The Complex Exponential and Trig Functions

Complex Numbers. For many, many purposes in advanced mathematics it is useful to enlarge the real number system to include the square roots of -1, conventionally denoted by $\pm i$. On combining these with the real numbers by addition and multiplication, one obtains the system $\mathbb{C}$ of *complex numbers*, that is, the set of all quantities of the form $a + ib$ where a and b are real numbers. a and b are called the *real* and *imaginary parts* of $a + ib$.

The four operations of arithmetic extend easily to the complex numbers. Addition and subtraction are child's play:

$$(a + ib) \pm (c + id) = (a + c) \pm i(b + d).$$

Multiplication is also easy, using the fact that $i^2 = -1$:

$$(a + ib)(c + id) = ac + ibc + iad + i^2 bd = (ac - bd) + i(bc + ad).$$

Division is slightly less obvious, but it can be reduced to real division by a trick that is familiar from manipulations with square roots of real numbers:

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

The complex numbers tend to be surrounded with an air of mystery that is totally undeserved. There are two psychological barriers to using them: the fact that they don't fit on the real number line, and the name "imaginary" that is traditionally attached to them. The solution to the first problem finally dawned on people about 200 years ago: complex numbers should be represented not as points on a line but as points in a plane. That is, we identify the complex number $a+ib$ with the point (a, b) in the Cartesian plane. Once this geometric picture is at hand, the word "imaginary" loses its sting: complex numbers are no more (or less) imaginary than real numbers.

The *complex conjugate* of the complex number $z = a+ib$ is the number $a-ib$ (geometrically, the reflection of $a + ib$ in the x-axis). Mathematicians like to denote the conjugate of z by $\overline{z}$; physicists tend to prefer z^*. The *absolute value* or *modulus* of $z = a + ib$ is

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2},$$

which is geometrically the distance from z to the origin. It's an important fact that for any complex numbers $z = a + ib$ and $w = c + id$ we have

$$|zw| = |z|\,|w|,$$

which is just a restatement of the algebraic identity

$$(ac - bd)^2 + (bc + ad)^2 = (a^2 + b^2)(c^2 + d^2).$$

A series $\sum c_k$ of complex numbers $c_k = a_k + ib_k$ (a_k and b_k real) is said to converge if the corresponding series of real and imaginary parts, $\sum a_k$ and $\sum b_k$, both converge. In this case the sum of the series is the obvious thing:

$$c_k = a_k + ib_k \qquad \Longrightarrow \qquad \sum c_k = \sum a_k + i \sum b_k.$$

Since $|a| \le \sqrt{a^2+b^2}$ and $|b| \le \sqrt{a^2+b^2}$, we see that

$$\begin{aligned} \sum |c_k| \text{ converges } &\Longrightarrow \sum |a_k| \text{ and } \sum |b_k| \text{ converge} \\ &\Longrightarrow \sum a_k \text{ and } \sum b_k \text{ converge } \Longrightarrow \sum c_k \text{ converges.} \end{aligned}$$

Thus the fact that *an absolutely convergent series converges* continues to hold for complex series.

The Complex Exponential Function. The series $\sum_0^\infty z^n/n!$ converges absolutely for any complex number z, by the ratio test (since $|z^n| = |z|^n$). This series equals e^z when z is real, and we use it to *define* e^z for z complex:

$$e^z = \sum_0^\infty \frac{z^k}{k!} \qquad (z \in \mathbb{C}). \tag{1}$$

The main step in dispelling the mystery of this complex exponential function is showing that it still obeys the basic law of exponents.

Proposition. *For any complex numbers z and w,*

$$e^z e^w = e^{z+w}. \tag{2}$$

Proof: We have

$$e^z e^w = \left(\sum_{j=0}^\infty \frac{z^j}{j!}\right)\left(\sum_{k=0}^\infty \frac{w^k}{k!}\right) = \sum_{j,k=0}^\infty \frac{z^j w^k}{j!k!}.$$

We sum the double series on the right by first adding up the terms where $j+k$ is a fixed number n (that is, j runs from 0 to n and $k = n-j$), and then summing over all possible n (that is, $n = 0, 1, 2, \ldots$):

$$e^z e^w = \sum_{n=0}^\infty \sum_{j=0}^n \frac{z^j w^{n-j}}{j!(n-j)!} = \sum_{n=0}^\infty \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} z^j w^{n-j}.$$

By the binomial theorem, the sum over j gives $(z+w)^n$, so

$$e^z e^w = \sum_{n=0}^\infty \frac{(z+w)^n}{n!} = e^{z+w}. \qquad \blacksquare$$

(Actually, these manipulations with double series need some justification. I can give you a reference for the full proof if you're interested.)

Now, if $z = x + iy$, by (2) we have $e^z = e^x e^{iy}$. We know what e^x is; what about e^{iy}? Well since

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1, \ldots, i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i, \ldots,$$

from (1) we obtain

$$e^{iy} = \sum_0^\infty \frac{i^k y^k}{k!} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right),$$

or in other words,

$$e^{iy} = \cos y + i \sin y. \tag{3}$$

This marvelous formula, due to Euler, reveals the deep connection between exponential and trigonometric functions.

