Abelian Varieties - Farbod Shokrieh - Spring 2016
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1 Lecture - 01/28/2016

Origins of theory of abelian varieties comes from a basic question in calculus! We know we can compute integrals of the form

$$\int \frac{1}{\sqrt{1-x^2}}dx$$

by trig substitution, but not integrals

$$\int \frac{1}{\sqrt{f(x)}}dx$$

for f a polynomial with $\deg f \geq 3$. However, even though people couldn't compute these integrals, they could see that there were identities of the form

$$\int_0^a \frac{1}{\sqrt{f(x)}}dx + \int_0^b \frac{1}{\sqrt{f(x)}}dx = \int_0^{a*b} \frac{1}{\sqrt{f(x)}}dx$$

for some number $a * b$ obtained from a and b.

These are useful in the following areas:

1. Number theory - most of the serious things we know how to do in number theory involve working with moduli of abelian varieties (Fermat's last theorem, Faltings' theorem, etc.); also come up in rationality questions in class field theory.

2. Dynamical systems - solutions to certain Hamilton systems.

3. Algebraic geometry - they are "easy" algebraic geometric objects; if we're given a variety X it's hard to understand, but we can get a handle on it by associating a canonical abelian variety $A(X)$ to it (Picard, Albanese, intermediate Jacobian); we can then use "linear algebra" to study $A(X)$.

4. Physics - theta functions solve heat equations; also gets used in string theory.

For the most part we'll work over the base field $K = \mathbb{C}$; we see lots of very interesting ideas in this case, and we don't need any particularly hard theory to get a handle on it. One reason the theory over $\mathbb{C}$ is important is that if A is an abelian variety, then in a precise sense A is just $\mathbb{C}^g/\Lambda$ for a lattice Λ, i.e. A is a torus; so in other words we have

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0$$

where $\Lambda = \pi_1(A) \cong \mathbb{Z}^{2g}$.

Subtle remark: This identification makes sense in the "analytic category" but not in the algebraic category; the map $\mathbb{C}^g \to A$ is not algebraic. So we can't study abelian varieties in this way solely through algebraic methods. In the analytic setting it's "easy" to understand line bundles, theta functions, etc. by going to $\mathbb{C}^g$. (Remark: You can make sense of an analytification in nonarchimedean settings too, by using Berkovich spaces or formal schemes; this requires a lot more background but provides many important results). Fortunately, there are some things from the complex-analytic setting which can be mimicked in the algebraic setting (e.g. the lattice Λ can be related to the Tate module) and by using those algebraic analogues you can take the complex-analytic results over $\mathbb{C}$ and try to reproduce them over other fields.

Some more advanced things we could cover in detail later in the class (depending on audience interest):

- The theory over general fields.
- Theta functions.
- Neron models.
- Non-archimedean uniformizations.
- Moduli and compactifications.

- Heights and metrized line bundles.
- Degenerating families.

Now, getting to actual math. In scheme-theoretic language - for K any field, a K-variety is a geometrically integral K-scheme of finite type, and an abelian variety is a proper K-variety endowed with a structure of a K-group scheme. Remark: We will be able to prove that abelian varieties are automatically abelian (easy to show) and projective (much harder).

A theorem that shows us abelian varieties are ubiquitous. More generally we can define an algebraic group G over K as a connected, smooth k-group scheme. Examples: (1) affine algebraic groups (automatically a subgroup of GL_n/K, i.e. a linear algebraic group) and (2) abelian varieties. Chevalley's theorem says these are the only building blocks: if K is a perfect field and G is an algebraic group over K, there exists a unique short exact sequence

$$0 \to H \to G \to A \to 0$$

with H linear and $A = G/H$ abelian.

Weierstrass: Looked at $\mathbb{C}/\Lambda$ as a 2-dimensional torus. Asked if they are always algebraic, and showed the answer is actually yes. Let $E^{an} = \mathbb{C}/\Lambda$; theorem is E^{an} is algebraic. More precisely: E has the structure of a smooth projective curve of genus 1. Its affine equation is

$$y^2 = 4x^3 - 60G_4x - 140G_6 \qquad G_4 = \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^4} \qquad G_6 = \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^6}.$$

Even more precisely, you can write down the Weierstrass p-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\neq 0}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

and compute

$$\wp'(z) = \sum_{\lambda\in\Lambda} \frac{-2}{(2-\lambda)^3},$$

see that the pair $(x, y) = (\wp(z), \wp'(z))$ satisfies the affine equation above. So the theorem really says that $z \mapsto (\wp(z), \wp'(z))$ is a group isomorphism $\mathbb{C}/\Lambda \cong E$ where E is the projectivized elliptic curve.

What about higher dimensions? This direction is false: a general torus $\mathbb{C}^g/\Lambda$ will not be algebraic. However, the converse is true; a general abelian variety A will be (after analytification) of the form $\mathbb{C}^g/\Lambda$. (Proof later).

Theorem: The Weierstrass parametrization gives a bijection between lattices Λ in $\mathbb{C}$, and isomorphism classes of pairs (E, ω) with $E/\mathbb{C}$ an elliptic curve and ω a holomorphic differential form.

2 Lecture - 02/02/2016

Last time: ended by saying we can get a bijection between lattices Λ and pairs (E, ω) where E is an elliptic curve (with $E^{an} \cong \mathbb{C}/\Lambda$) and a holomorphic differential form ω which corresponds to some $f(z)fz$ where f is periodic with respect to Λ and holomorphic.

Some GAGA principles. Algebraic varieties vs. (complex) analytic spaces. If X is an algebraic variety over $\mathbb{C}$ this passes to a complex analytification X^{an}; if X is locally described by some set of equations in affine space pass that open set the zero locus as a subset of $\mathbb{C}^n$ (with its usual topology) and glue.

Facts: This construction is functorial (an algebraic morphism $X \to Y$ of varieties passes to a holomorphic map $X^{an} \to Y^{an}$). Also, X is proper/complete iff X^{an} is compact, and X is smooth or connected iff X^{an} is smooth or connected, respectively. A complex analytic space $\mathfrak{X}$ is called algebraic/algebraizable if there exists a variety $X/\mathbb{C}$ with $\mathfrak{X} \cong X^{an}$. Last time explicitly showed $\mathbb{C}/\Lambda$ is algebraic via Weierstrass p-functions.

Vector bundles (and more generally, their associated locally free sheaves): If L is a vector bundle on X, then it passes to an analytic vector bundle L^{an} on X^{an}. This is functorial in the sense that if $f : F \to G$ is a morphism of vector bundles it passes to $f^{an} : F^{an} \to G^{an}$. It is not true that all holomorphic vector bundles over X^{an} are algebraizable! But we have:

Theorem 1 (Serre). *Let X be a proper algebraic variety over $\mathbb{C}$. If $\mathcal{F}$ is a coherent sheaf over X^{an}, then there exists a unique algebraic coherent sheaf F over X with $F^{an} \cong \mathcal{F}$. If $\mathfrak{f} : \mathcal{F} \to \mathcal{G}$ is a homomorphism of holomorphic coherent sheaves on X^{an} then there exists a unique $f : F \to G$ with $f^{an} = \mathfrak{f}$. Moreover if F is a coherent sheaf on X then the natural map $H^i(X, F) \to H^i(X^{an}, F^{an})$ are isomorphisms of $\mathbb{C}$-vector spaces.*

Complex tori. Let V be a vector space on $\mathbb{C}$, and $\Lambda \subseteq V$ a lattice (full rank discrete subgroup). Have Λ act by addition, and then the quotient $X = V/\Lambda$ is a complex torus. This is a complex manifold, which inherits the structure of an abelian complex Lie group over $\mathbb{C}$, and is compact because Λ has full rank. Moreover, meromorphic functions on X correspond to meromorphic Λ-periodic functions on V.

Definition: A complex analytic abelian variety is a complex torus with "sufficiently many" meromorphic functions. (Enough to give a closed embedding to projective space). We'll see that this is exactly what makes X algebraizable and thus an algebraic abelian variety.

Compactness implies abelian. Theorem: Any connected compact complex Lie group X is a complex torus.

Proof: First, we see X is abelian. Consider the commutator map $\Phi : X \times X \to X$ given by $\Phi(x, y) = xyx^{-1}y^{-1}$; this is continuous. Fix an open neighborhood U of 1 (isomorphic to a bounded open set in $\mathbb{C}$). Then since $\Phi(x, 1) = 1$ there's a neighborhood V_x of x and $\widetilde{V}_x$ of 1 such that $\Phi(V_x, \widetilde{V}_x) \subseteq U$. Then X is covered by the V_x's, and by compactness X is some finite union $V_{x_1} \cup \cdots \cup V_{x_n}$. Set $W = \bigcap \widetilde{V}_{x_i}$; this is an open neighborhood of the identity such that $\Phi(V_{x_i}, W) \subseteq U$ and thus $\Phi(X, W) \subseteq U$. This in fact implies $\Phi(X, W) = 1$ because each $x \mapsto \Phi(x, w)$ lands inside U and thus is bounded and therefore constant by Louiville's theorem. Since $\Phi(1, w) = 1$ we get $\Phi(x, w) = 1$ for all $x \in X$ and $w \in W$. Since W is open and nonempty, and since X is connected, $\Phi(x, y) = 1$ for all $x, y \in X$.

Then, if $V \to X$ is a universal cover, V inherits the structure of a simply connected abelian Lie group and thus must be $\mathbb{C}^g$. Moreover π is a homomorphism with discrete kernel, and by compactness of X the kernel must be full rank.

Remarks: Once we have $X = V/\Lambda$ we see that V is a universal cover of X. Moreover, $\Lambda = \pi_1(X, 0)$, and since this is already abelian it's isomorphic to $H_1(X, \mathbb{Z})$. Since X is locally isomorphic to V, we can view V as the tangent space at 0, T_0X; then $\pi : T_0X \to X$ is the exponential map.

The period matrix. (Due to Lefschetz, generalizing original work of Riemann for Riemann surfaces). Given $X = V/\Lambda$, associate to it a $g \times 2g$ complex matrix Π. To get this fix a $\mathbb{C}$-basis $e_1, \ldots, e_g$ for V, and fix $\lambda_1, \ldots, \lambda_{2g}$ a $\mathbb{Z}$-generating set for Λ. Then decompose $\lambda_j = \sum \lambda_{ij} e_i$ and take $\Pi = (\lambda_{ij})$.

So, Π determines X but depends on the choices. Question: Given $\Pi \in M_{g\times 2g}(\mathbb{C})$, is there a complex torus with Π the period matrix of X?

