Our goal is to find an idempotent $W \in M_n(A_i)$ which is close to U in the given nonarchimedean topology. The fact that W is idempotent would show that its image is projective, and the hope is that W being close to U can be leveraged into showing they have isomorphic images (after base change to A). We first note that if V is close to U, then U being idempotent should imply that V is pretty close to being idempotent, that is, $||V^2 - V||$ is small. This can be thought of as a consequence to the fact that $x^2 - x$ is a continous function on $M_n(A)$. Let's prove this. Choose V such that $||V - U|| < \varepsilon$. Then:

$$\begin{aligned}
||V^2 - V|| &= ||V^2 - V - (U^2 - U)|| \\
&= ||V^2 - U^2 - (V - U)|| \\
&\leq \max\{||V^2 - U^2||, ||V - U||\}.
\end{aligned}$$

We already know that $||V - U|| < \varepsilon$, which should imply that $||V^2 - U^2||$ is small too. Indeed:

$$\begin{aligned}
||V^2 - U^2|| &= ||(V - U)^2 + UV + VU - 2U^2|| \\
&= ||(V - U)^2 + U(V - U) + (V - U)U|| \\
&\leq \max\{||V - U||^2, ||U|| \cdot ||V - U||\} \\
&= \max\{\varepsilon^2, \varepsilon \cdot ||U||\}.
\end{aligned}$$

Since $||U||$ will never change, this means we can make this as small as we want. Later, we will want $||V^2 - V|| < ||U||^{-3}$, so once and for all we fix $V \in M_n(A_i)$ such that $||V - U||| < ||U||^{-4}$.

Notice that $||U|| \geq 1$. Indeed, if $||U|| < 1$ then U would be topologically nilpotent, that is, U^n would converge to 0. We also know it converges to U by idempotence, so that $U = 0$. So for any interesting case we have $||U|| \geq 1$. Since the norm is submultiplicative we cannot insist $||U|| = 1$. We do have that $||V - U|| < ||U||^{-4} \leq 1$.

Recall from Section 2.2 that all triangles in any nonarchimedean group are isosceles. Indeed, according to Lemma 2.22, $||a + b|| \leq \max\{||a||, ||b||\}$, with equality if $||a|| \neq ||b||$. In particular, if $||a + b|| < \max\{||a||, ||b||\}$, then we conclude that $||a|| = ||b||$. We can think about this as saying two things that are close together must have the same size. In particular, since $||V - U|| < 1 \leq ||U|| \leq \max\{||U||, ||V||\}$, we can conclude that $||V|| = ||U||$.

We must produce matrix W over A_i, which is near to V (and hence near to U), and which is also idempotent. To find such a W we will use Newton's method of approximation. For functions of 1 variable, Newton's method is the following: given a function $f(x)$ and a guess x_0 for a root, we generate closer and closer guesses according to the formula

$$x_{l+1} = x_l - \frac{f(x_l)}{f'(x_l)},$$

which is the zero of the tangent line to f at x_l. If x_0 was a good guess and f is well enough behaved, then the x_l will converge to a root of f. It turns out that there is a formulation of Newton's method for matrices, or more generally for functions between Banach spaces. To make this formulation we need the following definition. We do not need the full strength of the definition, but we include it for completeness.

Definition 6.26. Let M and N be Banach spaces, and $x \in M$. A function $F : M \to N$ is said to be *Frèchet differentiable* if there exists a bounded linear operator $L : M \to N$ such that

$$\lim_{h \to 0} \frac{||F(x+h) - F(x) - Lh||}{||h||} = 0.$$

Then L is called the *Frèchet derivative* of F at x and is denoted $F'(x)$.

Certainly $F'(x)$ is a linear approximation of F near x, and thus its roots are more easily discoverable. Therefore we can continue as with classical Newton approximation. Suppose we are trying to find a root of a function F on $M_n(A)$. If we have a guess X_0 we can iterate along,

$$X_{l+1} = X_l - F'(X_l)^{-1} F(X_l),$$

where $F'(X_l)$ is the Frechèt derivative of F at X_l, and hope that the X_l converge to a root of F. We are trying to find a zero of the function $F(X) = X^2 - X$, whose roots are precisely the idempotent matrices, and we would like to find one near U (and therefore near V). So let's try starting from a guess $W_0 = V$, and then impliment Newton's method to try and find a root of F near V.

A first thing to notice is that $F'(X)$ is the linear functional $H \mapsto XH + HX - H$. Since we are applying this to $H = X^2 - X$, we have that X and H commute, so that $F'(X) = 2X - 1$. Therefore we are iterating along

$$X - \frac{X^2 - X}{2X - 1}.$$

There is no reason to believe that $2X - 1$ is invertible. This may seem like an obstacle, but since we are approximating, we don't need a perfect inverse, just an approximation of one. Since all the matrices we are studying are idempotent, or close to it, we guess that $2X - 1$ is its own inverse. Indeed

$$(2X - 1)(2X - 1) = 4X^2 - 4X + 1 = 4(X^2 - X) + 1.$$

Since X will be close to U, it will be close to being idempotent, so that $X^2 - X$ is small, and if Newton's method works, then X converges to an idempotent matrix, and so $2X - 1$ converges to its own inverse. Therefore we will try finding W using Newton's method as follows.

$$\begin{aligned} W_0 &= V \\ W_{l+1} &= W_l - (2W_l - 1)(W_l^2 - W_l) = 3W_l^2 - 2W_l^3. \end{aligned}$$

Amazingly, this works.

Proposition 6.27

The W_l converge to an idempotent $W \in M_n(A_i)$ such that $|W - U| < 1$.

Proof. Let us make a few initial computations.

$$\begin{aligned} W_{l+1} - W_l &= 3W_l^2 - 2W_l^3 - W_l \\ &= (W_l^2 - W_l)(1 - 2W_l). \end{aligned}$$

If the W_l converge an idempotent matrix, then this converges to 0. Note also that

$$\begin{aligned} W_{l+1}^2 - W_{l+1} &= (3W_l^2 - 2W_l^3)^2 - 3W_l^2 + 2W_l^3 \\ &= 4W_l^6 - 12W_l^5 + 9W_l^4 + 2W_l^3 - 3W_l^2 \\ &= (W_l^2 - W_l)^2(4W_l^2 - 4W_l - 3). \end{aligned}$$

With this in hand, we establish the following fact by induction.

Claim 6.28

For any $l \geq 0$ we have

i) $||W_l - U|| < ||U||^{-2}$

ii) $||W_l^2 - W_l|| \leq ||U||^{-2} \left(||V^2 - V|| \cdot ||U||^2\right)^{2^l}$

Proof. The base case is immediate, because $W_0 = V$. For (i) we have

$$||V - U|| < ||U||^{-3} \leq ||U||^{-2}.$$

For (ii) we have

$$||V^2 - V|| = ||U||^{-2}||V^2 - V|| \cdot ||U||^2 = ||U||^{-2} \left(||V^2 - V|| \cdot ||(U)||^2\right)^{2^0}.$$

For the inductive step we assume both (i) and (ii) hold for $\leq l$. Then

$$||W_{l+1} - U|| = ||W_{l+1} - W_l + W_l - U|| \leq \max\{||W_{l+1} - W_l||, ||W_l - U||\}.$$

The latter is $< ||U||^{-2}$ by the inductive hypothesis, so it suffices to show this for the former. By our earlier computation

$$||W_{l+1} - W_l|| = ||(W_l^2 - W_l)(1 - 2W_l)|| \leq ||W_l^2 - W_l|| \cdot ||1 - 2W_l||.$$

By the inductive hypothesis we have

$$\begin{aligned} ||W_l^2 - W_l|| &\leq ||U||^{-2} \left(||V^2 - V|| \cdot ||U||^2\right)^{2^l} \\ &< ||U||^{-2} \left(||U||^{-3}||U||^2\right)^{2^l} \\ &= ||U||^{-2-2^l} \\ &\leq ||U||^{-3}. \end{aligned}$$

It is important that we have a strict inequality in the second line above, but our choice of V above ensured this. Now we also notice that

$$||1 - 2W_l|| = ||1 - 2U + 2U - 2W_l|| \leq \max\{||1 - 2U||, ||2U - 2W_l||\}.$$

The latter is $< ||U||^{-2}$ by the inductive hypothesis (and noting that $||2|| \leq 1$ by Lemma 2.21), and the former is $\leq ||U||$ by the strong triangle inequality (and again Lemma 2.21). Putting this all together we have shown that $||W_l^2 - W_l|| < ||U||^{-2}$, which proves statement (i).

For statement (ii) we compute the following, applying the inductive hypothesis in the third line.

$$\begin{array}{rcl} ||W_{l+1}^2 - W_{l+1}|| & = & ||(W_l^2 - W_l)^2(4W_l^2 - 4W_l - 3)|| \\ & \leq & ||W_l^2 - W_l||^2||4W_l^2 - 4W_l - 3|| \\ & \leq & \left(||U||^{-2}\left(||V^2 - V|| \cdot ||U||^2\right)^{2^l}\right)^2 ||4W_l^2 - 4W_l - 3|| \\ & = & ||U||^{-4}\left(||V^2 - V|| \cdot ||U||^2\right)^{2^{l+1}} ||4W_l^2 - 4W_l - 3||. \end{array}$$

So it suffices to show that $||4W_l^2 - 4W_l - 3|| \leq ||U||^2$. Well,

$$||4W_l^2 - 4W_l - 3|| = ||(4W_l^2 - 4W_l + 1) - 4|| \leq \max\{||4W_l^2 - 4W_l + 1||, ||4||\}.$$

Certainly $||4|| \leq 1 \leq ||U||^2$, and

$$||4W_l^2 - 4W_l + 1|| = ||(2W_l - 1)^2|| \leq ||2W_l - 1||^2 \leq ||U||^2.$$

This completes the proof of the claim.

Since $||V^2 - V|| \cdot ||U||^2 < ||U||^{-2} \leq 1$, we have $(||V^2 - V|| \cdot ||U||^2)^{2^l}$ converging to zero, so that $||W_l^2 - W_l||$ converges to zero as l increases. Therefore,

$$||W_{l+1} - W_l|| \leq ||W_l^2 - W_l|| \cdot ||1 - 2W_l|| \leq ||W_l^2 - W_l|| \cdot ||U||,$$

converges to zero. By Lemma 2.23 we only need the difference successive terms to converge to 0 in order for a sequence to converge, and since each W_l is over A_i, which is complete, we have shown that the W_l converge to some $W \in M_n(A_i)$. Not only that, since $||W_l^2 - W_l||$ converges to 0, we have that $||W^2 - W|| = 0$ so that $W^2 = W$. As above, the proof boils down to the fact that $X^2 - X$ is continuous, but we record it here for posterity.

Claim 6.29

W is idempotent.

Proof. For every l we have

$$||W^2 - W|| = ||W^2 - W - (W_l^2 - W_l) + W_l^2 - W_l|| \leq \max\{||W^2 - W - (W_l^2 - W_l)||, ||W_l^2 - W_l||\}.$$

We can make the latter as small as we want by increasing l, As for the former,

$$||W^2 - W - (W_l^2 - W_l)|| = ||(W^2 - W_l^2) - (W - W_l)|| \leq \max\{||W^2 - W_l^2||, ||W - W_l||\}.$$

Again, we can make the latter as small as we want by increasing l. So we'd like to say the same about the former. First, we notice that

$$||W_l|| \leq ||W_l - U + U|| \leq \max\{||W_l - U||, ||U||\} \leq ||U||.$$

Therefore,

$$\begin{array}{rcl} ||W^2 - W_l^2|| & = & ||(W - W_l)^2 - WW_l - W_lW - 2W_l|| \\ & = & ||(W - W_l)^2 - (W - W_l)W_l - W_l(W - W_l)|| \\ & \leq & \max\{||W - W_l||^2, ||W_l|| \cdot ||W - W_l||\} \\ & \leq & \max\{||W - W_l||^2, ||U|| \cdot ||W - W_l||\}. \end{array}$$

Since we can make both factors as small as we want by increasing l, we can conclude that $||W^2 - W|| = 0$ so that $W^2 = W$.

To complete the proof of Proposition 6.27, we must check that W is close to U.

$$||W - U|| = ||W - W_l + W_l - U|| \leq \max\{||W - W_l||, ||W_l - U||\}.$$

We can make the former term as small as we'd like by increasing l, and the latter is strictly less than $||U||^{-2} \leq 1$, so that $||W - U|| < 1$. This completes the proof.

This shows that the image of W is a projective module. To complete the proof of Proposition 6.25 we must show that $\operatorname{im} W \otimes_{A_i} A \cong \operatorname{im} U = P$. Since $M_n(A_i) \subseteq M_n(A)$, we can view W as a matrix over A, and then it suffices to show that $\operatorname{im} W \cong \operatorname{im} U$ as modules over A (since tensor product is extension of scalars).

