Math 425
Assignment 3

due Wednesday, January 27

1. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable.
   a. Suppose that $|f(x)| \le 1$ and $|f''(x)| \le 1$ for all x. Show that $|f'(x)| \le 2$ for all x. (Hint: Suppose to the contrary that $|f'(x_0)| > 2$ for some x_0. Use the bound on f'' to show that $|f'(x)| > 1$ on some reasonably large interval, and thence obtain a contradiction with the bound on f.)
   b. More generally, show that if $|f(x)| \le C_0$ and $|f''(x)| \le C_2$ for all x, then $|f'(x)| \le 2\sqrt{C_0 C_2}$ for all x. (Hint: apply part (a) to $g(x) = (1/C_0) f(\sqrt{C_0/C_2}\, x)$.)

2. Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ be a differentiable vector-valued function. Show that $\|\mathbf{f}(t)\|$ is constant if and only if $\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0$ for all t. (You should spend a couple of minutes thinking about the geometry here. Think of $\mathbf{f}(t)$ as the position of a particle at time t. What does the condition $\|\mathbf{f}(t)\| = C$ say about the motion? What about the condition $\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0$?)

The next four problems have to do with Newton's method for solving equations. Recall the basic idea: to find a solution of $f(x) = 0$, make an initial guess x_1. Replace f by its tangent line at x_1, $y = f(x_1) + f'(x_1)(x - x_1)$, and find *its* zero, namely $x_2 = x_1 - f(x_1)/f'(x_1)$. Take x_2 as the next approximation and iterate, obtaining the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (*)$$

which hopefully will converge to the zero of f. Obviously this doesn't always work — you can get in big trouble if f' is ever zero in the range of x's you're looking at, for example. But in favorable cases, where f is strictly increasing or decreasing and of constant convexity, it works like a charm.

More precisely, in problems 3–5 suppose that f is twice differentiable on $[a, b]$, $f(a) < 0 < f(b)$, and $f'(x) \ge \delta > 0$ and $0 \le f''(x) \le M$ on $[a, b]$. Thus f is strictly increasing, so there is a unique $c \in [a, b]$ such that $f(c) = 0$, and for $x \in [a, b]$, $f(x) > 0$ precisely when $x > c$. The object is to compute c. To do this, set $x_1 = b$ and define x_n recursively by (*).

3. Show that the sequence $\{x_n\}$ is decreasing and $f(x_n) > 0$ for all n. (The latter inequality depends on the assumption that $f'' \ge 0$, so the graph of f is concave up. Draw a picture to see how it works.)

4. Show that $\lim_{n\to\infty} x_n = c$. (Show that $\lim_{n\to\infty} x_n$ exists — call it l — and then that $f(l) = 0$.)

5. Show that $x_{n+1} - c = [f''(t_n)/2f'(x_n)](x_n - c)^2$ for some $t_n \in (c, x_n)$, so that $0 \leq x_{n+1} - c \leq (M/2\delta)(x_n - c)^2$. (Hint: Expand $f(x)$ about $x = x_n$ and then set $x = c$.) This shows that once $x_n - c$ is small, the rate of convergence of x_n to c is extremely rapid: if $x_N - c < \delta/M$, for example, then $x_{N+j} - c < (\delta/M)2^{1-2^j}$.

Under the above hypotheses, if you take x_1 to be to the left of c, so that $f(x_1) < 0$, you'll find that $x_2 > c$ (again because f is concave up), so the preceding analysis applies starting with x_2 provided that x_2 is still in the interval $[a, b]$ where the hypotheses are valid. If we assume f is decreasing and/or concave down instead of increasing and concave up, the analysis is similar, with some inequalities reversed. But:

6. Let $f(x) = x^3 - 5x$, $[a, b] = [-1, 1]$. Take $x_1 = 1$ as the initial guess for the zero (which, of course, is $c = 0$) and plug into (*) to compute x_n for $n > 1$. (This is easier than you might expect.) Why doesn't Newton's method find the zero? What hypothesis fails?