Theorem: Let P be the $2g \times 2g$ matrix consisting of one copy of Π on top of one copy of $\overline{\Pi}$ (the complex conjugate matrix). Then Π is the period matrix for some $\mathbb{C}^g/\Lambda$ iff P is nonsingular.

Proof: Π is a period matrix iff the columns of Π are $\mathbb{R}$-linearly independent...

Holomorphic maps, homomorphisms, and isogenies. Let $X = V/\Lambda$ and $X' = V'/\Lambda'$ be complex tori of dimensions g and g'. Two special examples: homomorphisms (holomorphic and respect group structure) and translations (maps $X \to X$ by $x \mapsto x + x_0$). Theorem: if $h : X \to X'$ is a holomorphic map between complex tori, there exists a unique homomorphism $f : X \to X'$ such that $h(x) = f(x) + h(0)$. Moreover, there exists a unique $\mathbb{C}$-linear map $F : V \to V'$ with $F(\Lambda) \subseteq \Lambda'$ inducing f.

3 Lecture - 02/04/2016

Theorem: If $X = V/\Lambda$ and $X' = V'/\Lambda'$ are complex tori and $h : X \to X'$ is a holomorphic map, then there exists a unique homomorphism $f : X \to X'$ such that $h = t_{h(0)} \circ f$ and there exists a unique $\mathbb{C}$-linear map $F : V \to V'$ with $F(\Lambda) \subseteq \Lambda'$ inducing F.

Proof idea: Take $f = t_{-h(0)} \circ h$, which satisfies $f(0) = 0$. Then if $\pi : V \to X$ is the projection, $f \circ \pi : V \to X'$ lifts to a holomorphic map $F : V \to V'$ satisfying $\pi' \circ F = f \circ \pi$ and $F(0) = 0$. Claim that F is a $\mathbb{C}$-linear map, which will give us everything.

How do we prove this claim? Fix $\lambda \in \Lambda$; by construction of F we have $F(v + \lambda) - F(v) \in \Lambda'$ for any v. Moreover, $v \mapsto F(v + \lambda) - F(v)$ is continuous from X (connected) into $\Lambda' \subseteq V'$ (discrete) and thus must be constant (at $F(\lambda) = F(0 + \lambda) - F(0)$). So we have $F(v + \lambda) = F(v) + F(\lambda)$ for every v, λ. This is a weaker form of what we want ($\mathbb{C}$-linearity); won't finish the proof but we will be able to conclude it by proving something similar for all derivatives of F and use Liouville's theorem...

Hom-sets . Let $\mathrm{Hom}(X, X')$ be the set of all homomorphisms $X \to X'$. This is an abelian group under addition of maps. If $X = X'$ we let $\mathrm{End}(X) = \mathrm{Hom}(X, X')$; this becomes a ring under composition.

Corollary to the above theorem: we have an injective homomorphism

$$\rho_{an} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$$

sending $f : X \to X'$ to its lift $F : V \to V'$. Moreover we have an injective homomorphism

$$\rho_{int} : \mathrm{Hom}(X, X') \to \mathrm{Hom}(\Lambda, \Lambda')$$

by sending f to $F|_\Lambda$. Remark: Both of these homomorphisms respect endomorphism ring structures if $X = X'$.

Theorem: $\mathrm{Hom}(X, X') \cong \mathbb{Z}^m$ for some $m \leq 4gg'$. (Proof: use the second isomorphism in the corollary, since $\Lambda \cong \mathbb{Z}^{2g}$ and $\Lambda' \cong \mathbb{Z}^{2g'}$ so $\mathrm{Hom}(\Lambda, \Lambda') \cong \mathbb{Z}^{4gg'}$ and $\mathrm{Hom}(X, X')$ embeds in this).

How do these relate to period matrices? Let Π and Π' be period matrices for X, X'. If we have $f : X \to X'$, then by picking bases we get that $\rho_{an}(f)$ is given by $A \in M_g(\mathbb{C})$ and $\rho_{int}(f)$ is given by $R \in M_{2g}(\mathbb{Z})$. If Π, Π' are the corresponding period matrices, the conditions $F(\Lambda) \subseteq \Lambda'$ means $A\Pi = \Pi' R$. (Conversely, given four matrices with this property then they correspond to a morphism between complex tori). What if $X = X'$? Can get

$$\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{\Pi} \end{bmatrix} = \begin{bmatrix} \Pi \\ \overline{\Pi} \end{bmatrix} R$$

and thus $\rho_{Int} \otimes 1 \cong \rho_{an} \oplus \overline{\rho}_{an}$ in $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

Kernels and images. Given a homomorphism $f : X \to X'$ we have:

Lemma: (a) $\mathrm{img}(f)$ is a complex subtorus of X'.
(b) $\ker(f)$ is a closed subgroup of X with finitely many components; the connected component of 1 is a complex torus.

Proof is fairly easy; for part (b) we're claiming that we have an extension

$$1 \to X_0 \to G \to \Gamma \to 1$$

with X_0 a complex torus and Γ a finite abelian group. Exercise: Describe Γ as a direct sum of cyclic groups in terms of Π, Π', A, R (need to compute a Smith normal form somewhere).

Isogenies. A homomorphism $f : X \to X'$ is called an *isogeny* if f is surjective with finite kernel; equivalently f is surjective and $\dim X = \dim X'$. Essential example: If $X = V/\Lambda$ is a complex torus and $\Gamma \subseteq X$ is a finite subgroup (i.e. $\pi^{-1}[\Lambda]$ is a lattice containing Λ with finite index), then X/Γ is a complex torus and $X \to X/\Gamma$ is an isogeny. Easy exercise: All isogenies over $\mathbb{C}$ are of this form.

Easy lemma (Stein factorization): Any surjection $f : X \to X'$ of complex tori factors as a surjection $X \to X/(\ker f)_0$ (a quotient of X by a complex subtorus) and an isogeny $X/(\ker f)_0 \to X'$.

If $f \in \mathrm{Hom}(X, X')$ we set the degree of f to be $|\ker f|$ if this is finite and 0 if $\ker f$ is infinite. Easy to check $\deg(f) = [\Lambda' : \rho_{Int}(f)\Lambda]$. (Remark: If $X = X'$ then this index is $\deg(\rho_{int}(f))$; note that this determinant is ≥ 0 since $\rho_{int} \otimes 1 = \rho_{an} \oplus \overline{\rho}_{an}$, and is 0 iff the kernel is infinite).

Lemma: if $f : X \to X'$ and $f' : X' \to X''$ are isogenies, then so is $'f \circ f$. Proof: $\deg(f' \circ f) = \deg(f')\deg(f)$.

A very important example: $n_X : X \to X$ given by $x \mapsto nx$. Then $\ker n_X$ (usually denoted $X[n]$) is the n-torsion elements in A, and we can see it is isomorphic to $(\frac{1}{n}\Lambda)/\Lambda \cong \Lambda/n\Lambda \cong Z_n^{2g}$. So $\deg(n_X) = n^{2g}$, and n_X is an isogeny. (Corollary: a complex torus X is a divisible group).

Example: If ℓ is prime, multiplication-by-ℓ gives a map $X[\ell^{n+1}] \to X[\ell^n]$; so we can form $T_\ell(X) = \varprojlim X[\ell^n]$, and find (since Λ is finitely generated) this is $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Note that $T_\ell(X)$ makes sense over any fields (even if we don't have Λ when we're not over $\mathbb{C}$). Here it's easy to see that a morphism $X \to X'$ is determined by the induced map $T_\ell(X) \to T_\ell(X')$. Over general fields this is much, much harder! It's the *Tate conjecture*, which was only proven over number fields by Faltings as an essential part of his proof of the Mordell conjecture.

Importance of isogenies. They are "almost isomorphisms". Namely, we have a theorem that if $f : X \to X'$ is an isogeny and n is the exponent of the abelian group $\ker(f)$, then there exists an isogeny $g : X' \to X$, such that $g \circ f = n_X$ and $f \circ g = n_{X'}$. Moreover such a g is unique (up to isomorphism?)

Since n is the exponent of $\ker f$, we have $\ker f \subseteq \ker(n_x) = X[n]$. Then there exists a unique $g : X' \to X$ with $g \circ f = n_X$, defined by $g(x') = nx$ for some (all) x where $f(x) = x'$. Use that $\deg(g)\deg(f) = \deg(n_X)$ and $\deg(f), \deg(n_X) \neq 0$ so $\deg(g) \neq 0$ to get that g is an isogeny; then just need to check $g \circ f = n_{X'}$. Corollary: "isogenous to" is an equivalence relation.

Now, define $\mathrm{End}_{\mathbb{Q}}(X) = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Q}}(X, X') = \mathrm{Hom}(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$. Degree function extends to these. Why are we doing this? Well, we have $f \in \mathrm{End}(X)$ is an isogeny iff it's invertible in $\mathrm{End}_{\mathbb{Q}}(X)$.

Cohomology. Have lots of cohomology theories (Betti, de Rham, Dobleault, Hodge decomposition, ...). Betti cohomology is just singular cohomology of $X(\mathbb{C})$; if $X = V/\Lambda$ we know $\Lambda = \pi_1(X, 0) = H_1(X, \mathbb{Z})$ and then by the universal coefficient theorem we have $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$. If $n \geq 1$, we have a map $\bigwedge_{i=1}^n H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ induced by cup product, and this is an isomorphism (this follows from Künneth formula). If we let $\mathrm{Alt}^n(\Lambda, \mathbb{Z}) = \bigwedge_{i=1}^n \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the $\mathbb{Z}$-valued alternating n-forms, this means $H^n \cong \mathrm{Alt}^n$, which gives a very explicit way of thinking about cohomology. Also, $H_n(X, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2g}{n}$. Furthermore if we set $H^n(X, \mathbb{C}) \cong H^n(X, \mathbb{Z}) \otimes \mathbb{C}$ we have

$$H^n(X, \mathbb{C}) \cong \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C}) = \bigwedge_{i=1}^n \mathrm{Hom}_{\mathbb{R}}(\Lambda, \mathbb{C}) \cong \bigwedge_{i=1}^n H^1(X, \mathbb{C}),$$

and the de Rham theorem tells us $H^n(X, \mathbb{C}) \cong H^n_{\mathrm{dR}}(X, \mathbb{C})$, and the latter can be explicitly described as a complex vector space of invariant n-forms with basis $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ with $i_1 < \cdots < i_n$.