Let $u = Ux$ be an arbitrary nonzero element in $\operatorname{im} U$. Since U acts as the identity on u, we have

$$||Wu - u|| = ||(W - U)(u)|| < ||u||.$$

Since all triangles are isosceles (Lemma 2.22), this implies that $||Wu|| = ||u||$. Therefore the restriction of W to $\operatorname{im} U \subseteq A^n$ is injective. Indeed, if $||Wu|| = 0$ then $||u|| = 0$ and since the norm on A^n is the maximum of the coordinates, this would imply $u = 0$. A symmetric argument shows that the restriction of U to the image of W is also injective. In summary, we have

$$\begin{tikzcd} \operatorname{im} U \arrow[r, hook, "W"] & \operatorname{im} W \arrow[r, hook, "U"] & \operatorname{im} U. \end{tikzcd}$$

Since $||W - U|| < 1$, it is topologically nilpotent as a map from A^n to itself, but in fact, I claim that $(W - U)^2$ is a topologically nilpotent in the ring of bounded linear maps from $\operatorname{im} U \to \operatorname{im} U$. That it is bounded and topologically nilpotent has already been established, so it just suffices to show that that the image lands in $\operatorname{im} U$. Indeed, because U acts as the identity on u and W acts as zero on $W - 1$ we have

$$\begin{aligned} (W - U)^2 u &= (W - U)(W - U)u \\ &= (W - U)(W - 1)u \\ &= -U(W - 1)(u) \\ &= (1 - UW)u \in \operatorname{im} U. \end{aligned}$$

This not only shows that it is a topologically nilpotent map from $\operatorname{im} U$ to itself, but in fact, that it is equal to $1 - UW$ when restricted to $\operatorname{im} U$. Therefore, by the geometric series (Lemma 2.26) we conclude that $1 - (1 - UW) = UW$ is a unit and is therefore invertible, i.e., an isomorphism. So we have

$$\begin{tikzcd} \operatorname{im} U \arrow[r, hook, "W"] \arrow[rr, bend left, "UW"] & \operatorname{im} W \arrow[r, hook, "U"] & \operatorname{im} U. \end{tikzcd}$$

The composition is an isomorphism so that $U : \operatorname{im} W \to \operatorname{im} U$ surjects. We have already established injectivity, so this completes the proof of Proposition 6.25. We have the following immediate consequence.

Theorem 6.30

A finite projective module on the perfectoid Tate algebra is free. A finite projective module on Laurent series ring of a perfectoid Tate algebra (given by inverting any number of the indeterminates) is free.

Proof. Both of these are completed unions of Tate algebras (or Laurent series Tate algebras), on which these properties hold by [25].

7 Line Bundles and Cohomology on Projectivoid Space

In classical algebraic geometry, the notion of *projective geometry* is a very powerful tool to study properties of varieties and schemes. Indeed, one can learn a lot about a scheme by understanding its maps to various projective spaces, and this theory is intimately connected to the theory of line bundles on that space. In this and the following section we develop an analogous theory for perfectoid spaces.

In Example 4.23 we defined a perfectoid analog of projective space, which we call *projectivoid space*, and denote by $\mathbb{P}^{n,\mathrm{perf}}$. Proposition 4.24 stated that the construction of projectivoid space is compatible with the tilting functor. In this section we begin our exploration of so called *projectivoid geometry* by developing the theory of line bundles on projectivoid space. In particular, we compute the Picard group of $\mathbb{P}^{n,\mathrm{perf}}$, as well as the sheaf cohomology of all line bundles (as invertible sheaves). We continue developing the theory in the following section, where we will show how an arbitrary perfectoid space's maps to projectivoid space is intimately connected to its theory of line bundles, reflecting the situation in classical algebraic geometry but with an extra arithmetic twist.

Recall that for any ringed space X, there is a natural isomorphism $\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathscr{O}_X^*)$ (see for example [14] Exercise III.4.5). We will frequently use this isomorphism in what follows.

7.1 Reductions Using Čech Cohomology

In Example 4.23 we constructed n-dimensional projectivoid space by gluing together $n+1$ copies of the perfectoid unit disk. We will henceforth refer to the cover of $\mathbb{P}^{n,\mathrm{perf}}$ by these disks as the *standard cover*. Furthermore, Theorem 6.30 shows that any line bundle on $\mathbb{P}^{n,\mathrm{perf}}$ becomes trivial on the standard cover and its various finite intersections. It therefore seems reasonable to use Čech cohomology with respect to this cover to study line bundles on projectivoid space.

Theorem 7.1 (Čech-to-Derived Spectral Sequence: [1] Exposé V Théorème 3.2)

Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X, and $\mathfrak{U}$ a cover of X. Let $\mathscr{H}^q(\mathscr{F}) : U \mapsto H^q(U, \mathscr{F})$ be the cohomology presheaf. Then there is a spectral sequence:

$$E_2^{p,q} : \ \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}^q(\mathscr{F})) \implies \mathrm{H}^{p+q}(X, \mathscr{F}).$$

Let $X = \mathbb{P}^{n,\mathrm{perf}}$, and $\mathfrak{U} = \{U_0, \cdots, U_n\}$ be the standard cover by perfectoid disks. Theorem 6.24 tells us that

$$\mathscr{H}^1(\mathscr{O}_X^*)(U_i) \cong \mathrm{H}^1(\mathbb{D}^{n,\mathrm{perf}}, \mathscr{O}^*) \cong \mathrm{Pic}\,\mathbb{D}^{n,\mathrm{perf}} = 0,$$

for each i, so that

$$E_2^{0,1} = \check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{H}^1(\mathscr{O}_X^*)) = 0.$$

Therefore the sequence of low degree terms degenerates to

$$\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X^*) \cong \mathrm{H}^1(X, \mathscr{O}_X^*).$$

Thus Čech cohomology with respect to this cover computes sheaf cohomology, as desired. To summarize:

Lemma 7.2

With our notation as above, $\mathrm{Pic}(X) \cong \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X^*)$.

Recall from Definition 3.28 the sheaf $\mathscr{O}_X^+$ of integral elements:

$$U \mapsto \{f \in \mathscr{O}_X(U) : |f(x)| \leq 1 \text{ for all } x \in U\}.$$

Let $\mathscr{O}_X^{+*}$ be the subsheaf of integral units. We next reduce to computing the cohomology of this sheaf.

Lemma 7.3

With our notation as above, $\mathrm{Pic}(X) \cong \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X^{+*})$.

Proof. We have a short exact sequence of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & \prod_i \mathscr{O}_X^{+*}(U_i) \arrow[r] \arrow[d] & \prod_i \mathscr{O}_X^{*}(U_i) \arrow[r] \arrow[d] & \prod_i |K^*| \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{i,j} \mathscr{O}_X^{+*}(U_i \cap U_j) \arrow[r] \arrow[d] & \prod_{i,j} \mathscr{O}_X^{*}(U_i \cap U_j) \arrow[r] \arrow[d] & \prod_{i,j} |K^*| \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{i,j,k} \mathscr{O}_X^{+*}(U_i \cap U_j \cap U_k) \arrow[r] \arrow[d] & \prod_{i,j,k} \mathscr{O}_X^{*}(U_i \cap U_j \cap U_k) \arrow[r] \arrow[d] & \prod_{i,j,k} |K^*| \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The left and middle complexes are the Čech complexes for $\mathscr{O}_X^{+*}$ and $\mathscr{O}_X^{*}$ respectively, and the map on the right is $|\cdot|$ which is plainly surjective. Also the right hand complex has kernel $|K^*|$ and is otherwise exact, so that the long exact sequence on cohomology gives

$$\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^{+*}) \cong \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^{*}),$$

for all $i > 0$. Letting $i = 1$ completes the proof.

This lemma opens up the possibility of reducing modulo the topologically nilpotent functions. Recall from Definition 3.30 the sheaf of topologically nilpotent elements,

$$\mathscr{O}_X^{++} : U \mapsto \{f \in \mathscr{O}_X(U) : |f(x)| < 1 \text{ for all } x \in U\}.$$

This is an ideal in $\mathscr{O}_X^{+}$, and the quotient we denote by $\tilde{\mathscr{O}}_X$. We need the following lemma of commutative algebra.

Lemma 7.4

Let $R \to S$ be a surjection of rings whose kernel I is contained in the Jacobson radical of R. Then the induced map on unit groups, $R^* \to S^*$, remains surjective.

Proof. Fix $s \in S^*$ and $r \in R$ mapping to s. If $r \in \mathfrak{m}$ for any maximal ideal of r, then its image would be contained in $\mathfrak{m}/I \cdot \mathfrak{m}$, a proper ideal of S. Since its image is a unit this cannot be the case. Since r is not contained in any maximal ideal it must be a unit.

By Lemma 2.26, $1 + f$ is a unit for any topologically nilpotent f. In particular, $\mathscr{O}_X^{++}$ is contained in the Jacobson radical of $\mathscr{O}_X^{+}$, so that taking unit groups on the surjection $\mathscr{O}_X^{+} \to \tilde{\mathscr{O}}_X$ induces the following exact sequence of multiplicative groups.

$$1 \longrightarrow 1 + \mathscr{O}_X^{++} \longrightarrow \mathscr{O}_X^{+*} \longrightarrow \tilde{\mathscr{O}}_X^{*} \longrightarrow 1.$$

By Proposition 5.4, the right hand map remains surjective when evaluated on the standard cover $\mathfrak{U}$ and all finite intersections. Thus we get an exact sequence of Čech complexes

$$0 \longrightarrow \check{\mathrm{C}}^*(\mathfrak{U}, 1 + \mathscr{O}_X^{++}) \longrightarrow \check{\mathrm{C}}^*(\mathfrak{U}, \mathscr{O}_X^{+*}) \longrightarrow \check{\mathrm{C}}^*(\mathfrak{U}, \tilde{\mathscr{O}}_X^{*}) \longrightarrow 0,$$

which induces a long exact sequence of Čech cohomology groups. Let's analyze the relevant portion.

$$\check{\mathrm{H}}^1(\mathfrak{U}, 1 + \mathscr{O}_X^{++}) \longrightarrow \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X^{+*}) \longrightarrow \check{\mathrm{H}}^1(\mathfrak{U}, \tilde{\mathscr{O}}_X^{*}) \longrightarrow \check{\mathrm{H}}^2(\mathfrak{U}, 1 + \mathscr{O}_X^{++}).$$

The second term is Pic X by Lemma 7.3. Let's compute the third. We will make use of projective coordinates $[T_0 : \cdots : T_n]$ for $\mathbb{P}^{n,\mathrm{perf}}$, and as above we denote by k the residue field of K. We denote the differentials of the Čech complex $\check{\mathrm{C}}^*(\mathfrak{U}, \tilde{\mathscr{O}}_X)$ by d^i. Note that for all U_i we have

$$\tilde{\mathscr{O}}_X(U_i) = k\left[\left(\frac{T_0}{T_i}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_i}\right)^{1/p^\infty}\right],$$

so that $\tilde{\mathscr{O}}_X^*(U_i) \cong k^*$, since the only invertible polynomials are the constant functions. Therefore we have $C^0(\mathfrak{U}, \tilde{\mathscr{O}}_X^*(U_i)) \cong (k^*)^{n+1}$, and viewing the kernel as the intersection we have $\ker d^0 \cong k^*$, so that $\operatorname{im} d^0 \cong (k^*)^n$.

The rings $\tilde{\mathscr{O}}_X(U_i \cap U_j)$ consist of Laurent polynomials, and the only invertible Laurent polynomials are monomials in the invertible variable. That is,

$$\tilde{\mathscr{O}}_X^*(U_i \cap U_j) = \left\{\lambda\left(\frac{T_i}{T_j}\right)^\alpha : \lambda \in k^*, \alpha \in \mathbb{Z}[1/p]\right\} \cong k^* \oplus \mathbb{Z}[1/p].$$

Let $(f_{ij}) \in C^1(\mathfrak{U}, \tilde{\mathscr{O}}_X^*)$, and suppose $(f_{ij}) \in \ker d^1$. This means that for all $i < j < k$ we have $f_{ij}f_{jk} = f_{ik}$. That is,

$$\lambda_{ij}\left(\frac{T_i}{T_j}\right)^{\alpha_{ij}} \cdot \lambda_{jk}\left(\frac{T_j}{T_k}\right)^{\alpha_{jk}} = \lambda_{ik}\left(\frac{T_i}{T_k}\right)^{\alpha_{ik}}.$$

In particular, $\alpha_{ij} = \alpha_{jk} = \alpha_{ik}$, and so the degree of every factor in an element of the kernel must match. The fact that $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$ leaves n degrees of freedom for the coefficient, so that $\ker d^1 = (k^*)^n \oplus \mathbb{Z}[1/p]$, and since $\operatorname{im} d^0 = (k^*)^n$, we conclude that $\check{\mathrm{H}}^1(\mathfrak{U}, \tilde{\mathscr{O}}_X^*) \cong \mathbb{Z}[1/p]$. Thus the exact sequence above becomes

$$\check{\mathrm{H}}^1(\mathfrak{U}, 1 + \mathscr{O}_X^{++}) \longrightarrow \operatorname{Pic} X \xrightarrow{\varphi} \mathbb{Z}[1/p] \longrightarrow \check{\mathrm{H}}^2(\mathfrak{U}, 1 + \mathscr{O}_X^{++}).$$

We'd like to show that φ is an isomorphism. If we could show that $C^*(\mathfrak{U}, 1 + \mathscr{O}_X^{++})$ is acyclic, we would be done. Unfortunately this is not so easy from the Čech complex alone. Nevertheless, we can begin by constructing a section to φ.

