More About Taylor Polynomials

Suppose $f(x)$ has $n + 1$ continuous derivatives, and let $P_n(x)$ be the $n$th Taylor polynomial of $f$ (about $a = 0$). The estimate for the remainder $R_{n+1}(x) = f(x) - P_n(x)$ on p. 668 of Salas-Hille-Etgen (formula 11.5.3) can be restated as follows:

If $|f^{(n+1)}(x)| \leq C$ for $x$ in some interval $I$ containing 0, then

$$ |R_n(x)| \leq \frac{C|x|^{n+1}}{(n+1)!} \text{ for } x \in I. \quad (1) $$

“Big O” notation: If $g(x)$ is a function defined near $x = 0$, and there is a constant $C$ such that $|g(x)| \leq C|x|^k$ for $x$ near 0, we say that $g(x)$ is $O(x^k)$ (as $x \to 0$). $O$ bears much the same relation to $o$ as $\leq$ does to $<$. That is, “$g(x) = o(x^k)$” means that $g(x) \to 0$ faster than $x^k$ as $x \to 0$, whereas “$g(x) = O(x^k)$” means that $g(x) \to 0$ at least as fast as $x^k$ as $x \to 0$.

With this notation, according to (1) we have $R_n(x) = O(x^{n+1})$, or

$$ f(x) = P_n(x) + O(x^{n+1}). \quad (2) $$

Moreover $P_n$ is the only polynomial of degree $\leq n$ with this property. Indeed:

**Proposition 1.** Suppose $f$ has $n + 1$ continuous derivatives, and suppose $Q_n$ is a polynomial of degree $\leq n$ such that $f(x) = Q_n(x) + O(x^{n+1})$ as $x \to 0$. Then $Q_n$ is the $n$th Taylor polynomial of $f$.

**Proof:** Let $P_n$ be the $n$th Taylor polynomial of $f$. Subtracting the equations $f(x) - Q_n(x) = O(x^{n+1})$ and $f(x) - P_n(x) = O(x^{n+1})$, we obtain $P_n(x) - Q_n(x) = O(x^{n+1})$. In other words, if $P_n(x) = \sum_{0}^{n} a_kx^k$ and $Q_n(x) = \sum_{0}^{n} a_kx^k$,

$$ (a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n = O(x^{n+1}). \quad (3) $$

Setting $x = 0$, we see that $a_0 - b_0 = 0$, or $a_0 = b_0$. This being so, if we divide (3) by $x$ we get

$$ (a_1 - b_1) + (a_2 - b_2)x + \cdots + (a_n - b_n)x^{n-1} = O(x^n). $$

Setting $x = 0$, we see that $a_1 = b_1$. Now we can divide (3) by $x^2$:

$$ (a_2 - b_2) + (a_3 - b_3)x + \cdots + (a_n - b_n)x^{n-2} = O(x^{n-1}). $$

Setting $x = 0$ again, we get $a_2 = b_2$. Continuing inductively, we find that $a_k = b_k$ for all $k$, so $P_n = Q_n$. \[\blacksquare\]

Proposition 1 is useful for calculating Taylor polynomials. It shows that using the formula $a_k = f^{(k)}(0)/k!$ is not the only way to calculate $P_n$; rather, if by any means we can find a polynomial $Q_n$ of degree $\leq n$ such that $f(x) = Q_n(x) + O(x^{n+1})$, then $Q_n$ must be $P_n$. Here are two useful applications of this fact.
Taylor Polynomials of Products. Let \( P^f_n \) and \( P^g_n \) be the \( n \)th Taylor polynomials of \( f \) and \( g \), respectively. Then
\[
f(x)g(x) = \left[ P^f_n(x) + O(x^{n+1}) \right]\left[ P^g_n(x) + O(x^{n+1}) \right] = \left[ \text{terms of degree } \leq n \text{ in } P^f_n(x)P^g_n(x) \right] + O(x^{n+1}).
\]
Thus, to find the \( n \)th Taylor polynomial of \( fg \), simply multiply the \( n \)th Taylor polynomials of \( f \) and \( g \) together, discarding all terms of degree \( > n \).

Example 1. What is the 6th Taylor polynomial of \( x^3e^x \)? Solution:
\[
x^3e^x = x^3 \left[ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4) \right] = x^3 + x^4 + \frac{x^5}{2} + \frac{x^6}{6} + O(x^7),
\]
so the answer is \( x^3 + x^4 + \frac{1}{2}x^5 + \frac{1}{6}x^6 \).

Example 2. What is the 5th Taylor polynomial of \( e^x \sin x \)? Solution:
\[
e^x \sin x = \left[ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6) \right]\left[ x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7) \right]
\]
\[
= x + x^2 + x^3 \left[ \frac{1}{2} - \frac{1}{6} \right] + x^4 \left[ \frac{1}{6} - \frac{1}{6} \right] + x^5 \left[ \frac{1}{24} - \frac{1}{12} + \frac{1}{120} \right] + O(x^6),
\]
so the answer is \( x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 \).

Taylor Polynomials of Compositions. If \( f \) and \( g \) have derivatives up to order \( n + 1 \) and \( g(0) = 0 \), we can find the \( n \)th Taylor polynomial of \( f \circ g \) by substituting the Taylor expansion of \( g \) into the Taylor expansion of \( f \), retaining only the terms of degree \( \leq n \). That is, suppose
\[
f(x) = a_0 + a_1x + \cdots + a_nx^n + O(x^{n+1}).
\]
Since \( g(0) = 0 \) and \( g \) is differentiable, we have \( g(x) = O(x) \) and hence
\[
f(g(x)) = a_0 + a_1g(x) + \cdots + a_ng(x)^n + O(x^{n+1}).
\]
Now plug in the Taylor expansion of \( g \) on the right and multiply it out, discarding terms of degree \( > n \).