Now: use the $\mathbb{C}$-structure (really everything is true for Kähler manifolds, but proofs and constructions are much more elementary for complex tori). Here we have a very nice decomposition

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^q(\Omega^p_X).$$

Here, $H^q(\Omega^p_X)$ is isomorphic to the Dolbeault cohomology $H^{p,q}(X)$. In general, $H^q(\Omega^p_X)$ can be explicitly described as $\bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega}$ for $\Omega = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Set $\Omega^p_X = (\bigwedge^p \Omega) \otimes \mathcal{O}_X$.

4 Lecture - 02/09/2016

Remarks on stuff from previous classes: Recall we said that we had a bijection between lattices Λ and pairs (E, ω) of an elliptic curve and a differential form. Someone asked what happens if we replace Λ with $a\Lambda$; turns out we can say it corresponds to $(E, a\omega)$ (or $(E, a^{-1}\omega)$? need to check).

Remark 2: Stein factorization is a special case of a very general result (in complex theory by Stein and others, and in general in EGA III): any proper $f : X \to S$ factors as a map $X \to S'$ proper with connected fibers and $S' \to S$ finite.

Remark 3: Last time said that we had a Hodge decomposition

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^q(\Omega^p) = \bigoplus_{p+q=n} H^{p,q}(X);$$

right now aren't really saying anything about what these vector spaces $H^{p,q}(X)$ are (but will today / later on). May need to return to this theory to prove e.g. vanishing results later; will either do that, or just omit those proofs.

Sheaves on a topological space X. Define $\mathcal{O}(X)$ to be the category where objects are open subsets of X and morphisms are inclusions $V \to U$ (for $V \subseteq U$). Remark: Turns out you can generalize this and allow more general things for morphisms than just inclusions; this is how you get étale cohomology and other things.

Now, let $\mathcal{C}$ be any other category; a *presheaf* is a contravariant functor $F : O(X) \to \mathcal{C}$. (This means for each inclusion map $i : V \to U$, we get a "restriction" map $\mathrm{res}_{V,U} : F(U) \to F(V)$, and this assignment is functorial). A *sheaf* is a presheaf F with some "locality" and "gluing" properties. Assume $\mathcal{C}$ has products; then we require for any open cover $\{U_i\}$ of $U \in O(X)$ we have an exact sequence

$$0 \to F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

(where the two parallel maps are "restriction to the first index" and "restriction to the second index" respectively). Example: $X = \mathbb{C}$, sheaf of holomorphic functions ($F(U)$ is the set of all holomorphic functions $U \to \mathbb{C}$, restriction maps are restriction of functions!)

Morphisms of sheaves are natural transformations; this gives us a category of sheaves $\mathbf{Shf}_X$ where the objects are sheaves on X and morphisms are these. Next, a *ringed space* is a topological space X together with a sheaf of rings $\mathcal{O}_X$; we call $\mathcal{O}_X$ the *structure sheaf* (usually some sheaf of holomorphic functions in this class). Define a locally ringed space to be one such that all of the stalks $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} F(U)$ are local rings. We may want to consider sheaves of $\mathcal{O}_X$-modules, i.e. each $F(U)$ is a $\mathcal{O}_X(U)$-module respecting restriction maps.

Abelian categories and cohomology. (Grothendieck's Tôhoku paper). There are 4 main examples of abelian categories to keep in mind:

1. The category of abelian groups (with homomorphisms).
2. The category of R-modules for R a commutative ring (with homomorphisms).
3. The category of G-modules for G a group (with G-equivariant homomorphisms).
4. The category of $\mathcal{O}_X$-modules for $(X, \mathcal{O}_X)$ a ringed space (with morphisms of sheaves the morphisms).

In general there's an abstract definition of an abelian category; not going to say it precisely but roughly it's a category $\mathcal{A}$ in which addition of morphisms, a zero object, and kernel and cokernels make sense.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories. If we start with an exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, we can hit it with the functor F and get maps in $\mathcal{B}$, but no guarantee for exactness. If, for every SES we do get an exact sequence

$$0 \to F(A) \to F(B) \to F(C)$$

(but not exact on the right side), say the functor is left exact. (Example: $\mathcal{A}$ the category of R-modules, $\mathcal{B}$ the category of abelian groups, and F the functor $F(-) = \mathrm{Hom}_R(D, -)$ for a fixed object D). Cohomology lets us study the failure of exactness of this sequence, i.e. the failure of surjectivity of $F(B) \to F(C)$. Use it to continue the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

to the right and write down a long exact sequence.

In general, there are many ways to write down such a sequence. But if $\mathcal{A}$ has "enough injectives" (a statement that holds for the categories we care about) then there is a canonical and "minimal" (in the sense that there's a universal property) way to do this. This is the *right derived functors*; i.e. functors $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ (with $R^0F = F$) giving us a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2(A) \to \cdots$$

plus satisfying some universal properties (an "effaceable δ-functor"). This is the covariant version; there's a similar one for contravariant functors.

Examples: For $F = \mathrm{Hom}_R(-, D)$ and $G(-) = \mathrm{Hom}_R(D, -)$ (both left-exact functors from R-modules to abelian groups, one contravariant and one covariant). These are both left-exact, which means that if $0 \to L \to M \to N \to 0$ is a SES then we have exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D)$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N).$$

In both cases, the derived functors are Ext groups: get long exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D) & \\ \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^1_R(M, D) \to \mathrm{Ext}^1_R(L, D) \to \mathrm{Ext}^2_R(N, D) \to \cdots \end{aligned}$$

So $R^i \mathrm{Hom}_R(-, D) = \mathrm{Ext}^i_R(-, D)$ and $R^i \mathrm{Hom}_R(D, -) = \mathrm{Ext}^i_R(D, -)$ (nontrivial fact that these agree!).

Important examples: G-modules - let A be an abelian group with a G-action $\varphi : G \to \mathrm{Aut}(A)$, i.e. a $\mathbb{Z}G$-module, and morphisms respect the group actions. The functor we want is $F(A) = A^G = \{x \in A : \forall g, gx = x\}$, which takes is from G-modules to abelian groups. This functor is the same as $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$, so its derived functors are (abstractly) Ext-functors, but this doesn't really give us a good way to compute it. Better known as the "group cohomology of G with coefficients in A", denoted $H^i(G, A)$.

How do we compute this? It turns out that $\mathbb{Z}$ has a very nice "standard resolution" (also called the "bar resolution"), given by $F_n = \bigotimes_{i=0}^n \mathbb{Z}G$. Using this we get the following recipe: for G a group and A a G-module, we let $C^n(G, A)$ be the group of maps $G^n \to A$ (for $C^0(G, A)$ this is just identified with A itself); this is the group of n-cochains with values in A. Define differential operators $d_n : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\begin{aligned} & d_n(f)(g_1, \ldots, g_{n+1}) = \\ & \qquad g_1 \cdot f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2} \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n). \end{aligned}$$

This is a bit of a strange formula but it's what pops out of the general theory. The cases of most interest are $n = 0, 1, 2$. For $n = 0$ we have $f = a \in C^0(G, A) = A$, and then we get

$$d_0(f)(g) = g \cdot a - a.$$

For $n = 1$ then f is a function with one input and we have

$$d_1(f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1).$$

For $n = 2$ similarly we have the following (which amazingly people wrote down correctly before the general theory):

$$d_2(f)(g, h, k) = gf(h, k) - f(gh, k) + f(g, hk) - f(g, h).$$

Fact: $d_n \circ d_{n+1} = 0$. Thus, if we define the group of n-cocycles as $Z^n(G, A) = \ker(d_n)$ and the group of n-coboundaries as $B^n(G, A) = \mathrm{img}(d_{n-1})$ (or as the trivial group for $n = 0$), we have $B^n(G, A) \subseteq Z^n(G, A)$. We define the n-th cohomology group as $H^n(G, A) = Z^n(G, A)/B^n(G, A)$; note $H^0(G, A) = A^G$.

Example 4: Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ a $\mathcal{O}_X$-module. Let $\Gamma(X, -)$ be the "global sections" functor, from $\mathcal{O}_X$-modules to R-modules for $\mathcal{O}_X(X)$, i.e. $\mathcal{F} \mapsto \mathcal{F}(X)$. Here there's the Grothendieck derived cohomology functor, but there's also a more concrete one, the Čech cohomology. These are not equal in general! Fortunately they are equal in the settings we'll be interested in (and many other common ones).

5 Lecture - 02/11/2016

Continuing from last time, with Example (4): $(X, \mathcal{O}_X)$ a ringed space, $\mathcal{F}$ a $\mathcal{O}_X$-module. Then $\Gamma(\mathcal{F}, X) = \mathcal{F}(X)$ is the module of global sections (an R-module for $R = \mathcal{O}_X(X)$). This functor is left-exact (to make sense of this we need to make sure we know what exact sequences of sheaves are - defining "kernels" is easy but to define "images" we need to sheafify).

So, we can define the right derived functor $R^i\Gamma(-, X)$, which we denote $H^i(X, \mathcal{F})$ (the sheaf cohomology of X with values in $\mathcal{F}$); in particular $H^0(X, -) = \Gamma(-, X)$. This is given abstractly by the theory we talked about last time; but like group cohomology we'll really need to work with sheaf cohomology, so we need a way to compute them explicitly.

One way to compute sheaf cohomology (in some cases) is by Čech cohomology, $\check{H}^i(X, \mathcal{F})$. Čech cohomology is more explicitly computable, and always gives us a map $\varphi : H^i(X, \mathcal{F}) \to \check{H}^i(X, \mathcal{F})$. We have:

1. For $i = 0$ and $i = 1$, φ is an isomorphism.
2. (Grothendieck): If X is a Noetherian, separated scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, φ is an isomorphism.
3. (Godement): If X is a paracompact and Hausdorff topological space, φ is an isomorphism.

But φ is not an isomorphism in general! Counterexamples:

- In his Tôhoku paper (p. 177) Grothendieck provides a counterexample where $H^2 \neq \check{H}^2$, for $X = \mathbb{A}^2$ with the Zariski topology and $\mathcal{F}$ comes from taking $\underline{\mathbb{Z}}$ and modifying it based on a space Y that's a union of two circles. Example is explicit but proof is deep!
- Recent paper (Schröer, arxiv post 1309.2524) gives a Hausdorff (but not paracompact) topological space constructed from 2-dimensional discs that's a counterexample.

Čech cohomology. So how do we define Čech cohomology? The idea is that if $\mathcal{U}$ is an open cover on X, the "nerve" of $\mathcal{U}$ approximates X. We define a q-simplex σ of $\mathcal{U}$ as an ordered collection of $q+1$ elements in $\mathcal{U}$ with nontrivial intersection that we call $|\sigma|$. If $\sigma = (U_i)$ (for $0 \leq i \leq q$) we define $\partial_j \sigma = (U_i)_{i \neq j}$ and then $\partial\sigma = \sum_{j=0}^{q} (-1)^{j+1} \partial_j \sigma$. Take $|\sigma| = \bigcap U_i$.