By Corollary 6.17, finite projective $\tilde{T}_n^{\mathrm{perf}}$-modules are free. In particular, for each U_i, we have that invertible $\tilde{\mathscr{O}}_{U_i}$-modules are free. Together with the Čech-to-derived spectral sequence, we can conclude that

$$\mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*) \cong \check{\mathrm{H}}^1(\mathfrak{U}, \tilde{\mathscr{O}}_X^*) \cong \mathbb{Z}[1/p].$$

With this in mind, we look more closely at,

$$\varphi : \operatorname{Pic}(X) \to \mathbb{Z}[1/p].$$

The $\mathbb{Z}[1/p]$ on the right can be interpreted as follows. For each $\alpha \in \mathbb{Z}[1/p]$ we can build an invertible $\tilde{\mathscr{O}}_X$-module starting with $\tilde{\mathscr{O}}_{U_i}$ on each open set in the standard cover, and gluing along transition maps $\left(\frac{T_i}{T_j}\right)^\alpha$ on $U_i \cap U_j$. Doing the same construction, but starting with $\mathscr{O}_{U_i}^+$ gives us a section of φ (call it σ). In particular, φ is surjective and we have an embedding $\mathbb{Z}[1/p] \hookrightarrow \operatorname{Pic}(X)$. Therefore we can consider twisting sheaves $\mathscr{O}(d) \in \operatorname{Pic}(X)$ for every $d \in \mathbb{Z}[1/p]$. Furthermore, our construction implies the following lemma.

Lemma 7.5
Let $A = K\left\langle T_0^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle$. *Then,*

$$\Gamma(U_{i_1} \cap \cdots \cap U_{i_r}, \mathscr{O}(d)) \cong \left(\widehat{A_{T_{i_1}\cdots T_{i_r}}}\right)_d$$

the degree d *part of the completion of the localization of* A.

In the next section we show that σ is surjective.

7.2 The Picard Group of Projectivoid Space

For the first part of this section we let X be a Tate adic space over K with pseudouniformizer ϖ. We make the following standing assumptions.

- $\mathrm{H}^i(X, \mathscr{O}_X^+) = 0$ for all $i > 0$.
- $\mathrm{H}^1(X, \mathscr{O}_X^*) \cong \mathrm{H}^1(X, \mathscr{O}_X^{+*})$.

The first has something to do with deformation theory, whereas the second seems more arithmetic in nature. We will show that projectivoid space satisfies both assumptions. First, let's enumerate a few useful exact sequences.

$$0 \longrightarrow \mathscr{O}_X^{++} \longrightarrow \mathscr{O}_X^{+} \longrightarrow \tilde{\mathscr{O}}_X \longrightarrow 0. \tag{2}$$

Because $\mathscr{O}_X^{++}$ consists of toplogically nilpotent functions, it is contained in the Jacobson radical of $\mathscr{O}_X^+$, so that the right hand map of the sequence remains surjective on unit groups by Lemma 7.4.

$$1 \longrightarrow 1 + \mathscr{O}_X^{++} \longrightarrow \mathscr{O}_X^{+*} \longrightarrow \tilde{\mathscr{O}}_X^* \longrightarrow 1. \tag{3}$$

If $1 + \mathscr{O}_X^{++}$ were acyclic, we could reduce finding line bundles on X to finding invertible $\tilde{\mathscr{O}}_X$ modules, which has in practice been much easier. But this acyclicity has so far been rather elusive (and seems unlikely in general). That being said, there is a filtration of $\mathscr{O}_X^{++}$ by sheaves of (principal) ideals (ϖ^d) for $d > 0$, so let's explore these ideals.

$$0 \longrightarrow (\varpi^d) \longrightarrow \mathscr{O}_X^{+} \longrightarrow \mathscr{A}_d \longrightarrow 0. \tag{4}$$

Notice that $\mathscr{A}_d$ is a sheaf of (nonreduced) $\mathscr{O}_X^+$-algebras, and that for every $d' > d$, we have surjections $\mathscr{A}_{d'} \twoheadrightarrow \mathscr{A}_d$ with kernel $(\varpi^d)/(\varpi^{d'})$. As before, (ϖ^d) is contained in the Jacobson radical, so that we also have

$$1 \longrightarrow 1 + (\varpi^d) \longrightarrow \mathscr{O}_X^{+*} \longrightarrow \mathscr{A}_d^* \longrightarrow 1. \tag{5}$$

Notice that $\varinjlim(1 + (\varpi^d)) = \cup_d(1 + (\varpi^d)) = 1 + \mathscr{O}_X^{++}$. Since colimits are exact on sheaves abelian groups, this implies

$$\varinjlim \mathscr{A}_d^* \cong \tilde{\mathscr{O}}_X^*.$$

On the other hand, $\varprojlim(\mathscr{A}_d) \cong \mathscr{O}_X^+$ since $\mathscr{O}_X^+$ is ϖ-adically complete. Since the unit group functor commutes with inverse limits (indeed, it is left adjoint to the group ring functor), we have

$$\varprojlim \mathscr{A}_d^* \cong \mathscr{O}_X^{+*}.$$

Lemma 7.6

For all $i > 0$ and $d > 0$, $\mathrm{H}^i(X, \mathscr{A}_d) = 0$.

Proof. This follows from the long exact sequence on cohomology associated to Sequence 4 and the first standing assumption (about the cohomology of $\mathscr{O}_X^+$), noticing that $(\varpi^d) \cong \mathscr{O}_X^+$ since it is a principal ideal.

Lemma 7.7

For all $d, i > 0$, the natural map

$$\mathrm{H}^i(X, \mathscr{A}_{2d}^*) \longrightarrow \mathrm{H}^i(X, \mathscr{A}_d^*)$$

is an isomorphism.

Proof. Consider:

$$\begin{tikzcd}
0 \arrow[r] & 1 + (\varpi^{2d}) \arrow[r] \arrow[d, hook] & \mathscr{O}_X^{+*} \arrow[r] \arrow[d, equal] & \mathscr{A}_{2d} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & 1 + (\varpi^{d}) \arrow[r] & \mathscr{O}_X^{+*} \arrow[r] & \mathscr{A}_{d} \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma, we have

$$1 \longrightarrow 1 + (\varpi^d)/(\varpi^{2d}) \longrightarrow \mathscr{A}_{2d}^* \longrightarrow \mathscr{A}_d^* \longrightarrow 1. \tag{6}$$

Notice that $1 + (\varpi^d)/(\varpi^{2d})$ has a natural $\mathscr{A}_d$-module structure making it isomorphic to $\mathscr{A}_d$, given locally by the map $1 \mapsto \varpi^d$. Indeed, the map is well defined because $(\varpi^d)/(\varpi^{2d})$ is a square zero ideal, and the kernel is precisely (ϖ^d) (which is 0 in $\mathscr{A}_d$), while surjectivity is clear. In particular, by Lemma 7.6, $1 + (\varpi^d)/(\varpi^{2d})$ has no higher cohomology, so the conclusion follows from the long exact sequence on cohomology on Sequence 6.

Lemma 7.8

For all $d' > d > 0$ and $i > 0$, the natural map:

$$\mathrm{H}^i(X, \mathscr{A}_{d'}^*) \longrightarrow \mathrm{H}^i(X, \mathscr{A}_d^*),$$

is an isomorphism.

Proof. By Lemma 7.7, replacing d with $2^l d$ will not change cohomology, so we may assume $d < d' < 2d$. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & \mathrm{H}^i(X, \mathscr{A}_{2d}^*) \arrow[rr, "\sim"] \arrow[dr, "\psi"] & & \mathrm{H}^i(X, \mathscr{A}_d^*) \\
\mathrm{H}^i(X, \mathscr{A}_{2d'}^*) \arrow[ur] \arrow[rr, "\sim"] & & \mathrm{H}^i(X, \mathscr{A}_{d'}^*) \arrow[ur] &
\end{tikzcd}$$

In particular, ψ is injective and surjective, so an isomorphism, which implies the result.

Therefore we have the following sequence of morphisms, whose composition is the map φ from Section 7.1. We leverage the fact that colimits of abelian sheaves are exact.

$$\begin{array}{rcl}
\mathrm{Pic}(X) & \cong & \mathrm{H}^1(X, \mathscr{O}_X^*) \\
 & \cong & \mathrm{H}^1(X, \mathscr{O}_X^{+*}) \\
 & \cong & \mathrm{H}^1(X, \varprojlim \mathscr{A}_d^*) \\
 & \longrightarrow & \varprojlim \mathrm{H}^1(X, \mathscr{A}_d^*) \\
 & \cong & \mathrm{H}^1(X, \mathscr{A}_d^*) \\
 & \cong & \varinjlim \mathrm{H}^1(X, \mathscr{A}_d^*) \\
 & \cong & \mathrm{H}^1(X, \varinjlim \mathscr{A}_d^*) \\
 & \cong & \mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*).
\end{array}$$

Now let us specialize to the case where $X = \mathbb{P}^{n,\mathrm{perf}}$, so that φ has a section σ (defined at the end of Section 7.1).

Lemma 7.9

$X = \mathbb{P}^{n,\mathrm{perf}}$ satisfies both the standing assumptions from the beginning of this section. Explicitly:

- $\mathrm{H}^i(X, \mathscr{O}_X^+) = 0$ *for all $i > 0$.*
- $\mathrm{H}^1(X, \mathscr{O}_X^*) \cong \mathrm{H}^1(X, \mathscr{O}_X^{+*})$.

Proof. The first statement will be proven in the next section (see Remark 7.12). The second follows from Lemma 7.3, noticing that Čech cohomology with the standard cover is effective due to the Čech-to-derived spectral sequence (Theorem 7.1) together with Corollary 6.21.

We make the necessary identifications to view the map φ and its section σ as maps between the following groups.

$$\begin{tikzcd}
\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \arrow[r, "\varphi"] & \varprojlim \mathrm{H}^1(X, \mathscr{A}_d^*) \arrow[l, bend right, dashed, "\sigma"']
\end{tikzcd}$$

We can view the first group as isomorphism classes of invertible $\mathscr{O}_X^+$ modules, and the second as inverse systems of isomorphism classes of invertible $\mathscr{A}_d$ modules. Under these identifications we have:

$$\begin{aligned}
\varphi : &\quad \mathscr{L} \mapsto \{\mathscr{L}/\varpi^d \mathscr{L}\} \\
\sigma : &\quad \{\mathscr{M}_d\} \mapsto \varprojlim \mathscr{M}_d.
\end{aligned}$$

In particular, there is a natural map

$$\mathscr{L} \longrightarrow \varprojlim \mathscr{L}/\varpi^d \mathscr{L} = \sigma\varphi\mathscr{L}.$$

Locally, on an affinoid $\mathrm{Spa}(R, R^+)$, we associate $\mathscr{L}$ to an invertible R-module M. Then this map becomes,

$$M \longrightarrow \varprojlim M/\varpi^d M \cong \hat{M},$$

which is an isomorphism since M is already complete. We conclude that σ is surjective, and therefore an isomorphism. Putting all this together, we have proved the following theorem.

Theorem 7.10

$\mathrm{Pic}\, \mathbb{P}^{n,\mathrm{perf}} \cong \mathbb{Z}[1/p]$.

7.3 Cohomology of Line Bundles

In their Ph.D. thesis [2], Harpreet Bedi computed the cohomology of some of the twisting sheaves $\mathscr{O}(d)$ on projectivoid space. His proof was modeled on the computation for classical projective space in Ravi Vakil's algebraic geometry notes ([35] Chapter 18.2), but doesn't explicitly take into account the completions involved, and only includes the case for $n = 2$. Instead, we adapt the proof from EGA III [13] which relates Čech cohomology to the Koszul complex. We also fill in the result, computing the cohomology of every line bundle in every degree. Our rings are going to be $\mathbb{Z}[1/p]$ graded, and for a graded ring A we will denote by A_d the degree d part.

