Math 425
Assignment 5

due Wednesday, February 10

Reminder: *The Midterm Exam is on Friday, February 12.* It will be taken from all the material we've covered through this assignment: essentially, Chapters 5 and 12 of Apostol (omitting the bits about complex-valued functions), together with my supplementary notes on l'Hôpital's rule, Taylor's theorem, and higher order partial derivatives. Some problems will be theoretical, some will be computational.

1. Problem 12.13 (page 364).

2. Suppose $w = f(s^2 - t^2, 2st)$. Show that $\dfrac{\partial^2 w}{\partial s^2} + \dfrac{\partial^2 w}{\partial t^2} = 4(s^2 + t^2)(D_1^2 f + D_2^2 f)$. (The derivatives of f are evaluated at $(s^2 - t^2, 2st)$.)

3. State and prove a version of Rolle's theorem for functions of several variables whose hypothesis is the following: Suppose f is a real-valued function that is differentiable on a bounded open set $U \subset \mathbb{R}^n$, continuous on the closure $\overline{U}$, and constant on the boundary ∂U. (The 1-dimensional Rolle's theorem is the case where $U = (a, b)$, $\partial U = \{a, b\}$.)

4. Define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ by $\mathbf{f}(u, v) = (u^2 - 5v,\ ve^{2u},\ 2u - \log(1 + v^2))$.
   a. Compute the Jacobian matrix $\mathbf{Df}(u, v)$. What is $\mathbf{Df}(0, 0)$?
   b. Suppose $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^1, $\mathbf{g}(1, 2) = (0, 0)$, and $\mathbf{Dg}(1, 2) = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. Compute $\mathbf{D}(\mathbf{f} \circ \mathbf{g})(1, 2)$.

5. Suppose $\mathbf{f}$ and $\mathbf{g}$ are differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Show that their dot product $h(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})$ is a differentiable real-valued function on $\mathbb{R}^n$ and that
$$\nabla h(\mathbf{x}) = [\mathbf{Df}(\mathbf{x})]^T \mathbf{g}(\mathbf{x}) + [\mathbf{Dg}(\mathbf{x})]^T \mathbf{f}(\mathbf{x}),$$
where A^T denotes the transpose of the matrix A and we think of $\nabla h(\mathbf{x})$, $\mathbf{f}(\mathbf{x})$, and $\mathbf{g}(\mathbf{x})$ as column vectors.

6. Use Taylor's formula to expand $x^2 + y^3 + xy^2$ in powers of $x - 1$ and $y - 2$. (Compute partial derivatives and plug in. You can check your answer by pure algebra.)

7. Let $f(x, y) = e^{xy} \cos(x + y^2)$.
   a. Find the 4th-order Taylor polynomial of f about $(0, 0)$. Don't compute any derivatives, just work from the known Taylor series for e^t and $\cos t$. (You should find that there is a constant term, no linear term, two 2nd-order terms, one 3rd-order term, and four 4th-order terms.)
   b. If you know the Taylor expansion of a function about a certain point, you can read off the derivatives of the function at that point from the coefficients of the expansion. For example, if $f(t) = t - 7t^2 + 3t^5 +$ higher order, you know that $f^{(3)}(0) = 0$ (no cubic term) and $f^{(5)}(0) = 3 \cdot 5! = 360$. From your answer to part (a), read off $\partial^{(3,1)} f(0, 0)$ and $\partial^{(0,4)}(0, 0)$. I'm using multi-index notation here, as explained in the notes.