Then define q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$ to be the set $C^q(U, \mathcal{F})$ of functions $\sigma \mapsto f_\sigma \in \mathcal{F}(|\sigma|)$. Get boundary maps $C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta_q \omega)(\sigma) = \sum_{j=0}^{q+1} (-1)^j \operatorname{res}_{|\partial_j\sigma|, |\sigma|} \omega(\partial_j \sigma).$$

Can check $\delta_{q+1} \circ \delta_q = 0$ and that cocycles $Z^q(\mathcal{U}, \mathcal{F}) = \ker \delta_q$ and coboundaries $B^q(\mathcal{U}, \mathcal{F}) = \operatorname{img}(\delta_{q-1})$, and then the Čech cohomology is $\widehat{H}^q(\mathcal{U}, \mathcal{F}) = Z^q/B^q$.

So this gives us the cohomology of an open cover; want a cohomology associated to the whole space X! Two ways to solve this:

1. If X has a "good" cover $\mathcal{U}$ (with all finite intersections of U_i's to be contractible) then $\check{H}^i(\mathcal{U}, \mathcal{F})$ is canonical.
2. In general, can define $\check{H}^i(X, \mathcal{F})$ as $\varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$. (Need to make sense of this direct limit - it's might be over an index set that's a proper class?).

Remark: Let G be a topological group. Let $BG = K(G, 1)$ be the Eilenberg-Mac Lane space which has $\pi_1 = G$ and $\pi_n = 0$ for $n > 1$. Then if A is a G-module the sheaf cohomology $H^n(BG, \underline{A})$ (here $\underline{A}$ is the constant sheaf, which is the sheafification of the constant presheaf) is isomorphic to the usual CW complex cohomology $H^n(BG, A)$ and to the group cohomology $H^n(G, A)$.

Back to complex tori (sort of). Want to study line bundles on a locally ringed space $(X, \mathcal{O}_X)$ (probably a complex manifold, or a variety, for the purposes of this class). If $\mathcal{F}$ is a sheaf it's called (globally) *free* if it's the direct sum of copies of the structure sheaf $\mathcal{O}_X$. We call it *locally free* if there exists an open cover $\{U_i\}$ such that each $\mathcal{F}|_{U_i}$ is free. Fact: Locally free sheaves of rank n are in bijective correspondence with vector bundles of rank n. (If $\pi : E \to X$ is a vector bundle, get a locally free sheaf with $\mathcal{F}(U)$ being the sections of π over U; conversely if $\mathcal{F}$ is locally free we can construct an associated line bundle as $\coprod U_i \times \mathbb{C}^n$ modulo gluing data).

Line bundles are vector bundles of rank 1 (equivalently, locally free sheaves of rank 1). Goal: Show that the set of line bundles has a cohomological interpretation. Let $\pi : L \to X$ be a line bundle, and let $\{U_\alpha\}$ be an open cover with trivializations $\varphi_\alpha : \pi^{-1}[U_\alpha] \cong U_\alpha \times \mathbb{C}$. Define the transition functions (for L with respect to the $\{\varphi_\alpha\}$) $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ by

$$g_{\alpha\beta}(z) = \varphi_\alpha \circ \varphi_\beta^{-1}|_{\{z\}\times\mathbb{C}}.$$

By itself this is a linear map $\{z\} \times \mathbb{C} \to \{z\} \times \mathbb{C}$, which is determined by the complex number we're calling $g_{\alpha\beta}(z)$, and we check that $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ (so it's nonzero) and also $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$. Rewriting this latter condition gives a cocycle condition

$$g_{\alpha\beta}g_{\gamma\beta}^{-1}g_{\gamma\alpha} = 1.$$

Summarizing: a line bundle (trivialized by an open cover $\mathcal{U}$) determines a collection of holomorphic functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$, satisfying $g_{\alpha\beta}g_{\beta\alpha} = 1$ and $g_{\alpha\beta}g_{\gamma\beta}^{-1}g_{\gamma\alpha} = 1$. Conversely, given such a collection of $g_{\alpha\beta}$ we can construct a line bundle L with transition functions $g_{\alpha\beta}$ as a quotient of $\coprod U_\alpha \times \mathbb{C}$ by the appropriate gluing relations.

Choices involved: If $f_\alpha \in \mathcal{O}^\times(U_\alpha)$ is a nonvanishing holomorphic function on U_α, and we construct new trivializations $\varphi'_\alpha = f_\alpha\varphi_\alpha$, then the new transition functions $g'_{\alpha\beta} = (f_\alpha/f_\beta)g_{\alpha\beta}$ give the same bundle L.

Now, the collection of $\{g_{\alpha\beta}\}$ is a Čech 1-cochain. The conditions we wrote down that it satisfies implies it's actually a 1-cocycle. Moreover, the ambiguity mentioned above is exactly the 1-coboundaries. We can then conclude the set of (isomorphism classes of) line bundles $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X^\times)$ (and moreover this is a group homomorphism, for the group structure on line bundles coming from tensor product and the group structure naturally showing up on $H^1(X, \mathcal{O}_X^\times)$).

Line bundles vs. "factors of automorphy": Let X be a complex torus, and $\widetilde{X} = \mathbb{C}^g$ its universal cover, with $\pi : \widetilde{X} \to X$ the covering map. Theorem: There exists a canonical exact sequence

$$0 \to H^1(\pi_1(X), H^0(\widetilde{X}, \mathcal{O}_X^\times)) \to H^1(X, \mathcal{O}_X^\times) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^\times)$$

The first term is the group of "factors of automorphy" and the latter two are Picard groups as above.

Corollary: If $\pi^*(L)$ is trivial then it's completely described by a "factor of automorphy".

Corollary: If $\mathrm{Pic}(\widetilde{X}) = 0$ (e.g. if $\widetilde{X} = \mathbb{C}^g$) then the first map is an isomorphism.

6 Lecture - 02/18/2016

Some corrections/clarifications from last time: The isomorphism $H^n(BG, \underline{M}) \cong H^n(G, M)$ is fine if M has a trivial G-action; if M has a nontrivial G-action we can still make sense of this but the LHS needs to be reinterpreted in terms of "local coefficient systems". Also, in $H^1(X, \mathcal{O}_X^\times)$ the general definition of $\mathcal{O}_X^\times$ should be the sheaf of invertible elements in our rings; in our holomorphic setting this is equivalent to the sheaf of nonvanishing functions, because you can prove that if f is holomorphic and nonzero then $1/f$ is holomorphic.

Theorem stated at end of last time: we have an exact sequence

$$0 \to H^1(\pi_1(X), H^0(\widetilde{X}, \mathcal{O}_X^\times)) \to \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$$

e.g.

$$0 \to H^1(\pi_1(X), H^0(\widetilde{X}, \mathcal{O}_X^\times)) \to H^1(X, \mathcal{O}_X^\times) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^\times).$$

To prove this we need to start by analyzing what these cohomology groups are. First of all, the 1-cochains $C^1(G, M)$ are functions $G \to M$, which we need to analyze in the case of $G = \pi_1(X)$ and $M = H^0(\widetilde{X}, \mathcal{O}_X^\times)$. Thus M corresponds to holomorphic functions $g : \widetilde{X} \to \mathbb{C}^\times$; so functions $G \to M$ can be interpreted as holomorphic functions $f : G \times \widetilde{X} \to \mathbb{C}^\times$ (where G is given the discrete topology). Moreover, M is a G-module by the action coming from $\pi_1(X)$ acting by deck transformations on $\widetilde{X}$: given $g : \widetilde{X} \to \mathbb{C}^\times$ in M and $\lambda \in \pi_1(X)$ the action is by $(\lambda g)(\widetilde{x}) = g(\lambda\widetilde{x})$.

What are 1-cocycles? Those are twisted homomorphisms, i.e. (in our multiplicative setting) maps $h : G \to M$ satisfying

$$\lambda h(\mu) \cdot h(\lambda\mu)^{-1} \cdot h(\lambda) = 1.$$

Reinterpreting this in terms of viewing h as a holomorphic function $f : G \times \widetilde{X} \to \mathbb{C}^\times$ we're asking for it to satisfy

$$f(\lambda\mu, \widetilde{x}) = f(\mu, \lambda\widetilde{x}) f(\lambda, \widetilde{x})$$

for $\lambda, \mu \in \pi_1(X)$ and $\widetilde{x} \in \widetilde{X}$. The 1-coboundaries are then things of the form $(\lambda h)h^{-1}$ coming from $h_1 \in M = C^0(G, M)$; rephrased this means that the coboundaries are the functions

$$f(\lambda, \widetilde{x}) = \frac{g(\lambda\widetilde{x})}{g(x)}$$

for each $g : \widetilde{X} \to \mathbb{C}^\times$ in M.

So now the cohomology is the quotient of these two things. Informally, why should an element f in $Z^1(G, M)$ (described as a function $\pi_1(X) \times \widetilde{X} \to \mathbb{C}^\times$) give us a line bundle? The idea is we start with $L = \widetilde{X} \times \mathbb{C}$ and want to descend this to something on X using f. In particular we make $\pi_1(X)$ act on L by

$$\lambda \cdot (\widetilde{x}, t) = (\lambda \cdot \widetilde{x}, f(\lambda, \widetilde{x})t);$$

the cocycle condition makes this actually an action. Easy to check that this action is free (i.e. if $gx = x$ for some g and x then $g = \mathrm{id}$) and properly discontinuous (for any compact sets K_1, K_2 the set $\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$ is finite). Then can apply a standard theorem that the quotient of a complex manifold modulo a free properly discontinuous action we get a complex manifold. So our action on L gives us a quotient that's a holomorphic line bundle over X.

So we've gotten a map $Z^1(G, M) \to \mathrm{Pic}(X)$, and we could continue the argument in this topological way and prove that this map leads to the exact sequence we want. However, we'll restart the proof and do it algebraically.