Theorem 7.11

Let $X = \mathbb{P}^{n,\mathrm{perf}}$ be projectivoid space, and $\mathscr{O}_X(d) \in \mathrm{Pic}\, X$ an arbitrary line bundle. Then:

1. *If $d \geq 0$,*
$$\mathrm{H}^0\left(X, \mathscr{O}_X(d)\right) = K\left\langle T_0^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle_d.$$

2. *If $d < 0$, then $\mathrm{H}^n\left(X, \mathscr{O}_X(d)\right)$ is the completion of the K vector space generated by monomials of degree d, where the degree of each indeterminate is strictly negative, that is:*
$$\mathrm{H}^n\left(X, \mathscr{O}_X(d)\right) = \left\langle T_0^{\alpha_0} \cdots T_n^{\alpha_n} \;\middle|\; \alpha_i \in \mathbb{Z}[1/p]_{<0} \text{ and } \sum \alpha_i = d\right\rangle^{\wedge}$$

3. *In all other cases,*
$$\mathrm{H}^r\left(X, \mathscr{O}_X(d)\right) = 0.$$

In particular:

$$h^r\left(X, \mathscr{O}_X(d)\right) = \begin{cases} 1 & r = d = 0 \\ \infty & r = 0 \text{ and } d > 0 \\ \infty & r = n \text{ and } d < 0 \\ 0 & \text{all other cases} \end{cases}.$$

PROOF. We will leverage that colimits of abelian groups are exact, so that cohomology of finite complexes of abelian groups commute with arbitrary direct sums. We will therefore study the Čech sequence associated to the sheaf

$$\mathscr{H} = \bigoplus_{d \in \mathbb{Z}[1/p]} \mathscr{O}_X(d).$$

Let $A = K\left\langle T_0^{1/p^\infty}, \cdots, T_n^{1/p^\infty}\right\rangle$, and $\mathfrak{U} = \{U_i\}$ be the standard cover of X. Then by Lemma 7.5 we have

$$\check{\mathrm{C}}^r(\mathfrak{U}, \mathscr{H}) = \prod_{0 \le i_1 < \cdots < i_r \le n} \widehat{A_{T_{i_1} \cdots T_{i_r}}}.$$

Since the differentials commute with degree and cohomology commutes with direct sums, we can conclude that

$$\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X(d)) = \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{H})_d.$$

Furthermore, all finite intersections of the U_i are affinoid, and vector bundles on affinoids are acyclic ([19] Theorem 1.4.2). Then by Theorem 7.1, Čech cohomology computes the sheaf cohomology. We first consider the 0th cohomology.

$$\check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{H}) = \bigcap_{i=0}^{n} \widehat{A_{T_i}} = A.$$

This proves the first statement of the theorem. For the second, we consider the sequence

$$C^*(A): \quad 0 \longrightarrow \prod A_{T_i} \longrightarrow \prod A_{T_i T_j} \longrightarrow \cdots \longrightarrow A_{T_0 \cdots T_n} \longrightarrow 0.$$

In each case, there is a countable fundamental system of neighborhoods of zero, given by (ϖ^n), so that proceeding by induction from the left to the right, we see that completion on this sequence commutes with taking cohomology (see, for example, [34] tag 0AMQ). In particular,

$$\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{H}) = \mathrm{H}^i\left(\widehat{C^*(A)}\right).$$

Let's analyze $H^n(C^*(A))$. Notice that $A_{T_0 \cdots T_n}$ is the K vector space generated by monomials $T_0^{\alpha_0} \cdots T_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}[1/p]$. The image of the $(n-1)$st differential is the K vector space generated by monomials where at least one of the $\alpha_i \geq 0$. Therefore $H^n(C^*(A))$, which is the cokernel of this differential, is the K vector space generated by monomials where each $\alpha_i < 0$. Taking completions proves the second statement of the theorem.

For the third statement of the theorem, the cases of $r < 0$ and $r > n$ are trivial, so we assume $0 < r < n$. We will show that $H^r(C^*(A)) = 0$ since the completion of 0 is 0. We point out that for any $f \in A$,

$$A_f \cong \varinjlim \left(A \xrightarrow{\cdot f} A \xrightarrow{\cdot f} \cdots\right).$$

For all $s \geq 0$, let $T^s = (T_0^s, \cdots, T_n^s)$. Then T^s is an A-regular sequence, and the associated Koszul complex $K^*(T^s)$ is a free resolution of $A/(T_0^s, \cdots, T_n^s)$.

$$K^*(T^s): \quad 0 \longrightarrow \Lambda^{n+1} A^{n+1} \longrightarrow \cdots \longrightarrow \Lambda^2 A^{n+1} \longrightarrow A^{n+1} \xrightarrow{(T_0^s, \cdots, T_n^s)} A \longrightarrow 0.$$

In particular, the homology groups $\mathrm{H}_i(K^*(T^s)) = 0$ for all $i > 0$. For each s we can also look at the dual Koszul complex, and take the colimit as s goes to infinity, with the above identification to the localization in mind.

$$\begin{tikzcd}
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
C_s : & 0 \arrow[r] & A \arrow[r, "T^s"] \arrow[d] & A^{n+1} \arrow[r, "\cdot\wedge T^s"] \arrow[d, "\cdot T"] & \Lambda^2 A^{n+1} \arrow[r] \arrow[d, "\cdot(T\wedge T)"] & \cdots \arrow[r] & \Lambda^{n+1}A^{n+1} \arrow[r] \arrow[d, "\cdot T^{\wedge(n+1)}"] & 0 \\
C_{s+1} : & 0 \arrow[r] & A \arrow[r, "T^{s+1}"] \arrow[d] & A^{n+1} \arrow[r, "\cdot\wedge T^{s+1}"] \arrow[d] & \Lambda^2 A^{n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Lambda^{n+1}A^{n+1} \arrow[r] \arrow[d] & 0 \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\varinjlim C_s : & 0 \arrow[r] & A \arrow[r] & \prod A_{T_i} \arrow[r] & \prod A_{T_iT_j} \arrow[r] & \cdots \arrow[r] & A_{T_0\cdots T_n} \arrow[r] & 0
\end{tikzcd}$$

For $i > 0$, the bottom row is $C^*(A)[1]$. By the self-duality of the Koszul complex, we have that

$$\mathrm{H}^i(C_s) \cong \mathrm{H}_{n+1-i}(K^*(T^s)),$$

so that in particular, for $i < n+1$, we have $H^i(C_s) = 0$. Since colimits of finite complexes commute with cohomology, we conclude that for $0 < r < n$,

$$\begin{aligned}
\mathrm{H}^r(C^*(A)) &\cong \mathrm{H}^{r+1}(\varinjlim C_s) \\
&\cong \varinjlim \mathrm{H}^{r+1}(C_s) \\
&= \varinjlim 0 \\
&= 0.
\end{aligned}$$

Taking completions proves the third statement of the theorem, and so we are done.

Remark 7.12

An identical argument, but replacing K with K°, computes the cohomology of the integral line bundles $\mathscr{O}_X^+(d)$ for all $d > 0$. In particular we see that $\mathscr{O}_X^+$ is acyclic, which is what we need for projectivoid space to meet the first standing assumption in Section 7.2, thus we complete the proof of Lemma 7.9.

7.3.1 Koszul-to-Čech: The Details

We made some identifications in order to have the sequences of Koszul complexes converge to the (shifted) Čech complexes. To be safe, we make these identifications explicit and check that all the diagrams commute as asserted above.

As a first step, let's make explicit the identification

$$\varinjlim(A \xrightarrow{f} A \xrightarrow{f} A \xrightarrow{f} \cdots) \xrightarrow{\sim} A_f.$$

In fact, this identification works replacing A with any A-module. We get a map out of the colimit via the (module) homomorphisms $a \mapsto a/f^r$ from the rth factor, and this map is clearly surjective. Injectivity follows because $a/f^r = 0$ implies $f^s \cdot a = 0$ for some s so that a maps to 0 in the colimit.

For the following, we fix a basis $e_0, \cdots, e_n$ for A^{n+1}. Then $\Lambda^k A^{n+1}$ is the free module generated by $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 0 \le i_1 < \cdots < i_k \le n\}$. Then the k-th differential of the Koszul complex,

$$\partial_s^k : \Lambda^k A^{n+1} \xrightarrow{\wedge T^s} \Lambda^{k+1} A^{n+1},$$

associated to the regular sequence T^s is given coordinate wise by

$$(f_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}) \mapsto \left(\sum_j T_{i_j}^s f_{i_1 \cdots \hat{i_j} \cdots i_{k+1}} \right) e_{i_1} \wedge \cdots \wedge e_{i_{k+1}}.$$

The vertical maps, $T^{\wedge k} : \Lambda^k A^{n+1} \longrightarrow \Lambda^k A^{n+1}$ from the Koszul complex associated to T^s to the one given by T^{s+1} are given coordinate wise by multiplication,

$$f_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto T_{i_1} \cdots T_{i_k} f_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Finally, identifying the colimit of the multiplication by $T_{i_1} \cdots T_{i_k}$ maps with the localization $A_{T_{i_1} \cdots T_{i_k}}$, the map $\Lambda^k A^{n+1} \longrightarrow \prod A_{T_{i_1} \cdots T_{i_k}}$ from the Koszul complex associated to T^s to the colimit is given coordinatewise by

$$f_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \frac{f_{i_1 \cdots i_k}}{(T_{i_1} \cdots T_{i_k})^s}.$$

It is now an exercise in careful book keeping to check that the following diagram commutes, which completes the result.

$$\begin{tikzcd}
\Lambda^k A^{n+1} \arrow[r, "\cdot\wedge T^s"] \arrow[d, "\cdot T^{\wedge k}"] \arrow[dd, bend right] & \Lambda^{k+1} A^{n+1} \arrow[d, "\cdot T^{\wedge(k+1)}"'] \arrow[dd, bend left, "\cdot(T_{i_1}\cdots T_{i_{k+1}})^{-s}"] \\
\Lambda^k A^{n+1} \arrow[r, "\cdot\wedge T^{s+1}"] \arrow[d, "\cdot(T_{i_1}\cdots T_{i_k})^{-s-1}"] & \Lambda^{k+1} A^{n+1} \arrow[d] \\
\prod A_{T_{i_1}\cdots T_{i_k}} \arrow[r, "\partial"] & \prod A_{T_{i_1}\cdots T_{i_{k+1}}}.
\end{tikzcd}$$

Let's start with the top block, and see what happens going from the top left corner to the middle left side coordinatewise. Going right and then down is

$$(f_{i_1\cdots i_k}) \mapsto \left(\sum_j T_{i_j}^s f_{i_1 \cdots \hat{i_j} \cdots i_{k+1}} \right) \mapsto \left(T_{i_1} \cdots T_{i_{k+1}} \sum_j T_{i_j}^s f_{i_1 \cdots \hat{i_j} \cdots i_{k+1}} \right).$$

Meanwhile, down and then to the right is

$$(f_{i_1\cdots i_k}) \mapsto (T_{i_1} \cdots T_{i_k} f_{i_1 \cdots i_k}) \mapsto \left(\sum_j (T_{i_1} \cdots \widehat{T_{i_j}} \cdots T_{i_{k+1}}) T_{i_j}^{s+1} f_{i_1 \cdots \hat{i_j} \cdots i_{k+1}} \right).$$

Therefore the upper half of diagram commutes. For the bottom half, we relabel $s+1$ as s to conserve notational energy. Chasing as before, first to the right and down.

$$(f_{i_1\cdots i_k}) \mapsto \left(\sum_j T_{i_j}^s f_{i_1 \cdots \hat{i_j} \cdots i_{k+1}} \right) \mapsto \left(\sum_j \frac{f_{i_1} \cdots \hat{i_j} \cdots i_{k+1}}{T_{i_1}^s \cdots \widehat{T_{i_j}^s} \cdots T_{i_{k+1}}^s} \right).$$

The bottom horizontal map is the Čech differential, so that going down and then to the right gives

$$(f_{i_1\cdots i_k}) \mapsto \left(\frac{f_{i_1 \cdots i_k}}{T_{i_1}^s \cdots T_{i_k}^s} \right) \mapsto \left(\sum_j \frac{f_{i_1} \cdots \hat{i_j} \cdots i_{k+1}}{T_{i_1}^s \cdots \widehat{T_{i_j}^s} \cdots T_{i_{k+1}}^s} \right).$$

8 Maps to Projectivoid Space

Suppose S is a scheme over K. Then there is a well known correspondence between maps from $S \to \mathbb{P}^n$ and globally generated line bundles on S and together with a choice of $n+1$ generating global sections (see for example [14] Theorem II.7.1). In this chaopter we will prove an analog of this correspondence for perfectoid spaces.

Definition 8.1. To a perfectoid space X over K, we associate a groupoid $\mathfrak{L}_n(X)$ whose objects consist of tuples $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right)$ for $i \geq 0$ and $j = 0, \cdots, n$, where $\mathscr{L}_i$ are line bundles on X, $s_0^{(i)}, \cdots, s_n^{(i)} \in \Gamma(X, \mathscr{L}_i)$ are generating global sections, and $\alpha_i : \mathscr{L}_{i+1}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_i$ are isomorphisms mapping $\left(s_j^{(i+1)}\right)^{\otimes p} \mapsto s_j^{(i)}$. Morphisms are isomorphisms of line bundles which are compatible with the global sections and isomorphisms α_i.

If $f : X \to Y$ is a K-morphism, we get a pullback functor $f^* : \mathfrak{L}_n(Y) \to \mathfrak{L}_n(X)$, so that $\mathfrak{L}_n$ is a category fibered in groupoids.

Remark 8.2
Under a suitable Grothendieck topology on the category of perfectoid spaces over K, we could view $\mathfrak{L}_n$ as a stack. We will show that $\mathfrak{L}_n$ is actually representable by a perfectoid space.

