Second-Order Linear Differential Equations

We consider the general second-order linear differential equation

\[ N(x)y'' + P(x)y' + Q(x)y = G(x). \]  

(1)

We shall say that an equation of this sort is in **standard form** if the coefficient of \( y'' \) is 1:

\[ y'' + p(x)y' + q(x)y = g(x). \]  

(2)

(1) can be put into the form (2) by dividing through by \( N(x) \), and for the purposes of this discussion we shall assume that this has been done. The equation (1) (or (2)) is said to be **homogeneous** or **reduced** if \( G = 0 \) (or \( g = 0 \)), **inhomogeneous** if not.

The discussion in Salas-Hille-Etgen, Sections 18.4–5, is limited to equations with constant coefficients (i.e., where \( N, P, Q \) in (1) or \( p, q \) in (2) are constants), but most of the general structural features of the set of solutions carry over to the variable-coefficient case too. The purpose of these notes is to present the theory in that generality.

We assume that the functions \( p, q, \) and \( g \) in (2) are continuous on an interval \( I = (\alpha, \beta) \) (which might be the whole line), and we seek solutions \( y \) on this interval. (If the equation is given in the form (1), this means that we need the functions \( N, P, Q, \) and \( G \) to be continuous on \( I \) and \( N \) to be nonvanishing on \( I \). Points where \( N \) vanishes, or where \( p \) and/or \( q \) have singularities, are called **singular points** of the equation (1) or (2). The general theory does not apply on an interval that contains singular points — but there is more to be said about them.)

It is convenient to denote the left hand side of (2) by \( L[y] \); thus \( L \) is the **differential operator**

\[ L = \frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x). \]

\( L \) is a linear operator; that is,

\[ L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] \quad (c_1, c_2 \text{ constants}). \]  

(3)

In particular, if \( L[y_1] = L[y_2] = 0 \) then \( L[c_1y_1 + c_2y_2] = 0 \) for all constants \( c_1 \) and \( c_2 \). (This fact is often called the **superposition principle** in the physics literature.)

The fundamental existence and uniqueness theorem is as follows:

**Theorem 1.** Suppose \( p, q, \) and \( g \) are continuous on the interval \( I, \) and \( x_0 \in I. \) For any numbers \( a \) and \( b \) there is a unique solution \( y \) of the equation (1) that satisfies \( y(x_0) = a \) and \( y'(x_0) = b. \)

The proof of this theorem is beyond the scope of this course. It can be found, for example, in Appendix 5 of *Fourier Analysis and its Applications* by G. B. Folland, or (usually in a more general form) in many advanced books on ordinary differential equations.

Theorem 1 tells us that a solution \( y \) of (2) is completely determined by the two constants \( y(x_0) \) and \( y'(x_0), \) which may be freely chosen. We say that the **solution space** for (2) is two-dimensional.
We begin by analyzing the homogeneous equation $L[y] = 0$; we shall return to the general case later. As we observed above, if $y_1$ and $y_2$ are solutions of $L[y] = 0$ then so is $c_1y_1 + c_2y_2$ for any constants $c_1$ and $c_2$. Thus, if we can find two solutions that are genuinely different (one is not a constant multiple of the other), we obtain a two-parameter family of solutions this way, and according to the preceding paragraph, there is hope that all solutions will belong to this family. More precisely, the question is whether we can find $c_1$ and $c_2$ so that

$$c_1y_1(x_0) + c_2y_2(x_0) = a,$$
$$c_1y_1'(x_0) + c_2y_2'(x_0) = b,$$  

(4)

for any given numbers $a$ and $b$. This is a system of two linear equations for the two unknowns $c_1$ and $c_2$, and it has a unique solution for any $a$ and $b$ if and only if the determinant of the coefficient matrix, $y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)$, is nonzero. This determinant plays a central role in the theory; it is called the **Wronskian** of $y_1$ and $y_2$ at $x_0$:

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$  

(5)

**Theorem 2.** Suppose $p$ and $q$ are continuous on the interval $I$, and $y_1$ and $y_2$ are solutions of $L[y] = 0$ on $I$, The following conditions are equivalent:

a. $W(y_1, y_2)(x_0) \neq 0$ for every point $x_0 \in I$.

b. $W(y_1, y_2)(x_0) \neq 0$ for some point $x_0 \in I$.

c. Every solution of $L[y] = 0$ on $I$ is of the form $c_1y_1 + c_2y_2$.

d. $y_1$ and $y_2$ are not constant multiples of each other.

**Proof:**

(a) $\implies$ (b): This is trivial.

(b) $\implies$ (c): Suppose $L[y] = 0$. If (b) holds, we can solve the equations (4) with $a = y(x_0)$ and $b = y'(x_0)$. But then $y$ and $c_1y_1 + c_2y_2$ both solve the differential equation and both have the same value and the same slope at $x_0$. By the uniqueness in Theorem 1, they are equal.

(c) $\implies$ (a): By Theorem 1, if $x_0 \in I$ there is a solution of $L[y] = 0$ with any specified values of $y(x_0)$ and $y'(x_0)$. If (c) holds, so that $y = c_1y_1 + c_2y_2$, this means that we can always solve (4) for arbitrary $a$ and $b$. Hence the determinant $W(y_1, y_2)(x_0)$ is nonzero.