Proof of theorem: The main nontrivial fact we need is that we have an isomorphism $\varphi_1 : H^1(G, M) \to K$ where K is the kernel of $H^1(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_{\widetilde{X}}^\times)$. First we need to come up with the map φ_1. We gave one idea of how to do this above, but now a more explicit way, directly going from f to a Čech cocycle in $Z^1(X, \mathcal{O}_X^\times)$. To do this we need an open cover. For $\pi : \widetilde{X} \to X$ fix an open cover $\{U_i\}_I$ such that for each i, there exists $W_i \subseteq \pi_1^{-1}[U_i]$ such that $\pi_i : \pi|_{W_i} : W_i \to U_i$ is a biholomorphism. Then for all i, j there exists a unique $\lambda_{ij} \in \pi_1(X)$ such that for all $x \in U_i \cap U_j$ we have $\pi_j^{-1}(x) = \lambda_{ij}\pi_i^{-1}(x)$, and it's easy to see $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$.

Now, we want to define our map $Z^1(G, M) \to \check{Z}^1(X, \mathcal{O}_X^\times)$. We want to map a function $f : G \times \widetilde{X} \to \mathbb{C}^\times$ (in $Z^1(G, M)$) to the Čech cocycle $\{g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)\}$ given by

$$g_{ij}(x) = f(\lambda_{ij}, \pi_i^{-1}(x)).$$

Can then check the 1-cocycle condition: need to check that $g_{ij}g_{jk} = g_{ik}$. For this we see that if $x \in U_i \cap U_j \cap U_k$ we have

$$g_{ij}(x)g_{jk}(x) = f(\lambda_{ij}, \pi_i^{-1}(x))f(\lambda_{jk}, \pi_j^{-1}(x)).$$

Since f is a 1-cocycle $G \to M$, the 1-cocycle condition tells us that this is equal to $f(\lambda_{ij}\lambda_{jk}, \pi_i^{-1}(x))$, and the 1-cocycle condition for the λ's tels us this is $f(\lambda_{ik}, \pi_i^{-1}(x)) = g_{ik}(x)$.

So we have a well-defined map

$$Z^1(G, M) \to \check{Z}^1(X, \mathcal{O}_X^\times) \twoheadrightarrow \check{H}^1(X, \mathcal{O}_X^\times).$$

Claim that the kernel of this map contains $B^1(G, M)$, so this descends to a homomorphism $H^1(G, M) \to \check{H}^1(X, \mathcal{O}_X^\times)$. Checking this: if we have a 1-coboundary $f(\lambda, \widetilde{x}) = h(\lambda\widetilde{x})/h(x)$, this will go to a Čech 1-cocycle

$$g_{ij}(x) = \frac{h(\lambda_{ij}\pi_i^{-1}(x))}{h(\pi_i^{-1}(x))} = \frac{h(\pi_j^{-1}(x))}{h(\pi_i^{-1}(x))}$$

This tells us $g_{ij}(x)$ is a Čech 1-coboundary so is trivial in $\check{H}^1(X, \mathcal{O}_X^\times)$.

So we have a map $\varphi_1 : H(G, M) \to \check{H}^1(X, \mathcal{O}_X^\times)$. Easy to check it actually lands in the kernel of the map to $\check{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^\times)$. Claim it is an isomorphism with this kernel; to prove this we want to construct an inverse map. So given a line bundle L in the kernel of this map, we know π^*L is trivial, i.e. we can fix a trivialization $\alpha : \pi^*L \cong \widetilde{X} \times \mathbb{C}$. Have that G acts on the LHS via the line bundle structure on L and thus on the RHS. So this gives an action of G on $\widetilde{X} \times \mathbb{C}$, so for all $\lambda \in G$ we have an automorphism $\varphi_\lambda : \widetilde{X} \times \mathbb{C}$ which acts by λ on the first coordinate. We then write

$$\varphi_\lambda(\widetilde{x}, t) = (\lambda\widetilde{x}, f(\lambda, \widetilde{x})t).$$

Easy to check that (1) $f(\lambda, \widetilde{x})$ is a cocycle, (2) if we replace α by another α' we get f' that differs from f by a 1-coboundary, so this map does indeed give us something on homology. This finishes the proof.

Via a similar proof, we have a theorem: If $\mathcal{F}$ is a sheaf of abelian groups on X, $G = \pi_1(X)$, and $M = H^0(\widetilde{X}, \pi^*\mathcal{F})$ then for any $n \geq 0$ there exists a canonical homomorphism $\varphi_n : H^n(G, M) \to H^n(X, \mathcal{F})$, For $n = 1$ the cocycle $\varphi_1(f)_{ij}$ is given by $f(\lambda_{ij}, \pi_i^{-1})$ and for $n = 2$, $\varphi_2(f)_{ijk}$ is given by $f(\lambda_{ij}, \lambda_{jk}, \pi_i^{-1})$.

Next goal: Describe global sections of $H^0(X, L) = \Gamma(L, X)$ for L in the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$, using the isomorphism φ_1 to make L correspond to a 1-cocycle in $H^1(G, M)$ before. If we fix a trivialization $\alpha : \pi^*L \cong \widetilde{X} \times \mathbb{C}$ this gives us an explicit cocycle $f \in Z^1(G, M)$ (giving the cohomology class $\varphi_1^{-1}(L)$).

Observe that we have a canonical isomorphism $H^0(X, L) \cong H^0(\widetilde{X}, \pi^*L)^G$, and then we have (via α) an isomorphism of this with $H^0(\widetilde{X}, \widetilde{X} \times \mathbb{C})^G$, where here the action of λ is given by $\varphi_\lambda(\widetilde{x}, t) = (\lambda\widetilde{x}, f(\lambda, \widetilde{x})t)$. This then corresponds to the set of holomorphic functions $\vartheta : \widetilde{X} \to \mathbb{C}$ satisfying $\theta(\lambda\widetilde{x}) = f(\lambda, \widetilde{x})\theta(\widetilde{x})$. So $H^0(X, L)$ is in bijection with the set of theta functions ϑ with f as its factor of automorphy. (Remark: This does depend on the trivialization α and thus the specific form of f; but changing α chances f in a predictable way and thus changes the set of ϑ). Also, one can check functoriality of everything we've done.

Now: Back to the case of complex tori $X = V/\Lambda$. Proposition: Every holomorphic line bundle on X pulls back to a trivial bundle on V. Proof outline: Need (1) the exponential sequence, (2) the $\overline{\partial}$-Poincaré Lemma, and (3) need that $H^2(V, \mathbb{Z}) = 0$.

How will this work? The exponential sequence is a short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_V \to \mathcal{O}_V^\times \to 1$$

where $\mathcal{O}_V \to \mathcal{O}_V^\times$ is given by $z \mapsto e^{2\pi i z}$. The long exact sequence of cohomology for the exponential sequence gives us

$$\cdots \to H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^\times) \to H^2(V, \mathbb{Z}) \to \cdots,$$

and results (2) and (3) tell us that this is actually $0 \to H^1(V, \mathcal{O}_V^\times) \to 0$ and thus $H^1(V, \mathcal{O}_V^\times) = 0$. So all line bundles on V are trivial.

Corollary: The exact sequence we were proving today gives an isomorphism

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times) \cong H^1(\Lambda, H^0(V, \mathcal{O}_V^\times)).$$

Next time: we'll talk about the exponential exact sequence, and generalize it to X. Get a similar long exact sequence, but here $H^2(X, \mathbb{Z})$ is nontrivial, and the map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is extremely important! Our next major goal is the Appell-Humbert theorem (actually due to Lefschetz).

7 Lecture - 02/23/2016

Last time: gave a proof that every line bundle on a vector space V is trivial. Used three main ingredients: (1) $H^2(V,\mathbb{Z})=0$ (pretty easy, since V is contractible), (2) the exponential sequence, and (3) the ∂-Poincaré lemma.

The exponential sequence and the first Chern class. This is an analytic construction (doesn't work in the algebraic seting). On any complex manifold $(X,\mathcal{O}_X)$ the sequence

$$0\to\underline{\mathbb{Z}}\to\mathcal{O}_X\to\mathcal{O}_X^\times\to 1$$

is exact, where the map $\exp:\mathcal{O}_X\to\mathcal{O}_X^\times$ is the exponential map, given on open sets by $\exp(U):\mathcal{O}_X(U)\to\mathcal{O}_X^\times(U)$ by $f\mapsto e^{2\pi i f}$. Check this is actually a morphism of sheaves (easy).

Exactness of the sequence

$$0\to\underline{\mathbb{Z}}\to\mathcal{O}_X\to\mathcal{O}_X^\times$$

is straightforward; the kernel of $\exp:\mathcal{O}_X(U)\to\mathcal{O}_X^\times(U)$ are the locally constant $\mathbb{Z}$-valued functions, which are $\underline{\mathbb{Z}}(U)$. What about surjectivity of $\mathcal{O}_X\to\mathcal{O}_X^\times$? This asks about existence of logarithms. But complex logarithms are not always "globally" possible! However, they are "locally" possible; for any $f\in\mathcal{O}_X^\times(U)$ and any $x\in U$, we can find a contractible neighborhood V of x with $V\subseteq U$, and then $\mathrm{res}_V(f)$ does have a logarithm. So $\mathcal{O}_X(U)\to\mathcal{O}_X^\times(U)$ is surjective for small enough U, and this is enough for the sheaf map $\mathcal{O}_X\to\mathcal{O}_X^\times$ to be considered surjective (the cokernel is a presheaf which sheafifies to 0).

So now we have the long exact sequence for the exponential function, which will give us

$$0\to H^0(X,\underline{\mathbb{Z}})\to H^0(X,\mathcal{O}_X)\to H^0(X,\mathcal{O}_X^\times)\to H^1(X,\underline{\mathbb{Z}})\to\cdots$$

so surjectivity of the global exponential map is controlled by $H^1(X,\underline{\mathbb{Z}})$, a "generalized winding number". Continuing we get

$$H^1(X,\underline{\mathbb{Z}})\to H^1(X,\mathcal{O}_X)\to H^1(X,\mathcal{O}_X^\times)\to H^2(X,\underline{\mathbb{Z}})\to H^2(X,\mathcal{O}_X)\to\cdots,$$

and beyond this we won't really care.

The first Chern class. The map in the above long exact sequence

$$H^1(X,\mathcal{O}_X^\times)\to H^2(X,\underline{\mathbb{Z}})$$

is called the *first Chern class map* and denoted $c_1:\mathrm{Pic}(X)\to H^1(X,\mathbb{Z})$ (changing the notation a bit). Define the *Néron-Severi group* of X, denoted $NS(X)$, as either the image of the map c_1, or equivalently (by exactness) as the kernel of $H^2(X,\underline{\mathbb{Z}})\to H^2(X,\mathcal{O}_X)$.

Remark: (1) We can this write this as $NS(X)=\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, where $\mathrm{Pic}^0(X)$ is the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles L with $c_1(L)=0$. This is just rephrasing the definition above, but can also prove that $\mathrm{Pic}^0(X)$ is equal to the connected component of the origin in $\mathrm{Pic}(X)$.