Remark 8.3
Note that if some α_i exists, it is unique. Indeed, for each i the global sections $s_j^{(i)}$ generate $\mathscr{L}_i$, so that an isomorphism $\mathscr{L}_{i+1}^{\otimes p} \longrightarrow \mathscr{L}_i$ shows that the global sections $\left(\sigma_j^{(i+1)}\right)^{\otimes p}$ generate $\mathscr{L}_{i+1}^{\otimes p}$. In particular, the isomorphism is completely determined by the images of these global sections.

Remark 8.4
For each i, the data $\left(\mathscr{L}_i, s_j^{(i)}\right)$ corresponds to a map to a projective space (as a rigid analytic variety), so that in positive characteristic objects of the category $\mathfrak{L}_n(X)$ correspond to Frobenius compatible systems of maps to projective space.

The main result of this section is that the category $\mathfrak{L}_n(X)$ parametrizes K-morphisms $X \longrightarrow \mathbb{P}^{n,\mathrm{perf}}$. In particular, viewing $\mathfrak{L}_n$ as a functor to sets we construct a natural isomorphism $\mathrm{Hom}(\cdot, \mathbb{P}^{n,\mathrm{perf}}) \cong \mathfrak{L}_n$ of functors from perfectoid spaces over K to sets. First we introduce a bit of notation.

Definition 8.5. Denote by $m_i : \mathscr{O}(1/p^{i+1})^{\otimes p} \xrightarrow{\sim} \mathscr{O}(1/p^i)$ the isomorphism of line bundles on $\mathbb{P}^{n,\mathrm{perf}}$ coming from multiplying factors together.

We now state the main theorem of this section (compare to [14] Theorem II.7.1).

Theorem 8.6
The functor $\mathfrak{L}_n$ is represented by projectivoid space.

Explicitly, the natural transformation $\mathrm{Hom}(\cdot, \mathbb{P}^{n,\mathrm{perf}}) \to \mathfrak{L}_n$, *which evaluated on X takes $\phi : X \to$ $\mathbb{P}^{n,\mathrm{perf}}$ to the tuple $\left(\phi^*\mathscr{O}(1/p^i), \phi^* T_j^{1/p^i}, \phi^* m_i\right) \in \mathfrak{L}_n(X)$ is an isomorphism of functors.*

Since $\left\{T_j^{1/p^i}\right\}_{j=0}^n$ generates $\mathscr{O}(1/p^i)$, we have that $\left\{\phi^*\left(T_j^{1/p^i}\right)\right\}_{j=0}^n$ generates $\phi^*\left(\mathscr{O}(1/p^i)\right)$. Furthermore, the standard isomorphisms $\mathscr{O}(1/p^{i+1})^{\otimes p} \xrightarrow{\sim} \mathscr{O}(1/p^i)$ coming from multiplying factors together send $(T_j^{1/p^{i+1}})^{\otimes p}$ to T_j^{1/p^i}, so pulling back these isomorphisms along ϕ gives us an element of $\mathfrak{L}_n(X)$. We construct an inverse to this transformation in Section 8.2, but first we will need a bit of setup.

8.1 $\mathscr{L}$-Distinguished Open Sets

For this section we let X be an adic space, $\mathscr{L}$ a line bundle on X, and $s_1, \cdots, s_n$ global sections of $\mathscr{L}$ which generate it at every point. Let $D(s_i) = \{x \in X : s_i|_x \text{ generates } \mathscr{L}_x\}$ be the *doesn't vanish* set of the section s_i. Then the map $\mathscr{O}_X \to \mathscr{L}$ determined by s_i is an isomoprhism on the stalks of every point of $D(s_i)$, and therefore restricts to an isomorphism on it. We suggestively denote the inverse by $s \mapsto s/s_i$. Let's validate this notation with the following lemma.

Lemma 8.7
On $D(s_i) \cap D(s_j)$, we have the following relation.

$$\frac{s_i}{s_j} \cdot \frac{s_j}{s_i} = 1.$$

Proof. We have two isomorphisms,

$$\begin{tikzcd}\Gamma(D(s_i) \cap D(s_j), \mathscr{O}_X) \arrow[r, shift left, "s_i"] \arrow[r, shift right, "s_j"'] & \Gamma(D(s_i) \cap D(s_j), \mathscr{L}).\end{tikzcd}$$

Then we have

$$\frac{s_i}{s_j} = s_j^{-1} \circ s_i(1)$$
$$\frac{s_j}{s_i} = s_i^{-1} \circ s_j(1).$$

Since the maps $s_i^{-1} \circ s_j$ and $s_j^{-1} \circ s_i$ are inverses to each other, we win.

For every $x \in D(s_i)$, we can use the isomorphism s_i^{-1} to get a valuation on $\Gamma(X, \mathscr{L})$.

$$\begin{tikzcd}\Gamma(X, \mathscr{L}) \arrow[r, "res"] & \Gamma(D(s_i), \mathscr{L}) \arrow[r, "s_i^{-1}"] & \Gamma(D(s_i), \mathscr{O}_X) \arrow[r, "x"] & \Gamma_x \cup \{0\} \\ s \arrow[rrr, mapsto] & & & |(s/s_i)(x)|\end{tikzcd}$$

With this in hand, we can define the following open subsets of $D(s_i)$ for each i.

Definition 8.8. Let X be a perfectoid space, $\mathscr{L}$ a line bundle on X and $s_1, \cdots, s_n$ generating global sections of $\mathscr{L}$. An open set of X is called an *$\mathscr{L}$-distinguished open set* if it is of the form

$$X\left(\frac{s_1, \cdots, s_n}{s_i}\right) = \{x \in D(s_i) : |(s_j/s_i)(x)| \leq 1 \text{ for all } j\}.$$

For the case of classical projective space, we can build a map to projective space along the *doesn't vanish* sets of the given sections, and glue them together. Here with our analytic topology, we must use these smaller $\mathscr{L}$-distinguished open sets. Let's prove these smaller open sets cover X. Indeed, our notation suggests that one of $|(s_j/s_i)(x)|$ or $|(s_i/s_j)(x)|$ should be less than 1, let's check the details.

Lemma 8.9
The $\mathscr{L}$-distinguished open sets $X_i = X\left(\frac{s_1, \cdots, s_n}{s_i}\right)$ for $i = 1, \cdots, n$ are open and cover X.

Proof. The openness of X_i follows because it is in fact a *rational* open in the adic space $D(s_i)$, which is open in X. To show these cover X, fix some $x \in X$. We already know the $D(s_i)$ cover X, because the s_i generate $\mathscr{L}$. So $x \in D(s_i)$ for some i. Fix some j. If $x \notin D(s_j)$ then $|(s_j/s_i)(x)| = 0 < 1$. Suppose otherwise that $x \in D(s_j)$. By Lemma 8.7 together with the multiplicativity of the valuation given by x, we have either $|(s_i/s_j)(x)| \leq 1$ or $|(s_j/s_i)(x)| \leq 1$. If the former holds for every j, then $x \in X_i$.

Suppose the latter holds for some j, and suppose further that $x \in D(s_k)$ for some other k. Then arguing as above we have $|(s_i/s_k)(x)| \leq 1$ or $|(s_k/s_i)(x)| \leq 1$. If the first case holds, then also $|(s_j/s_k)(x)| \leq 1$. Indeed,

$$|(s_i/s_k)(x)| = |(s_i/s_j)(x)| \cdot |(s_j/s_k)(x)| \leq 1,$$

so that

$$|(s_j/s_k)(x)| \leq |(s_j/s_i)(x)| \leq 1.$$

In this case, what remains is to compare s_j to the rest of the sections excluding i. We continue in this way going through each section we will find some r such that for all l, $|(s_l/s_r)(x)| \leq 1$, and so $x \in X_r$.

Example 8.10

The standard cover of $\mathbb{P}^{n,\text{perf}}$ by perfectoid unit disks consists of the $\mathscr{O}(1)$-distinguished open sets $\mathbb{P}^{n,\text{perf}}\left(\frac{T_0,\cdots,T_n}{T_i}\right)$.

We finally prove a lemma which implies that if $\mathscr{L}^{\otimes p} \cong \mathscr{M}$ and s and t are global sections of $\mathscr{L}$ and $\mathscr{M}$ respectively, with $s^{\otimes p} = t$, then $D(s) = D(t)$. In particular, using the notation of Definition 8.1, this implies that if the $s_j^{(i)}$ generate $\mathscr{L}_i$, then the $s_j^{(i+1)}$ generate $\mathscr{L}_{i+1}$.

Lemma 8.11

Let $(R, \mathfrak{m})$ be a local ring, and M, N invertible R-modules such that $M^{\otimes r} \cong N$ for a positive integer r. Let $f \in M$, and $g \in N$ such that under this identification $f^{\otimes r} = g$. Then if g generates N, f generates M.

Proof. We show the contrapositive. If f does not generate M, then by Nakayama's lemma, $f \in \mathfrak{m}M$. Thus $f = a \cdot s$ for some $a \in \mathfrak{m}$ and $s \in M$. But then under the appropriate identification,

$$g = f^{\otimes r} = (a \cdot s)^{\otimes r} = a^r \cdot s^{\otimes r} \in \mathfrak{m}^r N \subseteq \mathfrak{m}N.$$

Therefore g cannot generate N.

8.2 Construction of the Projectivoid Morphism

We can now finish the proof of Theorem 8.6 by constructing an inverse to the natural tranformation from the theorem. The result follows from the following proposition.

Proposition 8.12

Let X be a perfectoid space over K and $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_n(X)$. There is a unique K-morphism $\phi : X \longrightarrow \mathbb{P}^{n,\text{perf}}$ such that

$$\left(\phi^*\mathscr{O}(1/p^i), \phi^* T_j^{1/p^i}, \phi^* m_i\right) \cong \left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right).$$

Proof. Let $X_j = X\left(\frac{s_0^{(0)},\cdots,s_n^{(0)}}{s_j^{(0)}}\right)$ be the cover of X by $\mathscr{L}_0$-distinguished opens. Let $U_j = \mathbb{P}^{n,\text{perf}}\left(\frac{T_0,\cdots,T_n}{T_j}\right) \subseteq \mathbb{P}^{n,perf}$ be the standard cover by affinoids, which are isomorphic to the perfectoid unit polydisk, given by

$$\text{Spa}\left(K\left\langle\left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty}\right\rangle, K^\circ\left\langle\left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty}\right\rangle\right).$$

We build ϕ locally from maps $\phi_j : X_j \to U_j$. Since U_j is affinoid, Proposition 3.43 implies that it is equivalent to build a map of Huber pairs,

$$\left(K\left\langle\left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty}\right\rangle, K^\circ\left\langle\left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty}\right\rangle\right) \xrightarrow{\gamma_j} \left(\mathscr{O}_X(X_j), \mathscr{O}_X^+(X_j)\right).$$

That is, a ring map

$$K\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle \xrightarrow{\gamma_j} \Gamma(X_j, \mathscr{O}_X),$$

satisfying

$$\gamma_j\left(K^\circ\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle\right) \subseteq \Gamma\left(X_j, \mathscr{O}_X^+\right).$$

We define γ_j on generators by the rule

$$\gamma_j\left(\left(\frac{T_r}{T_j}\right)^{1/p^i}\right) = \frac{s_r^{(i)}}{s_j^{(i)}}.$$

To make sure this actually gives a homomorphism, we must check that

$$\left(\frac{s_r^{(i+1)}}{s_j^{(i+1)}}\right)^p = \frac{s_r^{(i)}}{s_j^{(i)}}.$$

First notice that, under the identification $\alpha_i : \mathscr{L}_{i+1}^{\otimes p} \cong \mathscr{L}_i$, the following diagram commutes (keeping in mind that the horizontal maps are not homomorphisms).

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r, "x \mapsto x^p"] \arrow[d, "s_j^{(i+1)}"] & \mathscr{O}_X \arrow[d, "s_j^{(i)}"] \\
\mathscr{L}_{i+1} \arrow[r, "s \mapsto s^{\otimes p}"] & \mathscr{L}_i.
\end{tikzcd}$$

Indeed, the commutativity of this diagram follows directly from the multilinearity of tensor product together with the identification $\left(s_j^{(i+1)}\right)^{\otimes p} = s_j^{(i)}$. Chasing this diagram, we see that

$$\begin{aligned}
\left(\frac{s_r^{(i+1)}}{s_j^{(i+1)}}\right)^p &= \left(\left(s_j^{(i+1)}\right)^{-1}\left(s_r^{(i+1)}\right)\right)^p \\
&= \left(s_j^{(i)}\right)^{-1}\left(s_r^{(i)}\right) \\
&= \frac{s_r^{(i)}}{s_j^{(i)}},
\end{aligned}$$

as desired. Therefore γ_j is a homomorphism. Finally, the definition of X_j implies that for all $x \in X_j$,

$$\left|\gamma_j\left(\frac{T_i}{T_j}\right)(x)\right| = \left|\frac{s_i^{(0)}}{s_j^{(0)}}(x)\right| \leq 1,$$

so that

$$\gamma_j\left(\frac{T_i}{T_j}\right) \in \Gamma\left(X_j, \mathscr{O}_X^+\right).$$

The multiplicativity of the valuation associated to x shows the same holds for all pth power roots so that

$$\gamma_j\left(K^\circ\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \cdots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle\right) \subseteq \Gamma\left(X_j, \mathscr{O}_X^+\right).$$

Therefore we get a morphism $\phi_j : X_j \to U_j \subseteq \mathbb{P}^{n,\mathrm{perf}}$, for each j. Notice that this diagram chase also says that $s_r^{(i)}/s_j^{(i)}$ is a p^i-th root of $s_r^{(0)}/s_j^{(0)}$ which is what we'd like.