Replacing y by $-y$, we see that

$$e^{-iy} = \cos(-y) + i\sin(-y) = \cos y - i \sin y. \tag{4}$$

Adding and subtracting (3) and (4), we obtain formulas for the trig functions in terms of exponentials:

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \qquad \sin y = \frac{e^{iy} - e^{-iy}}{2i}. \tag{5}$$

These equations explain the formal similarity between trig and hyperbolic functions:

$$\cosh(iy) = \cos y, \qquad \sinh(iy) = i \sin y.$$

They also lead to an easy derivation of the addition formulas for sine and cosine:

$$\begin{aligned} \cos(a \pm b) &= (\cos a)(\cos b) \mp (\sin a)(\sin b), \\ \sin(a \pm b) &= (\sin a)(\cos b) \pm (\cos a)(\sin b). \end{aligned} \tag{6}$$

Namely, use (5) to express the factors on the right in terms of $e^{\pm ia}$ and $e^{\pm ib}$, multiply out according to (2), and simplify to obtain the expressions on the left.

Trig Functions Done Right. The high-school definitions of sine and cosine are unacceptably vague because they involve measuring of an angle without giving a precise algorithm for doing so. We are now in a position to remedy this defect. Namely, we take the Taylor expansions

$$\cos x = \sum_0^\infty \frac{(-1)^k x^{2k}}{(2k)!}, \qquad \sin x = \sum_0^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \tag{7}$$

or equivalently the formulas (5), as a *definition* of sine and cosine. This leads immediately to the differentiation formulas

$$\cos' = -\sin, \qquad \sin' = \cos \tag{8}$$

and also to the addition formulas (6), as explained above. From these identities, all the other properties of trig functions are easy to derive, for example,

$$\cos^2 x + \sin^2 x = \cos(x - x) = \cos 0 = 1. \tag{9}$$

The one thing that is not so obvious is the connections of cos and sin with the number π, and in particular their periodicity properties. These can be derived as follows. First, observe that the series $\sum_0^\infty (-1)^k 2^{2k}/(2k)!$ for $\cos 2$ is an alternating series whose terms decrease in size beginning with $k = 1$; so by the alternating series test,

$$\cos 2 = 1 - \frac{2^2}{2!} = -1 \text{ with error less than } \frac{2^4}{4!} = \frac{2}{3},$$

and in particular $\cos 2 < 0$. Since $\cos 0 = 1 > 0$, by the intermediate value theorem there is at least one number $a \in (0, 2)$ such that $\cos a = 0$. Call the *smallest* such number [of course it turns out that there is only one] $\frac{1}{2}\pi$. (This is to be taken as a *definition* of π, from which the usual one as the ratio of the circumference to the diameter of a circle can then be derived by calculus.) Now $\cos x > 0$ for $x \in (0, \frac{1}{2}\pi)$, so by (8) $\sin x$ is increasing for $x \in (0, \frac{1}{2}\pi)$. Also $\sin 0 = 0$, so $\sin(\frac{1}{2}\pi) > 0$, and by (9), $\sin^2(\frac{1}{2}\pi) = 1 - \cos^2(\frac{1}{2}\pi) = 1$. Conclusion: $\sin(\frac{1}{2}\pi) = 1$. Now use the addition formulas:

$$\begin{aligned} \cos(x + \tfrac{1}{2}\pi) &= (\cos x)(\cos \tfrac{1}{2}\pi) - (\sin x)(\sin \tfrac{1}{2}\pi) = 0 \cdot \cos x - 1 \cdot \sin x = -\sin x, \\ \sin(x + \tfrac{1}{2}\pi) &= (\sin x)(\cos \tfrac{1}{2}\pi) + (\cos x)(\sin \tfrac{1}{2}\pi) = 0 \cdot \sin x + 1 \cdot \cos x = \cos x. \end{aligned}$$

Iterating these identities gives

$$\begin{aligned} \cos(x + \pi) &= \cos(x + \tfrac{1}{2}\pi + \tfrac{1}{2}\pi) = -\sin(x + \tfrac{1}{2}\pi) = -\cos x, \\ \sin(x + \pi) &= \sin(x + \tfrac{1}{2}\pi + \tfrac{1}{2}\pi) = \cos(x + \tfrac{1}{2}\pi) = -\sin x, \end{aligned}$$

and hence

$$\cos(x + 2\pi) = \cos(x + \pi + \pi) = \cos x, \qquad \sin(x + 2\pi) = \sin(x + \pi + \pi) = \sin x.$$

Logarithms and Powers of Complex Numbers. If z is a nonzero complex number, a *logarithm* of z is a complex number w such that $e^w = z$. Logarithms can easily be found by writing $z = x + iy$ in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$):

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = e^{\log r + i\theta},$$

so $\log r + i\theta$ is a logarithm of z. We say *a* logarithm rather than *the* logarithm because the angle θ is only determined up to multiples of 2π, so each z has infinitely many logarithms.

If we fix a logarithm of z, call it $\log z$, we can then define complex powers of z by

$$z^a = e^{a \log z},$$

the quantity on the right being defined by (1). Different choices of $\log z$ will usually yield different answers. If a is an integer there is no ambiguity; if $a = p/q$ with p, q integers then there are q possibilities (each nonzero complex number has q distinct qth roots); and if a is irrational there are infinitely many. But how to sort this all out sensibly is a subject for another course ...