Theorem (Severi for $K=\mathbb{C}$, Néron in general): The group $NS(X)$ is a finitely generated abelian group; we call the rank $\rho(X)$, the Picard number. (In fact from Hodge theory we'll see that c_1 lands inside of $H^{1,1}\subseteq H^2$, and thus $\rho(x)\le h^{1,1}(X)=g^2$).

First Chern class for complex tori. For a complex torus $X=V/\Lambda$, recall

$$H^2(X,\mathbb{Z})\cong\bigwedge^2 H^1(X,\mathbb{Z})\cong\bigwedge^2\mathrm{Hom}(\Lambda,\mathbb{Z})=\mathrm{Alt}^2(\Lambda,\mathbb{Z}),$$

where $\mathrm{Alt}^2(\Lambda,\mathbb{Z})$ is the set of $\mathbb{Z}$-valued bilinear alternating maps $\Lambda\times\Lambda\to\mathbb{Z}$.

Now let's look at the start of the exponential long exact sequence for V: we have

$$H^0(V, \mathcal{O}_V) \to H^0(V, \mathcal{O}_V^\times) \to 0 = H^1(V, \underline{\mathbb{Z}}),$$

so any nonvanishing global form f on V is $e^{2\pi i g}$ for some g. Next, recall that we have

$$H^1(\Lambda, H^0(V, \mathcal{O}_V^\times)) \cong H^1(X, \mathcal{O}_X^\times),$$

so for a given L, one has a factor of automorphy $f \in Z^1(\Lambda, H^0(V, \mathcal{O}_V^\times))$, which was a function $\Lambda \times V \to \mathbb{C}^\times$. Since we have a contractible universal cover, we can have $f = e^{2\pi i g}$ for $g : \Lambda \times V \to \mathbb{C}^\times$.

Theorem: With the canonical isomorphism discussed before, the first Chern class

$$c_1 : \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times) \to H^2(X, \underline{\mathbb{Z}}) \cong \mathrm{Alt}^2(\Lambda, \mathbb{Z})$$

given by $c_1(L) = E_L$ where E_L is the alternating bilinear form

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v)$$

for any $v \in V$ and any g such that $f = e^{2\pi i g}$ is a factor of automorphy for L.

Easy parts of proof: (1) E_L is independent of choices of g and of v (for latter, need to use the cocycle condition for f). (2) E_L is an alternating 2-form which is $\mathbb{Z}$-valued and bilinear. Harder part, which we don't have the right Hodge theory tools for yet: Prove that this assignment actually agrees with the first Chern class map described above.

Theorem: Given an alternating $\mathbb{R}$-bilinear map $V \times V \to \mathbb{R}$, TFAE:

1. There is L such that E is equal to (the extension of scalars to $\mathbb{R}$ of) $c_1(L)$ for some $L \in \mathrm{Pic}(X)$.

2. We have $E[\Lambda \times \Lambda] \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for $v, w \in V$.

Proof: Doing (1) $\implies$ (2) is not too difficult (need to do some work), but (2) $\implies$ (1) needs some Hodge theory we don't have.

So assuming these theorems, we can consider ourselves to have a complete description of the Chern class map and the Néron-Severi group for a complex torus. But the condition about $E(iv, iw) = E(v, w)$ is kind of a weird one; we'd like to have a better understanding of what that means:

Easy lemma in linear algebra: There is a 1-to-1 correspondence between Hermitian forms H on V (i.e. bilinear $H : V \times V \to \mathbb{C}$ satisfying $\overline{H(v, w)} = H(v, w)$) and alternating bilinear $E : V \times V \to \mathbb{R}$ satisfying $E(iv, iw) = E(v, w)$. This correspondence is given in one direction by $H \mapsto \mathrm{Im}(H)$, and the other by $E \mapsto H(v, w) = E(iv, w) + iE(v, w)$.

Corollary: $NS(X)$ (which we showed before was bilinear alternating $E : V \times V \to \mathbb{R}$ with $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$) is also isomorphic to the set of Hermitian forms $H : V \times V \to \mathbb{C}$ satisfying $\mathrm{Im}\, H[\Lambda \times \Lambda] \subseteq \mathbb{Z}$.

Next goal: So far we have an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0.$$

We want to show that this exact sequence is isomorphic to the sequence

$$0 \to \mathrm{Hom}(\Lambda, S^1) \to P(\Lambda) \to NS(X) \to 0,$$

where $\mathrm{Hom}(\Lambda, S^1)$ is a space of characters, $NS(X)$ is the linear algebra thing with Hermitian forms, and $P(\Lambda)$ is something explicit with both characters and Hermitian forms. This is the Appell-Humbert theorem (due to Lefschetz).

Digression: If we're over $\mathbb{R}$ we have the classical Poincaré lemma, which lets us get the connection between De Rham cohomology and singular cohomology. If we're over $\mathbb{C}$, we have a $\overline{\partial}$-Poincaré lemma which tells us something about Dolbeault cohomology. Will say things about these, and then a bit more about Hodge theory (and what's the major simplification for abelian varieties vs. Kalher manifolds).

Classical Poincaré lemma: Let us look at $\mathbb{C}^n$, which is $\mathbb{R}^{2n}$ as a real manifold (or more generally any smooth manifold M). Let $T_0^*(\mathbb{C}^n)$ (or $T_z^*(M)$) be the cotangent space, and pick the usual basis $\{dx_j, dy_j\}_{j=1}^n$. Over $\mathbb{C}$ we'll also want to work with the complex basis $\{dz_j, d\overline{z}_j\}_{j=1}^n$ with $dz_j = dx_j + idy_j$ and $d\overline{z}_j = dx_j - idy_j$. For the tangent space $T_0\mathbb{C}^n$ or T_zM, let $\{\partial/\partial z_j, \partial/\partial \overline{z}_j\}$ denote the dual basis. Can check

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \qquad \frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right).$$

Remark: Over $\mathbb{C}$, the Cauchy-Riemann equations tell us that if $f \in C^\infty(U)$ then f is holomorphic iff $\partial f/\partial \overline{z} = 0$.

Total differential: Given $f \in C^\infty(U)$, define the total differential as

$$df = \partial f + \overline{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_i + \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_i.$$

Theorem: For $f \in C^\infty(U)$ with $U \subseteq \mathbb{C}^n$, f is holomorphic iff $\partial f = 0$. Define the holomorphic tangent bundle as the vector subbundle generated by the $\{\partial/\partial z_j\}$.

Definition: A differential form of degree k (i.e. a k-form) is a smooth section of $\bigwedge^k T^*(M)$. I.e. if $p \in M$, a k-form β gives an alternating multilinear form $\bigoplus_{i=1}^k T_p(M) \to \mathbb{R}$. We then get a sheaf of k-forms on a smooth manifold M, which we denote Ω_M^k (and $\Omega^0 = \mathcal{O}_M$).

8 Lecture - 02/25/2016

Definition: A differential form of degree k is a smooth (C^∞) global section of $\Omega^k = \bigwedge^k T^*$ (exterior power of the cotangent bundle, or its corresponding sheaf). A k-form at $p \in M$ is an alternating multilinear form $\bigoplus^k T_p(M) \to \mathbb{R}$. Let $\Omega^k(M)$ or $A^k(M)$ (or $\Gamma(\Omega^k, M)$ or $H^0(M, \Omega^k)$) denote the set of global sections (i.e. global k-forms).

Have a cochain complex

$$0 \to \Omega^0(M) = \mathcal{O}_M \to \Omega^1(M) \to \Omega^2(M) \to \cdots$$

which eventually ends ($\Omega^k(M) = 0$ for $k > \dim M$). The differential maps $d : \Omega^k \to \Omega^{k+1}(M)$ can be defined explicitly on coordinates, or alternatively characterized as the unique collection of $\mathbb{R}$-linear maps satisfying

1. If $f \in \Omega^0(M) = \mathcal{O}_M(M)$ then df is the total differential of f and $d(df) = 0$.
2. We have $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg p} \alpha \wedge (d\beta)$.

In local coordinates this is given on a k-form $\varphi = g dx_{i_1} \wedge \cdots \wedge dx_{i_k} = g dx_I$ by

$$d\varphi = \sum \frac{\partial g}{\partial x_i}(dx_i \wedge dx_I).$$

Facts about d:

1. $d^2 = 0$ (so we do have a complex). Can thus define cohomology:

   $$H^p_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{img}(d : \Omega^{p-1}(M) \to \Omega^p(M))}.$$

   Elements of this kernel are called *closed* and elements of the image are called *exact*; so de Rham cohomology measures failure of closed forms to be exact. (Example to keep in mind: If S^1 is the circle we have a closed form $d\theta$ which is not exact).

2. Poincaré lemma: If $U \subseteq \mathbb{R}^n$ is contractible then $H^p_{\mathrm{dR}}(U) = 0$ for all $p > 0$.

3. De Rham theorem: If M is a smooth manifold then $\check{H}^p(M, \underline{\mathbb{R}})$ (which is our usual singular cohomology) is isomorphic to $H^p_{\mathrm{dR}}(M)$. Proof idea: $0 \to \underline{\mathbb{R}} \to \Omega^0 \to \Omega^1 \to \cdots$ is an exact sequence of sheaves (by Poincaré); break it up into short exact sequences in the standard way, use the corresponding long exact sequences on cohomology and it formally follows.

Dolbeault cohomology. Now we want to start using the complex structure on a complex manifold. Using holomorphic charts, we locally have differentials $\{dz_i, d\overline{z}_i\}$ spanning the cotangent space, with dual elements $\{\partial/\partial z_i, \partial/\partial \overline{z}_i\}$ spanning the tangent space. Using these split up

$$T^*_p(M) = T^*_p(M)' \oplus T^*_p(M)''$$

with $T^*_p(M)$ spanned by the dz_i's and $T^*_p(M)''$ spanned by the $d\overline{z}_i$'s. Then we have a direct sum decomposition

$$\bigwedge^n T^*_p(M) = \bigoplus_{p+q=n} \left(\bigwedge^p T^*_p(M)' \right) \otimes \left(\bigwedge^p T^*_p(M)'' \right),$$

with the (p, q) part spanned by things of the form $dz_I \wedge d\overline{z}_J$ for $|I| = p$ and $|J| = q$. Using this decomposition define a sheaf $\Omega^{p,q}$ of C^∞ (p, q)-forms; we set

$$\Omega^{p,q}(M) = \left\{ \varphi \in \Omega^n(M) : \forall z, \varphi(z) \in \left(\bigwedge^p T^*_p(M)' \right) \otimes \left(\bigwedge^p T^*_p(M)'' \right) \right\}.$$

Also called $A^{p,q}(M)$ or $\Gamma(\Omega^{p,q}, M)$ or $H^0(M, \Omega^{p,q})$. This gives a filtration $\Omega^n(M) = \bigoplus_{p+q=n} \Omega^{p,q}(M)$.