(d) $\iff$ (a): We shall show that (d) is false (i.e., $y_2 = cy_1$) precisely when (a) is false (i.e., $W(y_1, y_2)(x_0) = 0$ for some $x_0$). Clearly, if $y_2 = cy_1$ then $W(y_1, y_2) = y_1 cy_1' - cy_1 y_1' \equiv 0$. On the other hand, if $W(y_1, y_2)(x_0) = 0$ then $y_2(x_0)/y_1(x_0) = y_2'(x_0)/y_1'(x_0)$. Denoting this common value by $c$, we have $y_2(x_0) = cy_1(x_0)$ and $y_2'(x_0) = cy_1'(x_0)$. By the uniqueness in Theorem 1, $y_2 = cy_1$. (The little extra argument to dispose of the case where $y_1(x_0) = 0$ or $y_1'(x_0) = 0$ is left to the reader.)

A pair $y_1, y_2$ of solutions of $L[y] = 0$ that satisfies the conditions in Theorem 2 is called a **fundamental set of solutions**, and condition (c) says that if we know a fundamental set of solutions then we know all solutions.

How do we find a fundamental set of solutions? This is a hard problem in general. For constant-coefficient equations ($N, P, Q$ all constant in (1)) it is easy and is discussed in Section 18.4 of Salas-Hille-Etgen. We will also briefly discuss a method that works for many of the most important variable-coefficient equations, namely, constructing solutions by means of power series.
Now let us turn to the inhomogeneous equation \( L[y] = g \). The first observation is that if \( y \) satisfies \( L[y] = g \) and \( \tilde{y} \) satisfies \( L[\tilde{y}] = 0 \), then \( L[y + \tilde{y}] = L[y] + L[\tilde{y}] = g + 0 = g \). Thus, if we can find a fundamental set of solutions of the homogeneous equation \( L[y] = 0 \) and then find one solution \( y_p \) (\( p \) for “particular”) of the inhomogeneous equation \( L[y] = g \), we immediately get a two-parameter family of solutions of \( L[y] = g \), namely, \( y = y_p + c_1 y_1 + c_2 y_2 \). Moreover, the constants \( c_1 \) and \( c_2 \) can be chosen to satisfy any initial conditions \( y(x_0) = a \), \( y'(x_0) = b \) just as in the homogeneous case. As before, it follows that every solution of \( L[y] = g \) is of the form \( y = y_p + c_1 y_1 + c_2 y_2 \).

Thus, the first step in solving \( L[y] = g \) is to find a fundamental set of solutions to \( L[y] = 0 \). Once we have done this, there is a straightforward procedure, called variation of parameters, for finding a particular solution of the inhomogeneous equation \( L[y] = g \). It is described in Section 18.5 of Salas-Hille-Etgen. The book considers only equations with constant coefficients, but the technique and the results work perfectly well in the general case. To wit, given a fundamental set of solutions \( y_1, y_2 \) of \( L[y] = 0 \), we look for a solution \( y \) of \( L[y] = g \) in the form

\[
y = v_1 y_1 + v_2 y_2
\]

(6)

where \( v_1, v_2 \) are functions of \( x \). This will work provided we choose \( v_1, v_2 \) to satisfy

\[
y_1 v_1' + y_2 v_2' = 0, \quad y_1' v_1' + y_2' v_2' = g.
\]

So we solve this pair of equations for \( v_1' \) and \( v_2' \), integrate to get \( v_1 \) and \( v_2 \), and substitute the result into (6). The end result is the formula (18.5.6) for \( y \) in the box on p. 1125 of the book. However, instead of using this formula, it may be just as easy to follow the procedure just outlined.

Warning. There is one possible source of error: to apply variation of parameters, the equation needs to be in the standard form (2). If it is in the form (1), the input for the variation-of-parameters machine is not \( G(x) \) but \( G(x)/N(x) \).

Another method for finding a particular solution of \( L[y] = g \), known as “undetermined coefficients” or (perhaps better) “judicious guessing,” is also described in Section 18.5. It works only for equations with constant coefficients and only when the function \( g \) is of certain special forms (polynomials, sines and cosines, and exponentials, or sums and products of these), but it is usually easier than variation of parameters when it does work. In brief, the idea is that derivatives of functions of these forms are other functions of the same form, so when \( g \) is of one of these forms, one should be able to find a particular solution of \( L[y] = g \) of the same form. To be specific:

If \( g \) is of the form

\[
a \text{ polynomial of degree } n
\]

\[e^{ax}
\]

\[
\cos bx \text{ or } \sin bx
\]

\[x^n e^{ax}
\]

\[e^{ax} \cos bx \text{ or } e^{ax} \sin bx
\]

...and so forth. Exception: if a term in the proposed solution is a solution of the homogeneous equation, the proposed solution should be multiplied by \( x \).

Still another method that works nicely for these cases is the use of the Laplace transform, for which there will be another set of notes.