Math 425
Assignment 7

due Wednesday, March 3

Concerning the derivatives in problems 2–3, see the remarks on implicit differentiation on the next page. Concerning problems 4–6, see the remarks on curves and surfaces there.

1. Investigate the possibility of solving the equation $x^2 - 4x + 2y^2 - yz = 1$ for each of its variables as differentiable functions of the other two near the point $(2, -1, 3)$. Do this both by checking the hypotheses of the implicit function theorem and by explicitly computing the solutions.

2. Let $f(x, y, z) = xe^{-z} + 5z + 2y$.

   a. Show that there is a differentiable function g defined in some neighborhood of $(2, -1) \in \mathbb{R}^2$ such that $g(2, -1) = 0$ and $f(x, y, g(x, y)) = 0$.

   b. Compute $D_1g(2, -1)$ and $D_2g(2, -1)$, and use the result to give an approximate value of $g(2.03, -1.06)$.

3. Consider the pair of equations $xy + 2yz - 3xz = 0$, $xyz + x - y = 1$.

   a. Show that these equations can (in principle) be solved for y and z as functions of x near $(x, y, z) = (1, 1, 1)$, and compute dy/dx and dz/dx as functions of x, y, and z.

   b. Can these equations be solved for x and y as functions of z near $(1, 1, 1)$? How about x and z as functions of y?

4. Let $f(x, y) = (y^3 - x^2)(y - 1)$, and let $S = \{(x, y) : f(x, y) = 0\}$.

   a. Draw a sketch of S. (Hint: Keep in mind that a product is zero precisely when at least one of its factors is zero.)

   b. Find the critical points of f (points where $\nabla f = \mathbf{0}$). There are four of them, three of which lie in the set S. You should find that they are located precisely where the sketch in (a) shows some peculiar behavior of the set S. (The approximate location of the fourth one is also predictable from the sketch; do you see why?)

   c. You'll find that $D_2 f$ is nonzero at $(2, 1)$ and $(2, 4^{1/3})$ (both of which lie in S), so the implicit function theorem says that the equation $f = 0$ determines y as a function of x near each of these points. What are these functions? (Don't work too hard; use your sketch.)

5. Let $\mathbf{f}(u, v) = (au \cos v,\, au \sin v,\, u)$, where a and b are positive numbers. Think of $\mathbf{f}$ as a parametric representation of the surface $S = \{\mathbf{f}(u, v) : (u, v) \in \mathbb{R}^2\}$ in $\mathbb{R}^3$.

   a. Using the identity $\sin^2 + \cos^2 = 1$, find a smooth function $g(x, y, z)$ so that $S = \{(x, y, z) : g(x, y, z) = 0\}$.

   b. Draw a sketch of the surface S. (It may be helpful to consider the curves obtained by intersecting S with the horizontal planes $z =$ constant.) You should find that S is a smooth surface everywhere except at the origin $(0, 0, 0)$.

   c. The singularity of S at the origin should be reflected in the behavior of $\mathbf{Df}(0, v)$ (since $\mathbf{f}(0, v) = (0, 0, 0)$) and of $\nabla g(0, 0, 0)$. Verify this; precisely what happens?

6. Curves in $\mathbb{R}^3$ are usually best represented parametrically, that is, as the range of a map $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$. As in the case of plane curves, the condition that guarantees that such a map yields a smooth curve (locally) is that $\mathbf{f}$ be of class C^1 and that $\mathbf{f}'(t) \neq \mathbf{0}$. However, another way of producing curves is as the intersection of two surfaces. Let S_1 and S_2 be the smooth surfaces defined by the equations $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, where we assume that $\nabla g_j \neq \mathbf{0}$ on S_j $(j = 1, 2)$. Then $S_1 \cap S_2$ is the set where $g_1(x, y, z) = g_2(x, y, z) = 0$. Show that if ∇g_1 and ∇g_2 are linearly independent — or, equivalently, that $\nabla g_1 \times \nabla g_2 \neq \mathbf{0}$ — at every point of $S_1 \cap S_2$, then the equations $g_1 = g_2 = 0$ can be solved (at least locally) to yield two of the variables x, y, and z as C^1 functions of the third, and hence a parametrization of (at least a small piece of) $S_1 \cap S_2$ using that third variable as the parameter. (Remarks: (1) Which one of x, y, and z ends up as the independent variable may depend on the nature of ∇g_1 and ∇g_2. (2) Geometrically, the condition that ∇g_1 and ∇g_2 be linearly independent on $S_1 \cap S_2$ means that S_1 and S_2 are nowhere tangent to each other.)