What happens when we apply d to something in $\Omega^{p,q}(M)$? We can see it maps into $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ because $\varphi(z)$ is in

$$\Big(\bigwedge^p T_z^*(M)'\Big) \otimes \Big(\bigwedge^p T_z^*(M)''\Big) \wedge T_z^*(M).$$

Split up $d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ as the sum of $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Then: easy to check $\overline{\partial}^2 = 0$ and we get a cochain complex

$$0 \to \Omega^{p,0}(M) \to \Omega^{p,1}(M) \to \cdots,$$

and define the Dolbeault cohomology as

$$H_{\overline{\partial}}^{p,q}(M) = \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\operatorname{img}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Now, want to relate this to some other sheaf cohomology; our exact-sequence-based proof of the de Rham theorem suggests what we need. We first need the $\overline{\partial}$-Poincaré lemma: if D is a polydisc in $\mathbb{C}^n$ (a product of discs in $\mathbb{C}$) then $H_{\overline{\partial}}^{p,q}(D) = 0$ for $q \geq 1$. (Aside: Poincaré was studying the problem that if $g \in C^\infty(D)$ for $D \subseteq \mathbb{C}$ then he wanted to find f with $\partial f/\partial \overline{z} = g$ can be solved on on a slightly smaller disc).

Analogue of the de Rham theorem: Poincaré lemma tells us we have an exact sequence

$$0 \to \Omega_{hol}^p \to \Omega^{p,0} \to \Omega^{p,1} \to \cdots$$

with $\Omega_{hol}^p \to \Omega^{p,0}$ the inclusion and the other maps $\overline{\partial}$ (note that Ω_{hol}^p is indeed the kernel of $\overline{\partial}$ on Ω_{hol}^p). By the same argument from our sheaf proof of the de Rham theorem we can prove the Dolbeault theorem:

$$H_{\overline{\partial}}^{p,q}(M) \cong H^q(M, \Omega^p)$$

for $p, q \geq 0$. Examples:

1. $H_{\overline{\partial}}^{0,q}(M) \cong H^q(M, \mathcal{O}_M)$ (so if $q \geq \dim M$ both are zero).

2. We have $H^q(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0$ for $q \geq 1$ by example 1 and the $\overline{\partial}$-Poincaré lemma.

3. If D is a polydisc, $H_{\overline{\partial}}^{p,0} = H^0(D, \Omega_D^p) \cong H^0(D, \mathcal{O}_D) \otimes \Omega_D^p$ is usually nontrivial, so the $q \geq 1$ hypothesis in the Poincaré lemma matters!

Hodge decomposition. Assume $X = V/\Lambda$ is a complex torus. Then $H^n(X, \mathbb{C})$ is isomorphic to the set $IF^n(X) = \bigoplus_{p+q=n} IF^{p,q}(X)$, where $IF^{p,q}(X)$ (the "invariant forms") are the things of the form

$$\sum_{|I|=p, |J|=q} a_{IJ} dv_I \wedge d\overline{v}_J$$

for $a_{IJ} \in \mathbb{C}$, where $v_1, \ldots, v_g$ are the coordinate functions for a fixed basis of V. Because we're on a torus, it's easy to write down this decomposition.

For more general (compact) X: there's a general theory that says if X has a "nice" metric (i.e. a Euclidean metric or a "degree 2 approximation" of one, i.e. Kähler metric). Example: $X = V/\Lambda$ (has the Euclidean metric) or any complex projective variety (since $\mathbb{P}^n$ has a Fubini-Study metric which is Kähler). If X has a nice metric then $H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$ (where $H^{p,q}(X)$ is something that's isomorphic to $H_{\overline{\partial}}^{p,q}(X)$ and is isomorphic to the space of harmonic forms; also we have $\overline{H^{p,q}} \cong H^{q,p}$). For our situation of $X = V/\Lambda$ one can show the $IF^{p,q}(X)$ we wrote down is isomorphic to $H^{p,q}(X)$ is isomorphic to $H_{\overline{\partial}}^{p,q}(X) \cong H^q(X, \Omega^p)$. (The first two "is isomorphic to"s are theorems that take a fair amount of work).

What's in the background of these isomorphisms? We start with our Euclidean metric $ds^2 = \sum dv_i \otimes d\overline{v}_i$. This has an associated (1,1)-form

$$\omega = -\frac{1}{2}\operatorname{Im}(ds^2) = \frac{i}{2}\sum_{i=1}^{g} dv_i \wedge d\overline{v}_i.$$

Then we get a volume form

$$dv = \frac{1}{g!}\bigwedge^{g}\omega = (-1)^{\binom{g}{2}}\left(\frac{i}{2}\right)^{g}(dv_1 \wedge d\overline{v}_1 \wedge dv_2 \wedge \cdots).$$

Now that we have a volume form we can define an inner product on $\Omega^{p,q}(M)$ (but not complete, so not a Hilbert space - a cause of a lot of trouble) by

$$(\varphi, \psi) = \sum_{|I|=p,|J|=q}\int_X \varphi_{IJ}\overline{\psi}_{IJ}dv,$$

for $\varphi = \sum \varphi_{IJ} dv_I \wedge d\overline{v}_J$. Then since we have an inner product we can talk about an adjoint map $\overline{\delta}$ of $\overline{\partial}$. Define Laplace-Beltrami operator

$$\Delta = \overline{\partial\delta} + \overline{\delta\partial} : \Omega^{p,q}(M) \to \Omega^{p,q}(M).$$

In our situation in coordinates can compute

$$\Delta(\varphi dv_I \wedge dv_J) = -\sum_i \frac{\partial^2\varphi}{\partial v_i \partial\overline{v}_i}(dv_I \wedge dv_J)$$

so this is really the usual Laplacian.

9 Lecture - 03/01/2016

Remarks from last time: The Dolbeault theorem holds for all $p, q \geq 0$, and also all of the things we said about Kälher manifolds always are only for compact manifolds. Also a remark about the first Chern class map: if L is a line bundle, can associate a divisor D to it (talk about later); this will actually give us a cycle in $H_{2g-2}(X, \mathbb{Z})$, and $c_1(L)$ is the Poincaré dual of that cycle. This is a more geometric way of talking about it.

Last time: for $X = V/\Lambda$ a complex torus, last time said we have

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} IF^{p,q}(X)$$

where we define $IF^{p,q}(X)$ last time. More generally: for X a compact complex manifold with a Kähler form, then we have a Hodge decomposition

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the space of harmonic (p,q)-forms (and these spaces satisfy $\overline{H^{p,q}} \cong H^{q,p}$) and we have

$$H^{p,q} \cong H_d^{p+q}(X) \cong H_{\overline{\partial}}^{p,q}(X).$$

This is all hard; once this hard stuff is done we can show that in the case of a complex torus, the harmonic things are just the invariant forms and we recover what we had above.

Roadmap for how all of this works (in the case $X = V/\Lambda$, but containing all of the ideas we need in general): let $ds^2 = \sum_{j=1}^g dv_j \otimes d\overline{v}_j$, where $v_1, \ldots, v_g$ are the coordinates of V. Then we have a $(1,1)$-form

$$\omega = -\frac{1}{2}\operatorname{Im}(ds^2) = \frac{i}{2}\sum_{j=1}^g dv_j \wedge d\overline{v}_j.$$

Then there's a volume form $dV = \frac{1}{g!}\bigwedge^g \omega$ (which we wrote down the expansion of last time). Can define a Hermitian inner product on $\Omega^{p,q}(M)$ by

$$(\psi, \varphi) = \sum_{|I|=p, |J|=q} \int \varphi_{IJ}\overline{\psi}_{IJ} dV$$

where $\varphi = \sum_{|I|=p,|J|=q} \varphi_{IJ} dv_I \wedge d\overline{v}_J$ and similarly for ψ. This makes it into a pre-Hilbert space (i.e. a non-complete inner product space) and can define an adjoint map $\overline{\delta}$ to $\overline{\partial}$ satisfying $(\varphi, \overline{\partial}\psi) = (\overline{\delta}\varphi, \psi)$. Then the Laplace-Beltrami operator is $\Delta = \overline{\partial}\overline{\delta} + \overline{\delta}\overline{\partial}$. Why do we write this down? First justification: it looks like the Laplacian in local coordinates.

Main point of all this: from this formal setup, can very easily show that a closed form ψ whose norm (WRT the inner product) is minimal in its class (in de Rham or Dolbeault cohomology) is the unique solution in that class to $\overline{\delta}\psi = 0$. Note: closed means $\overline{\partial}\psi = 0$ as well, so we immediately get that $\Delta\psi = 0$, i.e. a minimal-energy thing is harmonic. Can go in the other direction too, that any harmonic form satisfies $\overline{\delta}\psi = 0$. Upshot: we can identify the cohomology group $H_{\overline{\partial}}^{p,q}(X)$ with the space $H^{p,q}(X)$ of harmonic forms - each cohomology class has a unique harmonic representative.

So now we want to study solutions to $\Delta\psi = 0$, which means we need to solve a PDE on X to understand $H^{p,q}(X)$. To study this we need to define two more operators that help us solve the PDE. The first is

$$H : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$$

which is given by

$$\varphi dv_I \wedge d\overline{v}_J \mapsto \left(\frac{1}{\operatorname{vol}(X)}\int_X \varphi dV\right) dv_I \wedge d\overline{v}_J;$$

this projects any form onto an invariant thing (certainly satisfies $H^2 = H$) and in the abelian variety thing this projects onto $IF^{p,q}(M)$.

The second operator is G, a Green's function for Δ; should be an inverse to the extent we can have: $\Delta G = G\Delta = \mathrm{id} - H$ and $HG = GH = 0$. Finding this is hard! For a general Kähler manifold need the Sobolev lemma and other things (see Griffiths and Harris chapter 0). For complex tori can get away with using just Fourier analysis. Won't write down explicitly what the G is but the idea is we have a function φ on X, it lifts to a periodic function $\widetilde{\varphi}$ on $V \cong \mathbb{C}^g$, which has a Fourier expansion, and $G\widetilde{\varphi}$ comes from renormalizing the coefficients.

