Math 425A
Assignment 1

due Wednesday, January 13

The following exercises were inspired by problems 5.5, 5.6, 5.14, 5.16, 5.19, and 5.22 in Apostol, but I found that I wanted to modify or reword them. *Note:* "Differentiable" = "having a finite derivative".

1. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.
   a. Calculate $f'(x)$ for $x \neq 0$.
   b. Show that $f'(0)$ exists and equals 0. (You'll need to work directly from the definition of $f'(0)$.) Conclude that f is a function whose derivative exists everywhere but is not everywhere continuous. By the way, this provides a nice illustration of Theorem 5.16: f' has the intermediate value property even thought it isn't continuous.

2. Let $f(x)$ be as in Problem 1 and let $g(x) = f(x) + \frac{1}{2}x$. Thus $g'(x) = f'(x) + \frac{1}{2}$, and in particular, $g'(0) = \frac{1}{2} > 0$. Show, nonetheless, that g is not increasing in any neighborhood of 0; more precisely, every interval containing 0 has subintervals on which g is decreasing. (Compare this with Theorem 5.7!)

3. Prove the product rule for nth order derivatives (also called *Leibniz's formula*): If $h(x) = f(x)g(x)$, then

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x), \qquad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

   (Use induction on n. You can take for granted the identity $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.)

4. Suppose that f is differentiable in the interval $(0, 1)$ and that f' is bounded there (i.e., $|f'(x)| \leq C$ for some $C > 0$). Show that $\lim_{n\to\infty} f(1/n)$ exists. (Hint: Show that this sequence is Cauchy.)

5. Suppose f is continuous on the interval (a, b) and differentiable everywhere in (a, b) except possibly at the point $c \in (a, b)$. Show that if $\lim_{x\to c} f'(x) = L$, then $f'(c)$ exists and equals L. (Apply the mean value theorem to the difference quotients defining $f'(c)$.)

6. Suppose that f is continuous on $[a, b]$ and twice differentiable on (a, b), and suppose that the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersects the graph of f at a third point $(c, f(c))$. Prove that there is at least one point $z \in (a, b)$ such that $f''(z) = 0$.

7. Suppose that f is differentiable on the interval (a, ∞), and that $f(x) \to 1$ and $f'(x) \to c$ as $x \to \infty$. Show that $c = 0$.