In what follows, \( I \) always denotes an interval in \( \mathbb{R} \). Also, “increasing” means “monotonically increasing” and “decreasing” means “monotonically decreasing.” (This abbreviated terminology is very common.)

The following remarks are relevant to a couple of the problems. To say that a function \( f : I \to \mathbb{R} \) is not monotonic means that there are four points \( a, b, c, d \in I \) such that \( a < b \) and \( f(a) < f(b) \) (\( f \) is not decreasing), and also \( c < d \) and \( f(c) > f(d) \) (\( f \) is not increasing). There are six possible order relations among the points \( a, b, c, d \) if they are all distinct (\( a < b < c < d \), \( a < c < d < b \), etc.), plus six more if two of them coincide, which makes these conditions awkward to deal with directly. But it’s easy, though a bit tedious, to check that in all cases there are three points on which monotonicity fails. That is, \( f \) is not monotonic if and only if there exist \( a, b, c \in I \) with \( a < b < c \) and either (i) \( f(a) < f(b) \) and \( f(b) > f(c) \), or (ii) \( f(a) > f(b) \) and \( f(b) < f(c) \). You can use this fact.

1. Let \( S \) be the unit sphere in \( \mathbb{R}^3 \), \( S = \{ x = (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1 \} \).
   a. Show that \( S \) is connected. (One way is to consider the spherical coordinate map \( F(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \). Or show that \( S \) is pathwise connected by some explicit construction.)
   b. Suppose \( f : S \to \mathbb{R} \) is continuous. Show that there must be a pair of diametrically opposite points on \( S \) at which \( f \) assumes the same value — that is, there is an \( x \in S \) with \( f(x) = f(-x) \). (Hint: Consider \( g(x) = f(x) - f(-x) \).)

2. Show that if \( f : I \to \mathbb{R} \) is continuous and has no local maxima or minima in the interior of \( I \), then \( f \) is monotonic. (\( f \) has a local maximum at \( x_0 \in I \) if there exists \( \delta > 0 \) so that \( f(x) \leq f(x_0) \) when \( |x - x_0| < \delta \); similarly for local minimum.)

3. Show that if \( f : I \to \mathbb{R} \) is continuous and one-to-one, then \( f \) is strictly monotonic. (“Strictly” means that \( \leq \) should be replaced by \( < \) in the definition of “monotonic.”)

4. Suppose \( f : I \to \mathbb{R} \) is strictly increasing. Show that if the image \( f(I) \) is either (i) connected, (ii) open, or (iii) closed, then \( f \) is continuous. (Of course the same is true if “increasing” is replaced by “decreasing.”)

5. Suppose \( f : I \to \mathbb{R} \) has the property that each \( x \in I \) has a neighborhood \( N_{r(x)}(x) \) on which \( f \) is increasing. Show that \( f \) is increasing on \( I \). (This is one of those statements that seem totally obvious but require some proof. Hint: If \( x_1, x_2 \in I \) with \( x_1 < x_2 \), the sets \( N_{r(x)/2}(x) \) with \( x_1 \leq x \leq x_2 \) form an open cover of \( [x_1, x_2] \).)