Math 425
Assignment 5
due Wednesday, February 10

Reminder: The Midterm Exam is on Friday, February 12. It will be taken from all the
material we’ve covered through this assignment: essentially, Chapters 5 and 12 of Apostol
(omitting the bits about complex-valued functions), together with my supplementary notes
on l’Hôpital’s rule, Taylor’s theorem, and higher order partial derivatives. Some problems
will be theoretical, some will be computational.


2. Suppose $w = f(s^2 - t^2, 2st)$. Show that $\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial t^2} = 4(s^2 + t^2)(D_1^2 f + D_2^2 f)$. (The
derivatives of $f$ are evaluated at $(s^2 - t^2, 2st)$.)

3. State and prove a version of Rolle’s theorem for functions of several variables whose
hypothesis is the following: Suppose $f$ is a real-valued function that is differentiable on
a bounded open set $U \subset \mathbb{R}^n$, continuous on the closure $\overline{U}$, and constant on the boundary
$\partial U$. (The 1-dimensional Rolle’s theorem is the case where $U = (a, b)$, $\partial U = \{a, b\}$.)

4. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(u, v) = (u^2 - 5v, ve^{2u}, 2u - \log(1 + v^2))$.
   a. Compute the Jacobian matrix $Df(u, v)$. What is $Df(0, 0)$?
   b. Suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$ is of class $C^1$, $g(1, 2) = (0, 0)$, and $Dg(1, 2) = (\frac{1}{3}, \frac{2}{3})$. Compute
      $D(f \circ g)(1, 2)$.

5. Suppose $f$ and $g$ are differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Show that their dot
product $h(x) = f(x) \cdot g(x)$ is a differentiable real-valued function on $\mathbb{R}^n$ and that
     $\nabla h(x) = [Df(x)]^T g(x) + [Dg(x)]^T f(x),$
where $A^T$ denotes the transpose of the matrix $A$ and we think of $\nabla h(x)$, $f(x)$, and
$g(x)$ as column vectors.

6. Use Taylor’s formula to expand $x^2 + y^3 + xy^2$ in powers of $x - 1$ and $y - 2$. (Compute
partial derivatives and plug in. You can check your answer by pure algebra.)

7. Let $f(x, y) = e^{xy} \cos(x + y^2)$.
   a. Find the 4th-order Taylor polynomial of $f$ about $(0, 0)$. Don’t compute any deriva-
tives, just work from the known Taylor series for $e^t$ and $\cos t$. (You should find that
there is a constant term, no linear term, two 2nd-order terms, one 3rd-order term,
and four 4th-order terms.)
   b. If you know the Taylor expansion of a function about a certain point, you can
read off the derivatives of the function at that point from the coefficients of the
expansion. For example, if $f(t) = t - 7t^2 + 3t^5 + \text{higher order}$, you know that
$f^{(3)}(0) = 0$ (no cubic term) and $f^{(5)}(0) = 3 \cdot 5! = 360$. From your answer to part
(a), read off $\partial^{(3,1)} f(0, 0)$ and $\partial^{(0,4)}(0, 0)$. I’m using multi-index notation here, as
explained in the notes.