The Laplace Transform

Here we give a very brief introduction to the Laplace transform, a useful device for solving certain kinds of differential equations.

Suppose \( f(t) \) is a piecewise continuous function defined for \( t \geq 0 \) which grows at most exponentially as \( t \to \infty \), that is, for which there exist constants \( C > 0 \) and \( a > 0 \) such that

\[
|f(t)| \leq Ce^{at}.
\]

The \textit{Laplace transform} of \( f \) is the function \( F(s) \) defined for \( s > a \) by

\[
F(s) = \int_0^\infty e^{-st} f(t) \, dt.
\]

(In view of (1), the integral converges for \( s > a \) by comparison to \( \int_0^\infty e^{(a-s)t} \, dt \).) The \textit{Laplace transform} itself is the operation that takes \( f \) to \( F \). When it is useful to indicate this operation explicitly, we denote it by \( \mathcal{L} \) and write

\[
F = \mathcal{L}[f] \quad \text{or} \quad F = \mathcal{L}[f(t)].
\]

Let’s compute a few examples to get started:

\[
\mathcal{L}[1] = \int_0^\infty e^{-st} \, dt = \left. -\frac{1}{s} e^{-st} \right|_0^\infty = \frac{1}{s} \quad (s > 0).
\]

\[
\mathcal{L}[\cos bt] = \int_0^\infty e^{-st} \cos bt \, dt = \left. \frac{-e^{-st}(s \sin bt + \cos bt)}{s^2 + b^2} \right|_0^\infty = \frac{b}{s^2 + b^2} \quad (s > 0).
\]

\[
\mathcal{L}[\sin bt] = \int_0^\infty e^{-st} \sin bt \, dt = \left. \frac{-e^{-st}(s \cos bt - \sin bt)}{s^2 + b^2} \right|_0^\infty = \frac{s}{s^2 + b^2} \quad (s > 0).
\]

The utility of the Laplace transform is due to some general facts about the way it operates. First, it is obvious that the Laplace transform is a \textit{linear} operation:

\[
\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2].
\]

More interestingly, the Laplace transform converts differentiation into a simple algebraic operation: Indeed, suppose \( f \) is differentiable, \( f' \) is continuous, and both satisfy the estimate (1), so that \( \mathcal{L}[f] \) and \( \mathcal{L}[f'] \) are both defined for \( s > a \). Then we have:

\[
\text{If } F = \mathcal{L}[f], \text{ then } \mathcal{L}[f'] = sF(s) - f(0).
\]

The proof is a simple integration by parts:

\[
\mathcal{L}[f'](s) = \int_0^\infty e^{-st} f'(t) \, dt = e^{-st} f(t) \big|_0^\infty - \int_0^\infty (-se^{-st}) f(t) \, dt = -f(0) + s \int_0^\infty e^{-st} f(t) \, dt = -f(0) + \mathcal{L}[f](s).
\]

\((e^{-st} f(t) \text{ vanishes at } t = \infty \text{ because } f \text{ satisfies (1)} \text{ and } s > a.\)
If \( f \) has more derivatives, we can apply this result repeatedly to get formulas for their Laplace transforms in terms of \( F = \mathcal{L}[f] \). For the second derivative, we have

\[
L[f''](s) = sL[f'](s) - f'(0) = sL[f](s) - f(0) - f'(0) = s^2F(s) - sf(0) - f'(0).
\] (8)

The pattern should now be clear: the formula for \( L[f^{(n)}] \) is

\[
L[f^{(n)}](s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).
\] (9)

The next general formula is, in a sense, dual to (7). Just as \( \mathcal{L} \) converts differentiation into multiplication by \( s \) (with an adjustment for the initial value \( f(0) \)), it converts multiplication by \( t \) into differentiation (with a minus sign):

\[
If \ F = \mathcal{L}[f], \ then \ \mathcal{L}[tf(t)] = -F'.
\] (10)

The proof is easy, if we take for granted the fact that differentiation with respect to \( s \) commutes with integration with respect to \( t \) (not completely obvious, but true):

\[
F'(s) = \frac{d}{ds} \int_0^\infty e^{-st}f(t) \, dt = \int_0^\infty \frac{d(e^{-st})}{ds}f(t) \, dt = -\int_0^\infty e^{-st}tf(t) \, dt = -\mathcal{L}[tf(t)](s).
\]

Here’s one more useful general formula:

\[
If \ F = \mathcal{L}[f], \ then \ \mathcal{L}[e^{at}f(t)] = F(s - a).
\] (11)

This is obvious:

\[
\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t) \, dt = \int_0^\infty e^{(a-s)t}f(t) \, dt = F(s - a).
\]

This formula also has a dual, which I’ll let you figure out in a homework problem.

