1. Let \( f : \mathbb{R} \to \mathbb{R} \) be twice differentiable.
   a. Suppose that \(|f(x)| \leq 1\) and \(|f''(x)| \leq 1\) for all \( x \). Show that \(|f'(x)| \leq 2\) for all \( x \).
      (Hint: Suppose to the contrary that \(|f'(x_0)| > 2\) for some \( x_0 \). Use the bound on \( f'' \) to show that \(|f'(x)| > 1\) on some reasonably large interval, and thence obtain a contradiction with the bound on \( f \).)
   b. More generally, show that if \(|f(x)| \leq C_0\) and \(|f''(x)| \leq C_2\) for all \( x \), then \(|f'(x)| \leq 2\sqrt{C_0C_2}\) for all \( x \).
      (Hint: apply part (a) to \( g(x) = (1/C_0) f(\sqrt{C_0/C_2} x) \).)

2. Let \( f : \mathbb{R} \to \mathbb{R}^n \) be a differentiable vector-valued function. Show that \( \|f(t)\| \) is constant if and only if \( f(t) \cdot f'(t) = 0 \) for all \( t \). (You should spend a couple of minutes thinking about the geometry here. Think of \( f(t) \) as the position of a particle at time \( t \). What does the condition \( \|f(t)\| = C \) say about the motion? What about the condition \( f(t) \cdot f'(t) = 0 \)?)

The next four problems have to do with Newton’s method for solving equations. Recall the basic idea: to find a solution of \( f(x) = 0 \), make an initial guess \( x_1 \). Replace \( f \) by its tangent line at \( x_1 \), \( y = f(x_1) + f'(x_1)(x-x_1) \), and find its zero, namely \( x_2 = x_1 - f(x_1)/f'(x_1) \). Take \( x_2 \) as the next approximation and iterate, obtaining the sequence

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},
\]

which hopefully will converge to the zero of \( f \). Obviously this doesn’t always work — you can get in big trouble if \( f' \) is ever zero in the range of \( x \)’s you’re looking at, for example. But in favorable cases, where \( f \) is strictly increasing or decreasing and of constant convexity, it works like a charm.

More precisely, in problems 3–5 suppose that \( f \) is twice differentiable on \([a,b]\), \( f(a) < 0 < f(b) \), and \( f'(x) \geq \delta > 0 \) and \( 0 \leq f''(x) \leq M \) on \([a,b]\). Thus \( f \) is strictly increasing, so there is a unique \( c \in [a,b] \) such that \( f(c) = 0 \), and for \( x \in [a,b] \), \( f(x) > 0 \) precisely when \( x > c \). The object is to compute \( c \). To do this, set \( x_1 = b \) and define \( x_n \) recursively by (*)

3. Show that the sequence \( \{x_n\} \) is decreasing and \( f(x_n) > 0 \) for all \( n \). (The latter inequality depends on the assumption that \( f'' \geq 0 \), so the graph of \( f \) is concave up. Draw a picture to see how it works.)

4. Show that \( \lim_{n \to \infty} x_n = c \). (Show that \( \lim_{n \to \infty} x_n \) exists — call it \( l \) — and then that \( f(l) = 0 \).)
5. Show that $x_{n+1} - c = \frac{f''(t_n)}{f'(x_n)}(x_n - c)^2$ for some $t_n \in (c, x_n)$, so that $0 \leq x_{n+1} - c \leq (M/2\delta)(x_n - c)^2$. (Hint: Expand $f(x)$ about $x = x_n$ and then set $x = c$.)

This shows that once $x_n - c$ is small, the rate of convergence of $x_n$ to $c$ is extremely rapid: if $x_N - c < \delta/M$, for example, then $x_{N+j} - c < (\delta/M)^{2^{1-2^j}}$.

Under the above hypotheses, if you take $x_1$ to be to the left of $c$, so that $f(x_1) < 0$, you’ll find that $x_2 > c$ (again because $f$ is concave up), so the preceding analysis applies starting with $x_2$ provided that $x_2$ is still in the interval $[a, b]$ where the hypotheses are valid. If we assume $f$ is decreasing and/or concave down instead of increasing and concave up, the analysis is similar, with some inequalities reversed. But:

6. Let $f(x) = x^3 - 5x$, $[a, b] = [-1, 1]$. Take $x_1 = 1$ as the initial guess for the zero (which, of course, is $c = 0$) and plug into (*) to compute $x_n$ for $n > 1$. (This is easier than you might expect.) Why doesn’t Newton’s method find the zero? What hypothesis fails?