Stirling’s Formula (and Wallis’s Formula)

The purpose of this note is to derive a remarkable, and useful, asymptotic formula for $n!$. One of the ingredients is the following nifty formula for $\pi$:

**Wallis’s Formula.**

$$
\frac{\pi}{2} = \lim_{n \to \infty} \frac{2 \cdot 4 \cdot 4 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)} = \lim_{n \to \infty} \left[ \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \right]^2 \frac{1}{2n+1}.
$$

**Proof:** From the reduction formula

$$
\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx
$$

it is easy to prove by induction that

$$
\int_0^{\pi/2} \sin^2n x \, dx = \frac{2n-1}{2n} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \quad \int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2n}{2n+1} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{2}
$$

Since $0 < \sin x < 1$ for $x \in (0, \frac{1}{2}\pi)$, we have $\sin^{2n+2} x < \sin^{2n+1} x < \sin^{2n} x$ on this interval, so the same inequalities hold for the integrals of these functions over $(0, \frac{1}{2}\pi)$:

$$
\frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n+2)} \cdot \frac{\pi}{2} < \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)} < \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \cdot \frac{\pi}{2}.
$$

Dividing through by the coefficient of $\frac{1}{2}\pi$ on the right, we get

$$
\frac{2n+1}{2n+2} \cdot \frac{\pi}{2} < \left[ \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \right]^2 \frac{1}{2n+1} < \frac{\pi}{2}.
$$

Now let $n \to \infty$ and apply the pinching theorem to obtain (1).

We return to the problem of estimating $n!$. As a first step, we observe that

$$
\log(n!) = \log 1 + \log 2 + \cdots + \log n.
$$

This looks like a Riemann sum for $\int_1^n \log x \, dx$ and suggests that the latter integral should be a good approximation to $\log(n!)$. To make this precise, for $1 \leq k < n$ we compare $\int_k^{k+1} \log x \, dx$ with the area of the trapezoid obtained by replacing the graph of $\log x$ with the straight line joining $(k, \log k)$ to $(k+1, \log(k+1))$. On the one hand,

$$
\int_k^{k+1} \log x \, dx = \left[ x \log x - x \right]_k^{k+1} = (k+1) \log(k+1) - k \log k - 1
$$

$$
= (k+1) \log k + \log \left( \frac{k+1}{k} \right) - k \log k - 1
$$

$$
= \log k + (k+1) \log(1 + k^{-1}) - 1.
$$
On the other hand, the area of the trapezoid is
\[
\frac{1}{2} \left[ \log k + \log(k + 1) \right] = \log k + \frac{1}{2} \left[ \log(k + 1) - \log k \right] = \log k + \frac{1}{2} \log(1 + k^{-1}).
\]
The difference between these two quantities is therefore
\[
a_k \equiv \int_k^{k+1} \log x \, dx - \frac{1}{2} \left[ \log k + \log(k + 1) \right] = (k + \frac{1}{2}) \log(1 + k^{-1}) - 1.
\]
To estimate how big this difference is, we use Taylor’s theorem (Salas-Hille-Etgen, Theorem 11.5.1 and formula (11.5.3)):
\[
\log(1 + t) = t - \frac{1}{2} t^2 + R_2(t), \quad \text{where, for } t > 0,
\]
\[
|R_2(t)| \leq \left[ \max_{0 \leq s \leq t} \left| - 2(s + 1)^{-3} \right| \right] \frac{t^3}{3!} = \frac{t^3}{3}.
\]
Taking \( t = k^{-1} \), we obtain
\[
a_k = (k + \frac{1}{2}) \left[ k^{-1} - \frac{1}{2} k^{-2} + R_2(k^{-1}) \right] - 1 = -\frac{1}{4} k^{-2} + (k + \frac{1}{2}) R_2(k^{-1}),
\]
and hence
\[
|a_k| \leq \frac{1}{4} k^{-2} + (k + \frac{1}{2}) \frac{1}{3} k^{-3} < \frac{1}{4} k^{-2} + (2k) \frac{1}{3} k^{-3} < k^{-2}. \quad (2)
\]
Now we add things up. The total area under the graph of \( \log x \), \( 1 \leq x \leq n \), is
\[
\int_1^n \log x \, dx = [x \log x - x]_1^n = n \log n - n + 1,
\]
the sum of the areas of the trapezoids is
\[
\sum_{k=1}^{n-1} \frac{1}{2} \log k + \log(k + 1) = \frac{1}{2} \log 1 + \log 2 + \log 3 + \cdots + \log(n - 1) + \frac{1}{2} \log n = \log(n!) - \frac{1}{2} \log n
\]
(since \( \log 1 = 0 \)), and the difference of these two quantities is
\[
\sum_{k=1}^{n-1} a_k = [n \log n - n + 1] - \log(n!) + \frac{1}{2} \log n = (n + \frac{1}{2}) \log n - n + 1 - \log(n!).
\]
In view of (2), the series \( \sum_{k=1}^{\infty} a_k \) converges by comparison to \( \sum k^{-2} \); let us denote its sum by \( A \). Thus,
\[
A - 1 = \lim_{n \to \infty} \left[ (n + \frac{1}{2}) \log n - n - \log(n!) \right] = \lim_{n \to \infty} \log \left[ \frac{n^{n+1/2} e^{-n}}{n!} \right].
\]
In other words, if we set $C = e^{1-A}$,
\[
\lim_{n \to \infty} \frac{n!}{n^{n+(1/2)}e^{-n}} = C. \tag{3}
\]

This formula says that for large $n$, $n!$ is roughly a constant $C$ times $n^{n+(1/2)}e^{-n}$. This is sufficient for most purposes, but we can actually identify the constant $C$ by using Wallis’s formula. First, observe that
\[
2 \cdot 4 \cdot 6 \cdots (2n) = 2^n[1 \cdot 2 \cdot 3 \cdots n] = 2^n n!,
\]
and
\[
1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{1 \cdot 2 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} = (2n)! \over 2^n n!.
\]
so Wallis’s formula (1) can be rewritten as
\[
\frac{\pi}{2} = \lim_{n \to \infty} \left[ \frac{2^{2n}(n!)^2}{(2n)!} \right]^2 \frac{1}{2n + 1}. \tag{4}
\]
Now, by (3),
\[
\lim_{n \to \infty} \frac{(n!)^2}{n^{2n+1}e^{-2n}} = C^2 \quad \text{and} \quad \lim_{n \to \infty} \frac{(2n)!}{(2n)^{2n+(1/2)}e^{-2n}} = C.
\]
Taking the quotient of these relations gives
\[
C = \lim_{n \to \infty} \frac{(n!)^2n^{-2n-1}e^{2n}}{(2n)!^2n^{-2n-(1/2)}e^{2n}2n} = \lim_{n \to \infty} \frac{2^{2n+(1/2)}(n!)^2}{n^{1/2}(2n)!},
\]
and hence by (4),
\[
C = \lim_{n \to \infty} \sqrt{\frac{4n+2}{n}} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n+1}} = \sqrt{4\sqrt{\frac{\pi}{2}}} = \sqrt{2\pi}.
\]
Combining this with (3), we have proved:

**Stirling’s Formula.**
\[
\lim_{n \to \infty} \frac{n!}{n^ne^{-n}\sqrt{2\pi n}} = 1.
\]