Math 425A
Assignment 2

due Wednesday, January 20

These problems involve l'Hôpital's rule and Taylor polynomials; see the notes linked to the class web site.

1. Recall that $f'(c)$ can be defined as $\lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$. Show that if f is of class C^2 on an open interval I containing c, then

$$\lim_{h\to 0} \frac{f(c+2h)-2f(c+h)+f(c)}{h^2} = f''(c),$$

and if f is of class C^3 on I, then

$$\lim_{h\to 0} \frac{f(c+3h)-3f(c+2h)+3f(c+h)-f(c)}{h^3} = f'''(c).$$

Can you guess what the corresponding formula for $f^{(k)}(c)$ is? (I'm not asking for the proof.)

2. Let $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$, noting that f is continuous at 0.

   a. Show that $\lim_{x\to 0+} f(x)/x^n = 0$ for all $n > 0$. (You'll find that a simple-minded application of l'Hôpital doesn't work. Try setting $y = 1/x$.)
   b. Show by induction on k that for $x > 0$, $f^{(k)}(x) = P(1/x)f(x)$ where P is a polynomial of degree $2k$.
   c. Deduce that $\lim_{x\to 0+} f^{(k)}(x) = 0$ for all k.
   d. Conclude that f is of class C^∞ (that is, of class C^k for all k), even at $x = 0$, and that $f^{(k)}(0) = 0$ for all k. (Hint: Problem 5 of last week's assignment.)

   Remarks: (1) The Taylor series $\sum_0^\infty f^{(k)}(0)x^k/k!$ of this function vanishes identically, so this is an example of a C^∞ function that is not the sum of its Taylor series. (2) The function $g(x) = f(x)f(1-x)$ is of class C^∞, positive on the interval $(0,1)$, and zero elsewhere. Such "smooth bump functions" are important technical tools in advanced analysis.

3. Let $f(x) = x^{1/5}$.

   a. Find the 2nd order Taylor polynomial $P_{2,32}(x)$ of $f(x)$ about $x = 32$ (i.e., expand $x^{1/5}$ in powers of $x - 32$).
   b. Show that if $x \ge 32$, the remainder $R_{2,32}(x)$ satisfies $|R_{2,32}(x)| \le (3\times 10^{-6})(x-32)^3$.
   c. Use the results of (a) and (b) to calculate $36^{1/5}$ approximately and estimate the error. Compare these results with the value of $36^{1/5}$ obtained on a calculator. How does the predicted error compare with the actual error?

4. Find the 3rd order Taylor polynomial of $e^{1-e^{-x}}$ without computing derivatives, and hence evaluate $\lim_{x\to 0}(e^{1-e^{-x}} - 1 - x)/x(\cos\frac{1}{2}x - 1)$ without using l'Hôpital.

5. You can form Taylor polynomials for a vector-valued function $\mathbf{f}$ just as for scalar-valued functions. (They are also vector-valued, the vectorial nature coming from the coefficients $\mathbf{f}^{(j)}(c)$.) What about the remainder? The Lagrange form (Theorem 3 of the notes) doesn't quite work; you can apply Theorem 3 to each component separately, but the intermediate points x_1 will probably be different for each different component, so you probably can't use the same x_1 for all of them at once. (The special case $k = 0$ of Theorem 3 is just the mean value theorem, and there's a counterexample for the vector-valued mean value theorem at the top of p.115 of Apostol.) Convince yourself, however, that the integral form of the remainder (Theorem 4) works just fine for vector-valued functions. (You don't have to hand this in.) Hence state and prove an analogue of Corollary 1 for vector-valued functions.

6. Suppose f is of class C^2 on $[a, b]$ and that $|f''(x)| \le C$ for all $x \in [a, b]$.

   a. Show that if $[c - (\delta/2),\, c + (\delta/2)] \subseteq [a, b]$, then $\int_{c-(\delta/2)}^{c+(\delta/2)} f(x)\,dx = \delta f(c)$ plus an error that is at most $C\delta^3/24$ in absolute value. (Hint: Taylor expansion of order 1 [i.e., tangent line approximation] about $x = c$, with remainder estimate.)

   b. Now take $\delta = (b - a)/n$ and conclude that the integral $\int_a^b f(x)\,dx$ is equal to the "midpoint Riemann sum" corresponding to n equal subdivisions of $[a, b]$, namely $\sum_1^n f(x_j)\Delta x$ with $\Delta x = \delta = (b - a)/n$ and $x_j = a + (j - \frac{1}{2})\delta$, plus an error that is at most $C(b - a)^3/24n^2$ in absolute value.

   (In contrast, the error in the left- or right-endpoint Riemann sum obtained by taking $x_j - a = (j - 1)\delta$ or $j\delta$ instead of $(j - \frac{1}{2})\delta$ is usually no better than $K(b - a)^2/2n$ where K is an upper bound for $|f'(x)|$, as the example of a linear function shows. The improvement from $1/n$ decay to $1/n^2$ decay as n increases is computationally significant! The error estimates for the trapezoidal rule and Simpson's rule can also be obtained by similar uses of Taylor's theorem.)

7. Use part (a) of the preceding problem to show that for $k = 1, 2, 3, \ldots$, $\int_{k-(1/2)}^{k+(1/2)} \log x\,dx = \log k + c_k$ where $|c_k| \le 1/2k^2$. (The coefficient 1/2 isn't very sharp when k is large, but that's OK.) Summing from $k = 1$ to $k = n$ and evaluating the integral by calculus, deduce that

$$\log(n!) = \int_{1/2}^{n+(1/2)} \log x\,dx + \sum_1^n c_k = (n + \tfrac{1}{2})\log n - n + C_n$$

   where the constant C_n approaches a finite limit L as $n \to \infty$. On exponentiating, this gives *Stirling's formula*

$$n! \sim e^L n^{n+(1/2)} e^{-n},$$

   where $A_n \sim B_n$ means that $A_n/B_n \to 1$ as $n \to \infty$. The constant e^L turns out to be $\sqrt{2\pi}$ — a pretty fact, but not needed to extract most of the power of Stirling's formula.