Math 425
Assignment 8

due Wednesday, March 10

1. Let $f(x,y) = xy(3-x-y)$.
   a. Draw a sketch of the set $\{(x,y) : f(x,y) = 0\}$ and indicate the regions where $f(x,y) > 0$ and $f(x,y) < 0$.
   b. Find the critical points of f.
   c. Tell whether each critical point is a local maximum, local minimum, or saddle point. You should be able to do this just by looking at your sketch for (a), without any second-derivative test.

2. Find the extreme values of $f(x,y) = 2x^2 + y^2 + 2x$ on the disc $\{(x,y) : x^2 + y^2 \leq 1\}$. (There's more than one way to look for extrema on the boundary. For practice, you might try a couple of different methods.)

3. Show that $f(x,y) = (x^2 - 2y^2)e^{-x^2-y^2}$ has an absolute minimum and maximum on $\mathbb{R}^2$, and find them.

4. Suppose f is differentiable on $\mathbb{R}^2$ and $f(x,y) \to +\infty$ as $\|(x,y)\| \to \infty$. (That is, for every $A > 0$ there is a $B > 0$ such that $f(x,y) > A$ whenever $\sqrt{x^2+y^2} > B$.) Show that f has an absolute minimum on $\mathbb{R}^2$ (but, of course, no absolute maximum).

5. Let $(x_1, y_1), \dots, (x_k, y_k)$ be points in the plane whose x-coordinates are all distinct. The linear function $f(x) = ax + b$ such that the sum of the squares of the vertical distances from the given points to the line $y = ax + b$ (namely, $\sum_1^k (y_j - ax_j - b)^2$) is minimized is called the *linear least-squares fit* to the points (x_j, y_j). Show that it is given by

$$a = \frac{k^{-1}\sum_1^k x_j y_j - \overline{x}\,\overline{y}}{k^{-1}\sum_1^k x_j^2 - \overline{x}^2}, \qquad b = \overline{y} - a\overline{x},$$

   where $\overline{x} = k^{-1}\sum_1^k x_j$ and $\overline{y} = k^{-1}\sum_1^k y_j$ are the averages of the x's and the y's. (Don't get confused: the x's and y's are fixed; a and b are the variables here.)

6. Given a positive constant c, let $P = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = c\}$, and let $S = \{(x_1, \dots, x_n) \in P : x_j \geq 0 \text{ for all } j\}$.
   a. Show that S is compact.
   b. Let $f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$. Use Lagrange's method to find the maximum value of f on S. (It exists, by part (a), and it doesn't occur on the boundary of S in P, since that set consists of points with at least one coordinate equal to 0, so $f = 0$ there.)
   c. Deduce the arithmetic-geometric mean inequality: For any positive $x_1, \dots, x_n$,

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$