Combining (10) and (11) with (3)–(5), we can enlarge our dictionary of Laplace transforms. For example,

\[
\mathcal{L}[e^{at}] = \frac{1}{s-a}
\] (12)

\[
\mathcal{L}[t^ne^{at}] = (-1)^n \frac{d^n}{ds^n} \frac{1}{s-a} = \frac{n!}{(s-a)^{n+1}}
\] (13)

\[
\mathcal{L}[e^{at}\cos bt] = \frac{s-a}{(s-a)^2 + b^2}
\] (14)

\[
\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2 + b^2}
\] (15)

One final, and absolutely crucial, general fact is that a function is completely determined by its Laplace transform. That is, if \( f \) and \( g \) are piecewise continuous functions that grow at most exponentially at infinity and \( \mathcal{L}[f] = \mathcal{L}[g] \), then \( f = g \). We shall not attempt to prove this here.
Let's see how the Laplace transform can be used to solve some differential equations.

**Example 1.** Let's start with a problem that we already know how to solve: the homogeneous second-order constant-coefficient equation with initial conditions

\[ y'' + ay' + by = 0, \quad y(0) = y_0, \quad y'(0) = y'_0. \]

Applying the Laplace transform turns this into the *algebraic* equation

\[ s^2 Y - y_0 s - y'_0 + a(sY - y_0) + bY = 0 \]

for \( Y = \mathcal{L}[y] \). The solution is

\[ Y = \frac{y_0 s + y'_0 + ay_0}{s^2 + as + b}. \]  \hspace{1cm} (16)

Note that the quadratic polynomial \( s^2 + as + b \) is the same one that enters into the method of solution in Section 18.4 of the text. If it has two real roots \( r_1, r_2 \), we can do a partial fraction decomposition of (16) to obtain

\[ Y = \frac{c_1}{s - r_1} + \frac{c_2}{s - r_2}, \]

where \( c_1 \) and \( c_2 \) depend on the initial values \( y_0 \) and \( y'_0 \). Undoing the Laplace transform by (12), we obtain

\[ y = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \]

If the quadratic in (16) has complex roots \( \alpha \pm i\beta \), (16) can be rewritten in the form

\[ Y = \frac{c_1(s - \alpha) + c_2\beta}{(s - \alpha)^2 + \beta^2} \quad (c_1 = y_0, \quad c_2 = (y'_0 + (\alpha + \alpha)y_0)/\beta), \]

and then from (14) and (15),

\[ y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t. \]

Finally, if \( a^2 = 4b \) so that the quadratic is \( (s - r)^2 \) with \( r = a/2 \), we rewrite (16) as

\[ Y = \frac{c_1(s - r) + c_2}{(s - r)^2} = \frac{c_1}{s - r} + \frac{c_2}{(s - r)^2} \quad (c_1 = y_0, \quad c_2 = y'_0 + (a + r)y_0), \]

and then from (13),

\[ y = c_1 e^{rt} + c_2 te^{rt}. \]

Thus we recover the results of Section 18.4; the nice thing is that the dependence of the constants \( c_1 \) and \( c_2 \) on the initial conditions comes out of the computation automatically.
Example 2. The same idea works for inhomogeneous constant-coefficient equations $L[y] = g$, provided that we can handle the Laplace transform of $g$. This is certainly the case when $g$ is of one of the types for which the method of judicious guessing works, which are the same types whose Laplace transforms have been computed above. To be specific, let’s solve

$$y'' + 5y' + 6 = \sin 5t, \quad y(0) = y'(0) = 0.$$

Applying the Laplace transform, with these initial conditions, yields

$$(s^2 + 5s + 6)Y = \frac{5}{s^2 + 25},$$

so that

$$Y = \frac{5}{(s^2 + 25)(s + 2)(s + 3)}.$$

A rather gruesome partial-fractions calculation yields

$$Y = \frac{1}{986} \left[ \frac{170}{s + 2} - \frac{145}{s + 3} - \frac{25s + 95}{s^2 + 25} \right],$$

so that, by (12), (14), and (15),

$$y = \frac{1}{986} \left[ 170e^{-2t} - 145e^{-3t} - 25\cos 5t - 19\sin 5t \right].$$