To get the map ϕ, we must check that these morphisms glue. This seems obvious with the notation we've selected, but let's be more careful. We must show that the restrictions of γ_j and γ_k are equal as maps from $\Gamma(U_j \cap U_k, \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}}) \longrightarrow \Gamma(X_j \cap X_k, \mathscr{O}_X)$. Therefore it suffices to show that,

$$\gamma_j\left(\left(\frac{T_k}{T_j}\right)^{1/p^i}\right) = \gamma_k\left(\left(\frac{T_j}{T_k}\right)^{1/p^i}\right)^{-1}.$$

With our notation, this boils down to

$$\frac{s_k^{(i)}}{s_j^{(i)}} \cdot \frac{s_j^{(i)}}{s_k^{(i)}} = 1.$$

But this is just Lemma 8.7.

The rest is immediate from the construction. Since $\mathscr{O}_{\mathbb{P}^n\mathrm{perf}}(d)$ is generated by the monomials of degree d, the construction shows that $\phi^*\mathscr{O}(1/p^i) = \mathscr{L}_i$, and that $\phi^*\left(T_j^{1/p^i}\right) = s_j^{(i)}$. Furthermore, any map $\psi : X \to \mathbb{P}^{n,\mathrm{perf}}$ with these properties must locally be given by the γ_i (composed with $s_j^{(i)}$), so that $\psi = \phi$. In fact, this could be an interpretation of the precise meaning of the data given by an element of $\mathfrak{L}_n(X)$. Indeed, the γ_i can be viewed as descent data for the $\mathscr{L}_i$ as the pullback of $\mathscr{O}(1/p^i)$.

8.3 The Positive Characteristic Case

If X is a perfectoid space of characteristic p, then the Frobenius morphism $F : \mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^p$ is an isomorphism. Therefore the pth power map on $\operatorname{Pic} X$ is an isomorphism as well, since it is $H^1(X, F)$. This means that given $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_n(X)$, the $\mathscr{L}_i$ for $i > 0$ are uniquely determined by $\mathscr{L}_0$. Similarly, since X is perfect, the map γ_i constructed in the proof of Proposition 8.2 is completely determined by where T_r/T_i goes for each $r \neq j$, because the pth roots of the image are unique. We summarize this in the following corollary.

Corollary 8.13

If X is a perfectoid space over K of characteristic p, a map $X \to \mathbb{P}^{n,\mathrm{perf}}$ is equivalent to a line bundle $\mathscr{L}$ on X and global sections $s_0, \cdots, s_n$ that generate $\mathscr{L}$, or equivalently, to a map to classical projective space $\mathbb{P}^n$ (where $\mathbb{P}^n$ can be viewed as an adic space as in Example 3.47).

We can now leverage the tilting equivalence to say that maps to $X \to \mathbb{P}^{n,\mathrm{perf}}$ in any characteristic are governed by single line bundles on $X^\flat$. Indeed, by the tilting equivalence (Theorem 4.11), we have that $\operatorname{Hom}(X, \mathbb{P}_K^{n,\mathrm{perf}}) = \operatorname{Hom}(X^\flat, \mathbb{P}_{K^\flat}^{n,\mathrm{perf}})$. This implies the following corollary to Theorem 8.6.

Corollary 8.14

If X is a perfectoid space over K of any characteristic, a map $X \to \mathbb{P}_K^{n,\mathrm{perf}}$ is equivalent to a single line bundle $\mathscr{L}$ on $X^\flat$ together $n+1$ global sections generating $\mathscr{L}$.

Using this corollary as an intermediary, we get a natural and geometric correspondence between certain inverse systems of line bundles on X and single line bundles on $X^\flat$.

Corollary 8.15

An element of $\mathfrak{L}_n(X)$ is equivalent to the a line bundle $\mathscr{L} \in \operatorname{Pic} X^\flat$ together with $n+1$ generating global sections.

This will be a useful tool in understanding the relationship between $\operatorname{Pic} X$ and $\operatorname{Pic} X^\flat$.

9 Untilting Line Bundles

The tilting equivalence (Theorem 4.11) is one of the most powerful tools perfectoid spaces provide us with. It allows us to pass back and forth between mixed characteristic and positive characteristic geometry and algebra, while maintaining much of the same information. In this section, we use the tilting equivalence as well as the tools of projectivoid geometry developed in Sections 7 and 8 in order to compare the Picard groups of a perfectoid space X and its tilt $X^\flat$. Indeed, the theory of maps to projectivoid space allows us to pass between line bundles on X and $X^\flat$ by choosing (compatible) generating sections, constructing the associated map to projectivoid space, and then using the tilting equivalence to pass across characteristics. We remark that the theory of *pro-étale cohomology* on perfectoid spaces allows us to make this comparison cohomologically, but the geometric theory we developed in the previous section gives us a firm geometric grasp.

9.1 Cohomological Untilting

In [3], Bhatt and Scholze introduce the *pro-étale* site for schemes and perfectoid spaces. We review the definition here.

Definition 9.1. A map $f : Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$ of affinoid perfectoid spaces is called *affinoid pro-étale* if it can be written as a cofiltered limit of étale maps $Y_i = \mathrm{Spa}(S_i, S_i^+) \to X$ of affinoid perfectoid spaces. More generally, a map $f : Y \to X$ of perfectoid spaces is *pro-étale* if is locally on the source and target affinoid pro-étale.

The (small) pro-étale site of X is the Grothendieck topology on the category of perfectoid spaces $f : Y \to X$ pro-étale over X on which a collection $\{f_i : Y_i \to X\}_{i\in I}$ is a covering if for each quasicompact open $U \subseteq X$ there exists a finite subset $J \subseteq I$ and quasicompact open subsets $V_i \subseteq Y_i$ for $i \in J$ such that $U = \cup_{i\in J} f_i(V_i)$.

If $\mathscr{F}$ is a pro-étale sheaf on X (that is a sheaf on the pro-étale site of X), the pro-étale cohomology groups $\mathrm{H}^i(X_{\text{pro-ét}}, \mathscr{F})$ are the derived functor sheaf cohomology groups on the pro-étale site.

Let X be a perfectoid space over K. The pro-étale sheaf $\mathbb{G}_{m,X}$ maps $U \mapsto \Gamma(U, \mathscr{O}_U)^*$. We have the following theorem.

Theorem 9.2

$\mathrm{H}^1(X_{\textit{pro-ét}}, \mathbb{G}_m) \cong \mathrm{Pic}\, X$.

Proof. For any site S, the cohomology group $\mathrm{H}^1(X_S, \mathbb{G}_m)$ parametrizes isomorphism classes of line bundles on X with respect to the topology of S. Furthermore, due to [22] Theorem 3.5.8, vector bundles (of any rank) on a perfectoid space with respect to the pro-étale, étale, and analytic topologies coincide.

We use the equivalence of the pro-étale topologies of X and $X^\flat$ to construct the tilt of $\mathbb{G}_m$ as a pro-étale sheaf on X:

$$\mathbb{G}^\flat_{m,X} : U \mapsto (\Gamma(U, \mathscr{O}_U)^\flat)^* = \Gamma(U^\flat, \mathscr{O}_{U^\flat})^* = \Gamma(U^\flat, \mathbb{G}_{m,X^\flat}).$$

The equivalence of the étale topologies on X and $X^\flat$ show that $\mathbb{G}^\flat_{m,X}$ is indeed a sheaf. Better yet, the effectiveness of Čech cohomology on the pro-étale site shows that

$$\mathrm{H}^i(X_{\text{pro-ét}}, \mathbb{G}^\flat_{m,X}) \cong \mathrm{H}^i(X^\flat_{\text{pro-ét}}, \mathbb{G}_{m,X^\flat}).$$

In particular, $\mathrm{H}^1(X_{\text{pro-ét}}, \mathbb{G}^\flat_{m,X}) \cong \mathrm{Pic}\, X^\flat$. Now consider the Kummer sequence for various powers of p.

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_{m,X} \longrightarrow \mathbb{G}_{m,X} \longrightarrow 0.$$

This is an exact sequence of sheaves on the pro-étale site of X. Indeed, this can be checked on the stalks, which on the pro-étale site are strictly Henselian local rings. Therefore we can form an inverse system of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mu_p \arrow[r] & \mathbb{G}_{m,X} \arrow[r] & \mathbb{G}_{m,X} \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u, equal] & \\
0 \arrow[r] & \mu_{p^n} \arrow[r] \arrow[u] & \mathbb{G}_{m,X} \arrow[r] \arrow[u] & \mathbb{G}_{m,X} \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & \mu_{p^{n+1}} \arrow[r] \arrow[u] & \mathbb{G}_{m,X} \arrow[r] \arrow[u] & \mathbb{G}_{m,X} \arrow[r] \arrow[u, equal] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u, equal] &
\end{tikzcd}$$

The vertical maps on the the left and middle sides are $x \mapsto x^p$. Taking this limit gives the following sequence.

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathbb{G}^\flat_{m,X} \xrightarrow{\sharp} \mathbb{G}_{m,X}.$$

The middle term is $\mathbb{G}^\flat_{m,X}$ by definition. Indeed, in construction the tilt of a perfectoid algebra R (Definition 2.56), we constructed an isomorphism of multiplicative monoids:

$$R^\flat \cong \varprojlim_{x \mapsto x^p} R,$$

which restricts to the desired isomorphism on unit groups. Finally, exactlness on the right can be checked explicitly in the pro-étale topology. Indeed, adjoining a pth power root is an étale cover so passing to the limit we get $\sharp$ to be surjective on a pro-étale cover. Therefore we get a short exact sequence of pro-étale sheaves:

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathbb{G}^\flat_{m,X} \xrightarrow{\sharp} \mathbb{G}_{m,X} \longrightarrow 0.$$

Remark 9.3

> If R is a perfectoid algebra we always get a map of monoids $\sharp : R^\flat \to R$ given by projection onto the first coordinate. Although it is not a ring homomorphism unless R already had characteristic p, its restriction to unit groups $(R^\flat)^* \to R^*$ is a group homomorphism. This construction is another way of building the map $\sharp : \mathbb{G}^\flat_{m,X} \to \mathbb{G}_{m,X}$. The advantage of our construction is that it explicitly exhibits the Tate module $\mathbb{Z}_p(1)$ as the kernel.

Taking long exact sequences in cohomology gives us the following diagram, where the rows are exact.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathrm{H}^1(X_{\text{pro-ét}}, \mu_{p^n}) \arrow[r] \arrow[u] & \operatorname{Pic} X \arrow[r] \arrow[u] & \operatorname{Pic} X \arrow[r] \arrow[u, equal] & \mathrm{H}^2(X_{\text{pro-ét}}, \mu_{p^n}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^1(X_{\text{pro-ét}}, \mu_{p^{n+1}}) \arrow[r] \arrow[u] & \operatorname{Pic} X \arrow[r] \arrow[u, "\mathscr{L} \mapsto \mathscr{L}^{\otimes p}"] & \operatorname{Pic} X \arrow[r] \arrow[u, equal] & \mathrm{H}^2(X_{\text{pro-ét}}, \mu_{p^{n+1}}) \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u, equal] & \vdots \arrow[u] & \\
\cdots \arrow[r] & \mathrm{H}^1(X_{\text{pro-ét}}, \mathbb{Z}_p(1)) \arrow[r] \arrow[u] & \operatorname{Pic} X^\flat \arrow[r, "\theta_0"] \arrow[u] \arrow[uu, bend right, "\theta_{n+1}"'] \arrow[uuu, bend right=60, "\theta_n"'] & \operatorname{Pic} X \arrow[r] \arrow[u, equal] & \mathrm{H}^2(X_{\text{pro-ét}}, \mathbb{Z}_p(1)) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Taking the inverse limit of the θ_n, we get a homomorphism of groups,

$$\theta : \operatorname{Pic} X^\flat \longrightarrow \varprojlim_{\mathscr{L} \mapsto \mathscr{L}^{\otimes p}} \operatorname{Pic} X, \tag{7}$$

and θ_0 is this map composed with the projection onto the first coordinate.

Remark 9.4

In Corollary 8.15 we established that inverse systems of pth roots of line bundles (with generating sections) on X correspond to individual line bundles (with generating sections) on $X^\flat$. This seems to suggest that θ could be an isomorphism in cases where we have nice maps to projective space.