Example 3. What is the 16th Taylor polynomial of \( e^{x^6} \)? Solution:
\[
e^x = 1 + x + \frac{x^2}{2} + O(x^3) \quad \implies \quad e^{x^6} = 1 + x^6 + \frac{x^{12}}{2} + O(x^{18}),
\]
so the answer is \( 1 + x^6 + \frac{1}{2}x^{12} \).

Example 4. What is the 4th Taylor polynomial of \( e^{\sin x} \)? Solution:
\[
e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + \frac{\sin^4 x}{24} + O(x^5)
\]
since \( \sin x = O(x) \). Now substitute \( x - \frac{1}{6}x^3 + O(x^5) \) for \( \sin x \) on the right and multiply out, throwing all terms of degree \( > 4 \) into the “\( O(x^5) \)” trash can:
\[
e^{\sin x} = 1 + \left[ x - \frac{x^3}{6} \right] + \frac{1}{2} \left[ x^2 - \frac{x^4}{3} \right] + \frac{x^3}{6} + \frac{x^4}{24} + O(x^5),
\]
so the answer is \( 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 \).
Taylor Polynomials and l’Hôpital’s Rule. Taylor polynomials can often be used effectively in computing limits of the form \(0/0\). Indeed, suppose \(f\), \(g\), and their first \(k-1\) derivatives vanish at \(x = 0\), but their \(k\)th derivatives do not both vanish. The Taylor expansions of \(f\) and \(g\) then look like

\[
f(x) = \frac{f^{(k)}(0)}{k!} x^k + O(x^{k+1}), \quad g(x) = \frac{g^{(k)}(0)}{k!} x^k + O(x^{k+1}).
\]

Taking the quotient and cancelling out \(x^k/k!\), we get

\[
\frac{f(x)}{g(x)} = \frac{f^{(k)}(0) + O(x)}{g^{(k)}(0) + O(x)} \to \frac{f^{(k)}(0)}{g^{(k)}(0)} \text{ as } x \to 0.
\]

This is in accordance with l’Hôpital’s rule, but the devices discussed above for computing Taylor polynomials may lead to the answer more quickly than a direct application of l’Hôpital.

Example 5. What is \(\lim_{x \to 0} (x^2 - \sin^2 x)/x^2 \sin^2 x\)? Solution:

\[
\sin^2 x = \left[x - \frac{x^3}{6} + O(x^5)\right]^2 = x^2 - \frac{x^4}{3} + O(x^5),
\]

so \(x^2 \sin^2 x = x^4 + O(x^5)\), and

\[
\frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{\frac{1}{3}x^4 + O(x^5)}{x^4 + O(x^5)} = \frac{\frac{1}{3} + O(x)}{1 + O(x)} \to \frac{1}{3}.
\]

Example 6. Evaluate

\[
\lim_{x \to 1} \left[\frac{1}{\log x} + \frac{x}{x - 1}\right].
\]

Solution: Here we need to expand in powers of \(x - 1\). First of all,

\[
\frac{1}{\log x} - \frac{x}{x - 1} = \frac{x - 1 - x \log x}{(x - 1) \log x} = \frac{(x - 1) - (x - 1) \log x - \log x}{(x - 1) \log x}.
\]

Next, \(\log x = (x - 1) - \frac{1}{2} (x - 1)^2 + O((x - 1)^3)\), and plugging this into the numerator and denominator gives

\[
\frac{(x - 1) - (x - 1)^2 - [(x - 1) - \frac{1}{2} (x - 1)^2] + O((x - 1)^3)}{(x - 1)^2 + O((x - 1)^3)} = \frac{-\frac{1}{2} + O(x - 1)}{1 + O(x - 1)} \to -\frac{1}{2}.
\]
Higher Derivative Tests for Critical Points. Recall that if $f'(a) = 0$, then $f(x)$ has a local minimum (resp. maximum) at $x = a$ if $f''(a) > 0$ (resp. $f''(a) < 0$). What happens if $f''(a) = 0$? Answer: The behavior of $f$ near $a$ is controlled by its first nonvanishing derivative at $a$.

**Proposition 2.** Suppose $f$ has $k$ continuous derivatives near $a$, and $f'(a) = f''(a) = \cdots = f^{(k-1)}(a) = 0$ but $f^{(k)}(a) \neq 0$. If $k$ is even, $f$ has a local minimum or maximum at $a$ according as $f^{(k)}(a) > 0$ or $f^{(k)}(a) < 0$. If $k$ is odd, $f$ has neither a minimum nor a maximum at $a$.

**Proof:** The $(k-1)$th Taylor polynomial of $f$ about $a$ is simply the constant $f(a)$ (all the other terms are zero), so Taylor’s formula of order $k-1$ with the Lagrange form of the remainder $R_k$ becomes

$$f(x) = f(a) + \frac{f^{(k)}(c)}{k!}(x-a)^k$$

for some $c$ between $x$ and $a$.

Now, if $x$ (and hence $c$) is close to $a$, $f^{(k)}(c)$ is close to $f^{(k)}(a)$. In particular, it is nonzero and has the same sign as $f^{(k)}(a)$. On the other hand, $(x-a)^k$ is always positive if $k$ is even but changes sign at $x = a$ if $k$ is odd. Thus, if $k$ is even, $f(x) - f(a)$ is positive or negative for $x$ near $a$ according to the sign of $f^{(k)}(a)$; but if $k$ is odd, $f(x) - f(a)$ changes sign as $x$ crosses $a$. The result follows. \qed