Implicit Differentiation. In freshman calculus you learn that if two variables x and y are related by an equation $f(x, y) = 0$, you can find dy/dx in terms of x and y by differentiating this equation with respect to x, assuming y to be a function of x and using the chain rule, to get $(D_1 f) + (D_2 f)(dy/dx) = 0$, then solving this for dy/dx. Similarly, suppose $\mathbf{f} : \mathbb{R}^{n+k} \to \mathbb{R}^n$ satisfies the conditions of the implicit function theorem so that the equation $\mathbf{f}(\mathbf{x}, \mathbf{t}) = \mathbf{0}$ can be solved (locally) to yield $\mathbf{x} = \mathbf{g}(\mathbf{t})$. One can differentiate the equation $\mathbf{f}(\mathbf{g}(\mathbf{t}), \mathbf{t}) = \mathbf{0}$ with respect to $\mathbf{t}$ to get $\mathbf{D_x f}(\mathbf{x}, \mathbf{t})\, \mathbf{Dg}(\mathbf{t}) + \mathbf{D_t f}(\mathbf{x}, \mathbf{t}) = \mathbf{0}$ (where $\mathbf{x} = \mathbf{g}(\mathbf{t})$), and then solve this system of (linear!) equations to find $\mathbf{Dg}(\mathbf{t})$ in terms of $\mathbf{t}$ and $\mathbf{x}$. Explicitly, $\mathbf{Dg}(\mathbf{t}) = -[\mathbf{D_x f}(\mathbf{x}, \mathbf{t})]^{-1} \mathbf{D_t f}(\mathbf{x}, \mathbf{t})$, but in specific examples it is usually easier to solve the linear system directly rather than computing the inverse matrix $[\mathbf{D_x f}(\mathbf{x}, \mathbf{t})]^{-1}$.

Curves and Surfaces (sketch of results to be discussed in class). There are 3 common ways of representing a curve C in the plane: (i) as a graph $y = f(x)$ or $x = g(y)$; (ii) as the set of (x, y) satisfying an equation $F(x, y) = 0$; (iii) as the range of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$. Note that (i) is a special case of (ii) (with $F(x, y) = y - f(x)$ or $x - g(y)$) and (iii) (with $\mathbf{f}(t) = (t, f(t))$ or $(g(t), t)$). Conversely, if F is of class C^1 and $\nabla F \neq 0$ on C, the implicit function theorem implies that the equation $F(x, y) = 0$ can be solved (locally) for y as a function of x or vice versa, so (ii) $\Rightarrow$ (i). Also, of $\mathbf{f}$ is of class C^1 and $\mathbf{f}'(t) \neq \mathbf{0}$, the inverse function theorem implies that one of the equations $x = f_1(t)$, $y = f_2(t)$ can be solved (locally) for t, which can then be substituted into the other to yield one of x and y as a function of the other, so (iii) $\Rightarrow$ (i) too. Analogous results hold for curves in $\mathbb{R}^3$; see Problem 6.

Similarly, a surface S in $\mathbb{R}^3$ can be represented (i) as a graph $z = f(x, y)$ (maybe with variables switched), (ii) as the set where $F(x, y, z) = 0$, or (iii) as the range of a function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$. Again (i) is a special case of (ii) and (iii); the implicit function theorem implies that if F is of class C^1 and $\nabla F \neq \mathbf{0}$ on S, then the equation $F = 0$ can be solved (locally) to yield an equation $z = f(x, y)$ (maybe with variables switched); and the inverse mapping theorem implies that if $\mathbf{f}$ is of class C^1 and $\mathbf{Df}(s, t)$ has rank 2 (i.e., if its columns $D_1\mathbf{f}$ and $D_2\mathbf{f}$ are linearly independent), the parameters (s, t) can be eliminated (locally) to yield an equation $z = f(x, y)$ (maybe with variables switched).