9.2 Untilting Via Maps to Projectivoid Space

We hope to give a geometric understanding of θ and θ_0 in terms of maps to projectivoid space. For a perfectoid space X over a perfectoid field K, we hope to understand whether the following correspondence holds.

$$\left\{ \begin{array}{l} \text{Globally generated} \\ \mathscr{L} \in \operatorname{Pic} X^\flat \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Systems of globally generated line} \\ \text{bundles } (\mathscr{L}_0, \mathscr{L}_1, \cdots) \text{ on } X \text{ such} \\ \text{that } \mathscr{L}_{i+1}^{\otimes p} \cong \mathscr{L}_i. \end{array} \right\}$$

We begin by constructing a map in the righthand direction. Given a globally generated invertible sheaf $\mathscr{L} \in \operatorname{Pic} X^\flat$, choose n sections which generate $\mathscr{L}$. Associated to this data there is a unique morphism $\phi^\flat : X^\flat \longrightarrow \mathbb{P}^{n,\mathrm{perf}}_{K^\flat}$, which is the tilt of a unique morphism $\phi : X \longrightarrow \mathbb{P}^{n,\mathrm{perf}}_K$. Let $\mathscr{L}_i = \phi^*(\mathscr{O}(1/p^i))$. This gives a system of $(\mathscr{L}_0, \mathscr{L}_1, \cdots)$ on the right hand side. As a first step we show that the sheaves $\mathscr{L}_i$ do not depend on the choices of global sections of $\mathscr{L}$.

Proposition 9.5

The construction in the previous paragraph is well defined, and $(\mathscr{L}_0, \mathscr{L}_1, \cdots) = \theta(\mathscr{L})$ where θ is the cohomological map defined above (Equation 7).

Proof. ϕ^* can be constructed cohomologically by applying cohomology to the unit of the adjunction, $u : \mathbb{G}_{m,\mathbb{P}^{n,\mathrm{perf}}_K} \to \phi_* \mathbb{G}_{m,X}$ and composing with the natural map $\mathrm{H}^1(\mathbb{P}^{n,\mathrm{perf}}, \phi_* \mathbb{G}_{m,X}) \to \mathrm{H}^1(X, \mathbb{G}_{m,X})$. Pulling u back along the pth power map gives $\phi^{\flat *}$ the same way. Since the pth power map commutes with pullback, we get the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}}_{K^\flat} \arrow[r, "\phi^{\flat *}"] \arrow[d, "\theta_{\mathbb{P}^{n,\mathrm{perf}}}"] & \operatorname{Pic} X^\flat \arrow[d, "\theta_X"] \\
\varprojlim \operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}}_K \arrow[r, "\phi^*"] & \varprojlim \operatorname{Pic} X.
\end{tikzcd}$$

Since $\mathscr{L} = \phi^{\flat *} \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1)$ and $\mathscr{L}_i = \phi^* \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_K}(1/p^i)$, we have reduced to proving the proposition for $\mathbb{P}^{n,\mathrm{perf}}_K$. Explicitly, we must show

$$\theta_{\mathbb{P}^{n,\mathrm{perf}}}(\mathscr{O}(1)) = (\mathscr{O}(1), \mathscr{O}(1/p), \mathscr{O}(1/p^2), \cdots).$$

Since $\operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}} = \mathbb{Z}[1/p]$, and is therefore uniquely p-divisible, it is enough to show that

$$\theta_{0,\mathbb{P}^{n,\mathrm{perf}}} \mathscr{O}(1) = \mathscr{O}(1).$$

Now θ_0 is the cohomological map associated to the Teichmuller map $\sharp : \mathbb{G}_m^\flat \longrightarrow \mathbb{G}_m$. We have seen in Proposition 2.59 that the Teichmuller map on the perfectoid Tate algebra maps $T_i \mapsto T_i$. View θ_0 as a map on Čech cohomology with respect to the standard affine covers, and view $\mathrm{H}^1(\mathbb{P}^{n,\mathrm{perf}}, \mathbb{G}_m)$ as descent data for building a line bundle (and similarly for the tilt). Then we see that $\sharp$ sends descent data for $\mathscr{O}(1)$ (which is monomials of degree one), to monomials of degree one, which build $\mathscr{O}(1)$ on $\mathbb{P}_K^{n,\mathrm{perf}}$.

This tells us that the geometric method of untilting line bundles is well defined because it agrees with the cohomological method which does not depend on the choice of sections.

In order to show this is a bijection, there are two questions that need answering (injectivity and surjectivity). Let's analyze them and see where the difficulties may lie.

- **Injectivity**: Suppose $\mathscr{L}, \mathscr{M} \in \operatorname{Pic}(X^\flat)$ are globally generated. Suppose choosing sections and untilting the associated maps to projectivoid space gives us maps $\phi : X \to \mathbb{P}_K^{n,\mathrm{perf}}$ and $\psi : X \to \mathbb{P}_K^{r,\mathrm{perf}}$. If $\phi^*(\mathscr{O}(1/p^i)) \cong \psi^*(\mathscr{O}(1/p^i) =: \mathscr{L}_i$ for all i, can we conclude that $\mathscr{L} \cong \mathscr{M}$? We can attack this using the methods of Section 8 by considering the tuples $(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X)$ and $(\mathscr{L}_i, t_j^{(i)}, \beta_i) \in \mathfrak{L}_r(X)$ associated to ϕ and ψ respectively. If the α_i and β_i agree, we can consider $\left(\mathscr{L}_i, \left\{s_j^{(i)}, t_k^{(i)}\right\}, \alpha_i\right) \in \mathscr{L}_{n+r+1}(X)$ and consider how the associated map $X \to \mathbb{P}_K^{n+r+1,\mathrm{perf}}$ tilts. We settle the case where $\alpha_i = \beta_i$ below. If α_i and β_i do not agree, they do differ by a global section of $\mathbb{G}_m$.
- **Surjectivity**: Suppose $(\mathscr{L}_0, \mathscr{L}_1, \cdots)$ are globally generated with $\mathscr{L}_{i+1}^{\otimes p} \cong \mathscr{L}_i$, and there are global sections $s_j^{(i)}$ generating $\mathscr{L}_i$ such that $\left(s_j^{(i)}\right)^{\otimes p} = s_j^{(i)}$. Then passing through the maps to projective space we get $\mathscr{L} \in \operatorname{Pic} X^\flat$ which maps to $(\mathscr{L}_0, \mathscr{L}_1, \cdots)$ under θ. But, can we always find sections $s_j^{(i)}$ and isomorphisms such that $\left(s_j^{(i+1)}\right)^{\otimes p} = s_j^{(i)}$? Restated, are there generating global sections of $\mathscr{L}_0$ all of whose pth power roots exist? If so, our correspondence surjects.

In the rest of this section we settle injectivity in the case where the isomorphisms $\mathscr{L}_{i+1}^{\otimes p} \cong \mathscr{L}_i$ agree for the two sets of sections.

Proposition 9.6

Let X be a perfectoid space over K. Suppose $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_n(X)$ and $\left(\mathscr{L}_i, t_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_r(X)$ correspond to maps $\phi : X \to \mathbb{P}_K^{n,\mathrm{perf}}$ and $\psi : X \to \mathbb{P}_K^{r,\mathrm{perf}}$ respectively. Then

$$\phi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}}(1) \cong \psi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{r,\mathrm{perf}}}(1).$$

Fix $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right)$ corresponding to a map $\phi : X \to \mathbb{P}_K^{n,\mathrm{perf}}$. As a first step, we show that we can add one global section to each $\mathscr{L}_i$ that are compatible with the α_i without changing the line bundle we get over $X^\flat$. Suppose that for each i, $t_i \in \Gamma(X, \mathscr{L}_i)$ is a global section such that $\alpha_i\left(t_{i+1}^{\otimes p}\right) = t_i$. For every $\lambda = (\lambda_0, \lambda_1, \cdots) \in \varprojlim K^* = K^{\flat *}$, we let $\psi_\lambda : X \longrightarrow \mathbb{P}_K^{n+1,\mathrm{perf}}$ be the projectivoid map corresponding to adding $\lambda_i t_i$ to the global sections defining the map. That is, ψ_λ corresponds to $\left(\mathscr{L}_i, \{s_j^{(i)}, \lambda_i t_i\}, \alpha_i\right)$. We hope to fit ϕ and ψ_λ in a commutative diagram. To do so we must develop an analog of rational maps in this analytic context.

If we want to define a map $\mathbb{P}^{n+1,\mathrm{perf}} \longrightarrow \mathbb{P}^{n,\mathrm{perf}}$ given by $\left(\mathscr{O}(1/p^i), \left\{T_0^{1/p^i}, \cdots, T_n^{1/p^i}\right\}, m_i\right)$ we would notice that this isn't defined wherever $|T_i/T_{n+1}| > 1$. In particular, it is only defined on the open set:

$$U = \bigcup_{j \neq n+1} \mathbb{P}_K^{n+1,\mathrm{perf}}\left(\frac{T_0, \cdots, T_{n+1}}{T_j}\right).$$

This is the projectivoid analog of projecting away from the hyperplane where T_{n+1} vanishes, (here we are projecting away from a polydisk at the 'north pole'). Unfortunately, the image of ψ_λ does not lie in U, because there may be points x where $\left|(s_j^{(0)}/t_0)(x)\right| > 1$ for all i, so that $|(T_i/T_{n+1})(\psi_\lambda(x))| > 1$. But, restricted to the open set

$$V_\lambda = \bigcup_j X\left(\frac{s_0^{(0)}, \cdots, s_n^{(0)}, \lambda_0 t_0}{s_j^{(0)}}\right),$$

the image of ψ_λ does lie in U. Thus we have the following commutative diagram for every λ.

$$\begin{tikzcd}
 & & \mathbb{P}_K^{n,\mathrm{perf}} \\
X \arrow[urr, "\phi"] \arrow[drr, "\psi_\lambda"'] & V_\lambda \arrow[l, hook'] \arrow[r, "\psi_\lambda"] & U \arrow[u, "\pi"'] \arrow[d, hook] \\
 & & \mathbb{P}_K^{n+1,\mathrm{perf}}
\end{tikzcd}$$

Lemma 9.7

The sets $V_{(\varpi^\flat)^r}$ form an open cover of X. As a consequence the sets $V^\flat_{(\varpi^\flat)^r}$ cover $X^\flat$.

Proof. Notice $(\varpi^\flat)^r = (\varpi^r, \varpi^{r/p}, \cdots)$. Fix $x \in X$. There is some j such that $x \in X\left(\frac{s_0^{(0)}, \cdots, s_n^{(0)}}{s_j^{(0)}}\right)$. Furthermore, since ϖ is topologically nilpotent, there is some r such that

$$\left|(\varpi^r t_0/s_j^{(0)})(x)\right| = |\varpi^r| \cdot \left|(t_0/s_j^{(0)})(x)\right| < 1$$

proving the first statement. The second is an immediate consequence of the tilting equivalence.

Lemma 9.8

For any $\lambda \in K^{\flat *}$,

$$\left(\phi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}}(1)\right)|_{V_\lambda^\flat} \cong \psi_\lambda^{\flat *}\left(\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1)|_{U^\flat}\right) \cong \left(\psi_\lambda^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1)\right)|_{V_\lambda^\flat}$$

Proof. This follows from the commutativity of the tilt of the diagram above, reproduced below, together with the fact that $\pi^\flat$ is given by the line bundle $\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1)|_U$ together with the sections $T_0, \cdots, T_n$.

$$\begin{tikzcd}
 & & \mathbb{P}_{K^\flat}^{n,\mathrm{perf}} \\
X^\flat \arrow[urr, "\phi^\flat"] \arrow[drr, "\psi_\lambda^\flat"'] & V_\lambda^\flat \arrow[l, hook'] \arrow[r, "\psi_\lambda^\flat"] & U^\flat \arrow[u, "\pi^\flat"'] \arrow[d, hook] \\
 & & \mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}
\end{tikzcd}$$

Lemma 9.9

Fix any $\lambda, \xi \in \varprojlim K^* = K^{\flat *}$. Then

$$\psi_\lambda^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1) \cong \psi_\xi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1).$$

Proof. Let $\tau : \mathbb{P}_K^{n+1,\mathrm{perf}} \to \mathbb{P}_K^{n+1,\mathrm{perf}}$ be the map determined by $\left(\mathscr{O}(1/p^i), \left\{T_0^{1/p^i}, \cdots, T_n^{1/p^i}, \frac{\gamma_i}{\xi_i}T_{n+1}^{1/p^i}\right\}, m_i\right)$. Then τ is an isomorphism, and $\tau^\flat$ is the map determined by $\mathscr{O}(1)$ and $T_0, \cdots, T_n, \frac{\gamma}{\xi}T_{n+1}$. We have the following two commutative diagrams, the right hand diagram being the tilt of the left.

$$\begin{tikzcd}
 & \mathbb{P}_K^{n+1,\mathrm{perf}} \arrow[dd, "\tau"] \\
X \arrow[ur, "\psi_\lambda"] \arrow[dr, "\psi_\xi"'] & \\
 & \mathbb{P}_K^{n+1,\mathrm{perf}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}} \arrow[dd, "\tau^\flat"] \\
X^\flat \arrow[ur, "\psi_\lambda^\flat"] \arrow[dr, "\psi_\xi^\flat"'] & \\
 & \mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}
\end{tikzcd}$$

Since $\tau^{\flat *}\mathscr{O}(1) = \mathscr{O}(1)$, we are done.