Example 3. The Laplace transform is also an efficient tool for solving systems of simultaneous differential equations. For example, here is a system of two equations in two unknown functions $y_1$ and $y_2$, with initial conditions:

$$y_1' = -8y_1 - 9y_2, \quad y_2' = 4y_1 + 4y_2; \quad y_1(0) = c_1, \quad y_2(0) = c_2.$$

Applying the Laplace transform yields a system of two linear algebraic equations for $Y_1 = \mathcal{L}[y_1]$ and $Y_2 = \mathcal{L}[y_2]$:

$$sY_1 - c_1 = -8Y_1 - 9Y_2, \quad sY_2 - c_2 = 4Y_1 + 4Y_2,$$

or

$$(s + 8)Y_1 + 9Y_2 = c_1, \quad -4Y_1 + (s - 4)Y_2 = c_2.$$

Solving these together yields

$$Y_1 = \frac{(s - 4)c_1 - 9c_2}{(s + 2)^2} = \frac{c_1}{s + 2} - \frac{6c_1 + 9c_2}{(s + 2)^2}, \quad Y_2 = \frac{4c_1 + (s + 8)c_2}{(s + 2)^2} = \frac{c_2}{s + 2} + \frac{4c_1 + 6c_2}{(s + 2)^2}.$$

Therefore, by (12) and (13),

$$y_1 = c_1e^{-2t} - (6c_1 + 9c_2)te^{-2t}, \quad y_2 = c_2e^{-2t} + (4c_1 + 6c_2)te^{-2t}.$$
Example 4. The Laplace transform is less effective in dealing with variable-coefficient equations, but sometimes it can be used there too. As an example, let us derive the fundamental set of solutions of the equation

\[ ty'' - (t + 2)y' + 2y = 0 \]  

(17)

that were used in Problem 2, Assignment 6 (with \( t \) relabeled as \( x \)). Applying the Laplace transform and setting \( Y = \mathcal{L}[y] \), \( y_0 = y(0) \), and \( y'_0 = y'(0) \), we have

\[ -\frac{d}{ds}[s^2Y - sy_0 - y'_0] + \frac{d}{ds}[sY - y_0] - 2[sY - y_0] + 2Y = 0. \]

The first term is \(-2sY - s^2Y' + y_0\) and the second one is \(Y + sY'\), so

\[(s - s^2)Y' + (3 - 4s)Y = -3y_0, \quad \text{or} \quad Y' + \frac{4s - 3}{s^2 - s}Y = \frac{3y_0}{s^2 - s}.\]

(Note that \(y'_0\) has dropped out! The point \( t = 0 \) is a singular point for equation (17), so we cannot specify initial conditions in the usual way.) This is a first-order linear equation for \( Y \). Since

\[ \frac{4s - 3}{s^2 - s} = \frac{s + 3(s - 1)}{s(s - 1)} = \frac{1}{s - 1} + \frac{3}{s}, \]

its integrating factor is \( \exp[\ln(s - 1) + 3 \ln s] = s^3(s - 1) \). Multiplying through by this yields

\[ [s^3(s - 1)Y]' = 3y_0s^2, \]

so

\[ Y = \frac{y_0s^3 + C}{s^3(s - 1)} = \frac{y_0}{s - 1} + \frac{C}{s^3(s - 1)} = \frac{y_0}{s - 1} + C \left[ \frac{1}{s - 1} - \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} \right]. \]

(Some partial-fractions algebra has been skipped here.) Inverting the Laplace transform now gives

\[ y = (y_0 + C)e^t - C(1 + t + \frac{1}{2}t^2). \]

Taking \( y_0 = 1, C = 0 \) gives the solution \( e^t \), and taking \( y_0 = 2, C = -2 \) gives the solution \( t^2 + 2t + 2 \), as claimed in the homework problem.

To harness the full power of the Laplace transform, one needs a way of inverting it that is more systematic than merely compiling a table of functions \( f \) and their Laplace transforms \( \mathcal{L}[f] \) and reading it from right to left, which is essentially what we have done. However, the general formula for the inverse Laplace transform involves the theory of functions of a complex variable, which is beyond the scope of this course. (It is this general inversion formula that establishes the fact that if \( \mathcal{L}[f] = \mathcal{L}[g] \) then \( f = g \), stated before Example 1.) If you’re interested, though, one reference is *Fourier Analysis and its Applications* by Folland.