Math 425
Assignment 4

due Wednesday, February 3

1. To warm up, compute the gradients $\nabla f(\mathbf{x})$ of the following functions $f : \mathbb{R}^n \to \mathbb{R}$:

   a. $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ b. $f(\mathbf{x}) = e^{-\|\mathbf{x}\|^2/2}$ c. $f(\mathbf{x}) = \sum_{i,j=1}^{n} a_{ij}x_ix_j \ (a_{ji} = a_{ij})$

   We could also write (c) as $f(\mathbf{x}) = A\mathbf{x}\cdot\mathbf{x}$ where A is the matrix (a_{ij}) and $\mathbf{x}$ is construed as a column vector.

2. Let $f(x, y) = \dfrac{x^2y}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. (Note that since $|xy| \leq \frac{1}{2}(x^2 + y^2)$, we have $|f(x, y)| \leq \frac{1}{2}|x|$, so f is continuous at $(0, 0)$.)
   a. Working directly from the definition of directional derivative, compute all the directional derivatives $f'((0, 0); (a, b))$ [the derivative of f at $\mathbf{0} = (0, 0)$ in the direction $\mathbf{u} = (a, b)$]. (In particular, these derivatives all exist!)
   b. Show that f is not differentiable at $\mathbf{0}$. (Hint: If it were, the directional derivatives would be given by $f'(\mathbf{0}, \mathbf{u}) = \nabla f(\mathbf{0})\cdot\mathbf{u}$.)

3. For $j = 1, \ldots, n$, let $f_j(t)$ be a function of one variable t that is differentiable at $t = c_j$, and let $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$. Show that f is differentiable at $(x_1, \ldots, x_n) = (c_1, \ldots, c_n)$ and compute its gradient there. (This is quite easy, but you have to proceed directly from the definition of differentiability for functions of one and several variables, as there is no assumption that the derivatives f_j' are continuous. This gives one way of exhibiting a family of functions of several variables that are differentiable but not of class C^1.)

4. Let $h(\mathbf{x}) = f(\|\mathbf{x}\|^2)$ where $\mathbf{x} \in \mathbb{R}^n$ and f is a differentiable function on $\mathbb{R}$. Show that $\|\nabla h(\mathbf{x})\|^2 = 4\|\mathbf{x}\|^2[f'(\|\mathbf{x}\|^2)]^2$.

5. A function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ is said to be *positively homogeneous of degree* $p \in \mathbb{R}$ if $f(\lambda\mathbf{x}) = \lambda^p f(\mathbf{x})$ for all $\lambda > 0$. Show that a C^1 function f on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is positively homogeneous of degree p if and only if $\mathbf{x}\cdot\nabla f(\mathbf{x}) = pf(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$. (This is known as *Euler's theorem.* Hint: Fix $\mathbf{x} \neq \mathbf{0}$ and let $g(\lambda) = f(\lambda\mathbf{x})$; compute $g'(\lambda)$.)

6. Suppose $\phi(x)$ is defined by a formula in which x occurs in several places (for example, there are three x's in $\sin(x + x^3)\int_0^x e^{-t^2}\,dt$.) Show that the derivative $\phi'(x)$ is obtained by differentiating with respect to each of the x's in turn, treating the others as constants, and adding the results. (Note that the sum and product rules for derivatives are special cases of this result. Hint: Let $F(x_1, \ldots, x_n)$ be the function of several variables obtained by replacing each of the x's in the formula for $\phi(x)$ by a different variable. Express ϕ in terms of F and use the chain rule.)