Putting these three lemmas together, we conclude that

$$\phi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}}(1) \cong \psi_1^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1).$$

Indeed, the pullback of $\mathscr{O}(1)$ along $\psi_1^\flat$ agrees with the pullback along $\psi_{(\varpi^\flat)^r}$, for any r, but this agrees with the restriction of $\phi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}}(1)$ to $V_{(\varpi^\flat)^r}^\flat$ for any r. Since these sets cover $X^\flat$, we are done.

In summary, we have proved the following proposition.

Proposition 9.10

Let $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_n(X)$, *correspond to a map* $\phi : X \to \mathbb{P}_K^{n,\mathrm{perf}}$. *Suppose* $t_i \in \Gamma(X, \mathscr{L}_i)$ *is a global section such that* $\alpha_i\left(t_{i+1}^{\otimes p}\right) = t_i$, *and let* $\psi : X \to \mathbb{P}_K^{n+1}$ *be the map associated to* $\left(\mathscr{L}_i, \{s_j^{(i)}, t_i\}, \alpha_i\right) \in \mathfrak{L}_{n+1}(X)$. *Then*

$$\phi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n,\mathrm{perf}}}(1) \cong \psi^{\flat *}\mathscr{O}_{\mathbb{P}_{K^\flat}^{n+1,\mathrm{perf}}}(1).$$

Adding sections one at a time by induction completes the proof of Proposition 9.6.

9.3 Injectivity of θ

With these tools in hand, we can prove the injectivity of θ for certain perfectoid spaces X. We will first need one more lemma.

Lemma 9.11

Let $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_n(X)$ *correspond to a map* $\phi : X \to \mathbb{P}_K^{n,\mathrm{perf}}$. *Fix* $\lambda = (\lambda_0, \lambda_1, \cdots) \in \Gamma(X, \mathscr{O}_X^{\flat *})$, *that is,* $\lambda_{i+1}^p = \lambda_i$, *so that* $\left(\mathscr{L}_i, \lambda_i s_0^{(i)}, \lambda_i \alpha_i\right) \in \mathfrak{L}_n(X)$ *corresponds to a map* $\psi : X \to \mathbb{P}_K^n$. *Then* $\phi = \psi$.

Proof. We can prove this in two ways.

In the proof of Theorem 8.6, we built ϕ from ring maps $\gamma_j : \frac{T_k^{1/p^i}}{T_j^{1/p^i}} \mapsto \frac{s_k^{(i)}}{s_j^{(i)}}$, and ψ from ring maps $\gamma_j' : \frac{T_k^{1/p^i}}{T_j^{1/p^i}} \mapsto \frac{\lambda_i s_k^{(i)}}{\lambda_i s_j^{(i)}} = \frac{s_k^{(i)}}{s_j^{(i)}}$. Since $\gamma_j = \gamma_j'$, we have $\phi = \psi$.

Alternatively, notice that multiplication by λ_i for each i gives us an isomorphism $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \xrightarrow{\sim} \left(\mathscr{L}_i, \lambda_i s_0^{(i)}, \lambda_i \alpha_i\right)$ in $\mathfrak{L}_n(X)$. Then we are done by Theorem 8.6.

Before we state the main theorem we make the following definition.

Definition 9.12. A line bundle $\mathscr{L}$ on a perfectoid space X is said to be *weakly ample* if for any other line bundle $\mathscr{M}$, there is some $N >> 0$ such that for all $r > N$ we have $\mathscr{M} \otimes \mathscr{L}^r$ globally generated.

Theorem 9.13

Suppose X is a perfectoid space over K. Suppose that X has a weakly ample line bundle and that $H^0(X_{\overline{K}}, \mathscr{O}_{X_{\overline{K}}}) = \overline{K}$, where $\overline{K}$ is a fixed algebraic closure of K. Then

$$\theta : \operatorname{Pic} X^\flat \hookrightarrow \varprojlim_{\mathscr{L} \mapsto \mathscr{L}^p} \operatorname{Pic} X.$$

In particular, if $\operatorname{Pic} X$ *has no p torsion, then*

$$\theta_0 : \operatorname{Pic} X^\flat \hookrightarrow \operatorname{Pic} X.$$

Proof. Fix $\mathscr{L}, \mathscr{M} \in \operatorname{Pic} X^\flat$ with $\theta(\mathscr{L}) = \theta(\mathscr{M})$. We first reduce to the case that $\mathscr{L}, \mathscr{M}$ are globally generated. Indeed, letting $\mathscr{F}$ be a weakly ample line bundle, we have $\theta(\mathscr{L} \otimes \mathscr{F}^N) = \theta(\mathscr{M} \otimes \mathscr{F}^N)$. If the result holds for globally generated line bundles, for large enough N we conclude that $\mathscr{L} \otimes \mathscr{F}^N \cong \mathscr{M} \otimes \mathscr{F}^N$ so that $\mathscr{L} \cong \mathscr{M}$.

Next we prove it for the case where K contains all pth power roots for all its elements. Choose generating sections $s_0, \cdots, s_n$ for $\mathscr{L}$ and $t_0, \cdots, t_r$ of $\mathscr{M}$, which give us maps

$$\phi^\flat : X^\flat \longrightarrow \mathbb{P}^{n,\mathrm{perf}}_{K^\flat},$$

and

$$\psi^\flat : X^\flat \longrightarrow \mathbb{P}^{r,\mathrm{perf}}_{K^\flat},$$

respectively. These untilt to

$$\phi : X \longrightarrow \mathbb{P}^{n,\mathrm{perf}}_{K},$$

and

$$\psi : X \longrightarrow \mathbb{P}^{r,\mathrm{perf}}_{K},$$

which in turn correspond to tuples $\left(\mathscr{L}_i, s_j^{(i)}, \alpha_i\right) \in \mathfrak{L}_n(X)$ and $\left(\mathscr{L}_i, t_j^{(i)}, \beta_i\right) \in \mathfrak{L}_r(X)$. Notice that α_i and β_i differ by an element

$$\lambda_i \in \operatorname{Isom}(\mathscr{L}_i, \mathscr{L}_i) = \Gamma(X, \mathscr{O}_X^*) = K^*.$$

That is, $\alpha_i = \lambda_i \beta_i$. Choose pth power roots λ_i^{1/p^j} for each i, j (these exist by assumption), and for all j define:

$$\begin{aligned}
\tilde{t}_j^{(0)} &= t_j^{(0)} \\
\tilde{t}_j^{(1)} &= \lambda_0^{-1/p} t_j^{(1)} \\
\tilde{t}_j^{(2)} &= \lambda_1^{-1/p} \lambda_0^{-1/p^2} t_j^{(2)} \\
&\vdots \\
\tilde{t}_j^{(i+1)} &= \lambda_i^{-1/p} \lambda_{i-1}^{-1/p^2} \cdots \lambda_0^{-1/p^{i+1}} t_j^{(i+1)} \\
&\vdots
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_i \left(\left(\tilde{t}_j^{(i+1)} \right)^{\otimes p} \right) &= \lambda_i \beta_i \left(\left(\lambda_i^{-1/p} \lambda_{i-1}^{-1/p^2} \cdots \lambda_0^{-1/p^{i+1}} t_j^{(i+1)} \right)^{\otimes p} \right) \\
&= \lambda_i \lambda_i^{-1} \lambda_{i-1}^{1/p} \cdots \lambda_0^{1/p^i} \beta \left(\left(t_j^{(i+1)} \right)^{\otimes p} \right) \\
&= \lambda_{i-1}^{1/p} \cdots \lambda_0^{1/p^i} t_j^{(i)} \\
&= \tilde{t}_j^{(i)}.
\end{aligned}$$

Therefore the tuple $(\mathscr{L}_i, \tilde{t}_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X)$, and it also corresponds to ψ by Lemma 9.11. Furthermore, the isomorphisms corresponding to this data are now α_i in both cases, so that by Proposition 9.6

$$\mathscr{L} = \phi^{\flat *}\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) \cong \psi^{\flat *}\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) = \mathscr{M}.$$

For the general case, we let L/K be the extension given by adjoining all pth power roots of all elements of K. We have the following diagram.

$$\begin{tikzcd}
\operatorname{Pic} X_L^\flat \arrow[r, hook, "\theta_L"] & \varprojlim \operatorname{Pic} X_L \\
\operatorname{Pic} X^\flat \arrow[r, "\theta"] \arrow[u] & \varprojlim \operatorname{Pic} X \arrow[u]
\end{tikzcd}$$

θ_L injects by the argument we just made. Furthermore, since $X_L^\flat \to X^\flat$ is a pro-étale cover of pth power degree, the kernel of $\operatorname{Pic} X^\flat \to \operatorname{Pic} X_L^\flat$ is pth power torsion. Since $X^\flat$ is perfect, $\operatorname{Pic} X^\flat$ has no pth power torsion, so the map injects. Therefore θ injects.

Open Problem 9.14

In which contexts does θ surject?

References

[1] Artin, M., Grothendieck, A., and Verdier, L. *Théorie des topos et cohomologie étale des schémas. (SGA 4 Tome 2).* Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] Bedi, H. *Line Bundles of Rational Degree Over Perfectoid Space.* PhD thesis, George Washington University, 2018.

[3] Bhatt, B., and Scholze, P. The pro-étale topology for schemes. *Astérisque*, 369 (2015), 99–201.

[4] Bosch, S., Guntzer, U., and Remmert, R. *Non-Archimedean Analysis.* Springer-Verlag, 1984.

[5] Bourbaki, N. *Topologie Général.* Springer, 2007.

[6] Buzzard, K., and Verberkmoes, A. Stably uniform affinoids are sheafy. *J. Reine Angew. Math. 740* (2018), 25–39.

[7] Das, S. *Vector Bundles on Perfectoid Spaces.* PhD thesis, University of California, San Diego, 2016.

[8] Fargues, L. Geometrization of the local langlands correspondence: An overview, 2016.

[9] Fontaine, J.-M., and Wintenberger, J.-P. Extensions algébrique et corps des normed des extensions apf des corps locaux. *C. R. Acad. Sci. Paris Sér. 288(8)* (1979), A441–A444.

[10] Fresnel, J., and van der Put, M. *Rigid analytic geometry and its applications*, vol. 218 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 2004.

[11] Glaz, S. *Commutative Coherent Rings.* Springer-Verlag, 1989.

[12] Greenberg, M. Perfect closures of rings and schemes. 313–317.

[13] Grothendieck, A., and Dieudonné, J. *Éléments de géométrie algébrique III*, vol. 11, 17 of *Publications Mathématiques.* Institute des Hautes Études Scientifiques., 1961–1963.

[14] Hartshorne, R. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[15] Huber, R. A generalization of formal schemes and rigid analytic varieties. 513–551.

[16] Huber, R. Continuous valuations. 455–477.

[17] Huber, R. *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces.* Vieweg, 1996.

[18] Joyal, B. Prime dividing the binomial coefficients.

[19] Kedlaya, K. Sheaves, stacks, and shtukas, March 2017.

[20] Kedlaya, K. S. Full faithfulness for overconvergent F-isocrystals. *ArXiv Mathematics e-prints* (Oct. 2001).

[21] Kedlaya, K. S., and Liu, R. Relative p-adic Hodge theory: foundations. *Astérisque*, 371 (2015), 239.

[22] Kedlaya, K. S., and Liu, R. Relative p-adic hodge theory: imperfect period rings, 2016.

[23] Lang, S. *Algebra.* Springer, 1993.

[24] Liu, Q. Un contre-exemple au "critère cohomologique d'affinoïdicité". *C. R. Acad. Sci. Paris Sér. I Math. 307*, 2 (1988), 83–86.

[25] Lütkebohmert, W. Vektorraumbündel über nichtarchimedischen holomorphen Räumen. *Math. Z. 152*, 2 (1977), 127–143.

[26] Quillen, D. Projective modules over polynomial rings.

[27] Scholze, P. Perfectoid spaces. *P. Publ. math. IHES 116*, 1 (2012), 245–313.

[28] Scholze, P. Perfectoid spaces and their applications, August 2014. Seoul.

[29] Scholze, P., and Weinstein, J. Moduli of p-divisible groups. *Camb. J. Math. 1*, 2 (2013), 145–237.

[30] Serre, J.-P. Faisceaux algebriques coherents. 197–278.

[31] Serre, J.-P. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble 6* (1955–1956), 1–42.

[32] Serre, J.-P. Sur la cohomologie des variétés algébriques. *J. Math. Pures Appl. (9) 36* (1957), 1–16.

[33] Serre, J.-P. Modules projectifs et espaces fibrés à fibre vectorielle. In *Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23*. Secrétariat mathématique, Paris, 1958, p. 18.

[34] Stacks Project Authors, T. *Stacks Project*, 2018.

[35] Vakil, R. MATH 216: Foundations of Algebraic Geometry.

[36] Vaseršteĭn, L. Vector bundles and projective modules. 749–755.

[37] Wedhorn, T. Adic spaces, June 2012.

[38] Weinstein, J. Adic spaces, March 2017.