Once we have this, we can then prove the Hodge theorem, that every class has a unique harmonic representative (ultimately the hard part of the theory is showing that we have enough harmonic forms!) because we can explicitly write down $\varphi = H\varphi + \overline{\delta\partial}\varphi + \overline{\partial\delta}\varphi$ as our Hodge decomposition of the form.

Appell-Humbert theorem. This gives an explicit description of $NS(X)$, $\mathrm{Pic}(X)$, and $\mathrm{Pic}^0(X)$ for a $X = V/\Lambda$ a complex torus. Recall that we had an explicit description of $NS(X)$ already: it corresponded to the set of Hermitian forms $H : V \times V \to \mathbb{C}$ with $\mathrm{Im}\, H(\Lambda, \Lambda) \subseteq \mathbb{Z}$. Define $\mathrm{Pic}^0(X)$ to be the kernel of the first Chern class map $c_1 : \mathrm{Pic}(X) \twoheadrightarrow NS(X)$, i.e. the collection of line bundles with $c_1(L) = 0$. So we have an SES

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0;$$

the Appell-Humbert theorem gives another (more explicit) SES isomorphic to this one.

Definition: Let $T_1 = \{z \in \mathbb{C}^\times : |z| = 1\}$ be the usual unit circle in $\mathbb{C}$. Given a Hermitian form H in $NS(X)$, define a *semi-character for* H to be a function $\chi : \Lambda \to T_1$ satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu) \exp(\pi i \,\mathrm{Im}\, H(\lambda, \mu)).$$

Let $P(\Lambda)$ denote the set of all pairs (H, χ) with $H \in NS(X)$ and χ a semi-character for H. Remarks: (1) for $H = 0$, semi-characters for H are exactly characters. (2) $P(\Lambda)$ is a group under $(H_1, \chi_1) \cdot (H_2, \chi_2) = (H_1 + H_2, \chi_1 + \chi_2)$. So we get an exact sequence

$$0 \to \mathrm{Hom}(\Lambda, T_1) \to P(\Lambda) \to NS(X)$$

with the first map given by $\chi \mapsto (0, \chi)$ and the second is $(H, \chi) \mapsto H$. Can also justify that the second map is surjective (by explicitly constructing a semi-character for each H?)

Now, we want to show this SES is isomorphic to the original one for $\mathrm{Pic}(X)$. For this we need an isomorphism $P(\Lambda) \cong \mathrm{Pic}(X) \cong H^1(\Lambda, M)$ (for $M = H^0(V, \mathcal{O}_V^\times)$). We define the map $P(\Lambda) \to H^1(\Lambda, M)$ by mapping (H, χ) to the class of a cocycle $a_{H,\chi}$ which we define by

$$a_{H,\chi}(\lambda, v) = \chi(\lambda) \exp(\pi H(\lambda, v) + \tfrac{\pi}{2} H(v, v)).$$

Straightforward to check the cocycle condition

$$a_{H,\chi}(\lambda + \mu, v) = a_{H,\chi}(\lambda, \mu + v) a_{H,\chi}(\mu, v).$$

Thus we get a map $P(\Lambda) \to \mathrm{Pic}(X)$ given explicitly by taking (H, χ) to a line bundle $L(H, \chi) = (V \times \mathbb{C})/\Lambda$ with Λ acting by

$$\lambda \circ (v, t) = (v + \lambda, a_{H,\chi}(\lambda, v)t).$$

Facts: (1) Can check explicitly that this is a group homomorphism. (2) Can check explicitly that the map $P(\Lambda) \to NS(X)$ from before equals the composition of this map $P(\Lambda) \to \mathrm{Pic}(X)$ with $c_1 : \mathrm{Pic}(X) \to NS(X)$. (3) This implies $P(\Lambda) \to NS(X)$ is surjective since c_1 is.

Calling this map $\alpha : P(\Lambda) \to NS(X)$, we get a morphism of SES's

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\Lambda, T_1) & \longrightarrow & P(\Lambda) & \longrightarrow & NS(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & NS(X) & \longrightarrow & 0
\end{array}$$

where α' is the restriction of α to $\mathrm{Hom}(\Lambda, T_1)$. Above we said the right-hand square commutes, and this implies α' restricts to what we want (and the left-hand square commutes arbitrarily). So we just need to show that α' and α are isomorphisms. It's sufficient to show α' is an isomorphism since then the five-lemma implies α is. (Remark: once we do this we have a very explicit way to talk about $\mathrm{Pic}(X)$ - not only have we parametrized line bundles by (H, χ), such a pair even leads to a distinguished cocycle in the cohomology class to work with).

Proof that α' is an isomorphism: Start with a diagram

$$\begin{array}{ccccccc} H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^\times) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\ & & \uparrow\!\!\uparrow & & \downarrow = & & \\ & & H^1(X,\mathbb{C}) & \xrightarrow{\varepsilon} & H^1(X,\mathcal{O}_X^\times) & & \end{array}.$$

The surjective map $H^1(X, \mathbb{C}) \twoheadrightarrow H^1(X, \mathcal{O}_X)$ comes from the Hodge decomposition and the Dolbeault theorem: have

$$H^1(X, \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X) \cong H^0(X, \Omega_X) \oplus H^1(X, \mathcal{O}_X)$$

and the map is projection to the second coordinate in this isomorphism. The map ε comes from taking the sheaf map $\underline{\mathbb{C}}^\times \to \mathcal{O}_X^\times$ taking a locally constant function f to the locally constant nonvanishing function $\exp(2\pi i f)$; this induces a map $\varepsilon : H^1(X, \underline{\mathbb{C}}) \to H^1(X, \mathcal{O}_X^\times)$ on cohomology. Can check this diagram commutes.

Now: the diagram tells us that $\mathrm{Pic}^0(X)$ is the image of

$$\varepsilon : H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X^\times) \cong H^1(\Lambda, M).$$

Therefore, any $L \in \mathrm{Pic}^0(X)$ can be represented by a Čech 1-cocycle (i.e. an element of $\check{Z}^1(X, \mathcal{O}_X^\times)$) with constant coefficients. From the formula we can see that the corresponding class in $H^1(\Lambda, M)$ is represented by a factor of automorphy (element of $Z^1(\Lambda, M)$) with constant coefficients, i.e. the cocycle $f(\lambda, v)$ is independent of v.

Claim 1: $\alpha' : \mathrm{Hom}(\Lambda, T) \to \mathrm{Pic}^0(X)$ is surjective. Proof: Let L be in $\mathrm{Pic}^0(X)$ have representative $f \in Z^1(\Lambda, M)$, which by what we've said has "constant coefficients". The cocycle condition is then

$$f(\lambda + \mu, \widetilde{x}) = f(\Lambda, \mu + \widetilde{x}) f(\mu, \widetilde{x});$$

since f is independent of the second entry and we define $f(\lambda) = f(\lambda, \widetilde{x})$ for any (and thus all) $\widetilde{x}$, this gives $f(\lambda + \mu) = f(\lambda) f(\mu)$ so f is a homomorphism $\Lambda \to \mathbb{C}^\times$. If we write $f = e^{2\pi i g}$ we get $g(\lambda + \mu) \equiv g(\lambda) + g(\mu) \pmod{\mathbb{Z}}$; so g is not a homomorphism itself, but the imaginary part is! So $\mathrm{Im}\, g : \Lambda \to \mathbb{R}$ is a homomorphism. Extend by $\mathbb{R}$-linearity to $\mathrm{Im}\, g : V \to \mathbb{R}$. Then define $\ell : V \to \mathbb{C}$ by $v \mapsto \mathrm{Im}\, g(iv) + i \,\mathrm{Im}\, g(v)$, which gives a $\mathbb{C}$-linear functional.

Now want to show that our $f(\lambda, \widetilde{x}) = f(\lambda) = e^{2\pi i g(v)}$ gets hit by some $\alpha(0, \chi) = a_{0,\chi}$. But we have $\exp(2\pi i \ell) \in H^0(V, \mathcal{O}_V^\times)$. We take $\chi_L(\lambda, v)$ to be defined by

$$\chi_L(\lambda, v) = f(\lambda) \exp(2\pi i \ell(v) - 2\pi i \ell(v + \lambda)).$$

Claim: $\chi_L(-, v)$ is in $\mathrm{Hom}(\Lambda, T_1)$, is independent of v, has values in T_1, and is in the same class as f in Z^1. Will check this next time; but basically the upshot is that we start with an arbitrary f and change it to get something that definitely is in the image of α' which is equivalent to f.

10 Lecture - 03/03/2016

Finishing up the Appell-Humbert proof: We needed to show that the map

$$\mathrm{Hom}(\Lambda, T_1) \to \mathrm{Pic}^0(X) \subseteq \mathrm{Pic}(X) \cong H^1(\Lambda, M)$$

for $M = H^0(V, \mathcal{O}_V^\times)$ (given by mapping χ to the cocycle $a_{0,\chi}(\lambda, v) = \chi(\lambda)$) is an isomorphism. For surjectivity, we started with a cocycle $f \in Z^1(\Lambda, M)$ that gave a class in $\mathrm{Pic}^0(X)$; such a cocycle has "constant coefficients", i.e. $f : \Lambda \times X \to \mathbb{C}^\times$ is constant in the second variable so really just gives a function $f : \Lambda \to \mathbb{C}^\times$ satisfying $f(\lambda) = f(\lambda)f(\mu)$, i.e. a character.

Given such an f, we want to change it by a coboundary to get a unital character. To do this, start by taking a global logarithm to get $f = \exp(2\pi i g)$ for g satisfying $g(\lambda + \mu) \equiv g(\lambda) + g(\mu) \pmod{\mathbb{Z}}$. Since this congruence is only for the real part, $\mathrm{Im}\, g : V \to \mathbb{R}$ is honestly $\mathbb{R}$-linear, and we can define a $\mathbb{C}$-linear functional $\ell : V \to \mathbb{C}$ in the standard way: $\ell(v) = \mathrm{Im}\, g(iv) + i\,\mathrm{Im}\, g(v)$. Then $e^{2\pi i \ell}$ is in M, and thus

$$(\lambda, v) \mapsto \exp(2\pi i \ell(v) - 2\pi i \ell(v + \lambda))$$

is a 1-coboundary which is a homomorphism and thus f is in the same cohomology class as the cocycle

$$\chi(\lambda, v) = f(\lambda) \exp(2\pi i \ell(v) - 2\pi i \ell(v + \lambda)).$$

But, we can check that

$$\chi(\lambda, v) = \exp(2\pi i\, \mathrm{Re}\, g(\lambda) - \mathrm{Im}\, g(i\lambda))$$

is independent of v and has magnitude 1, so it's the image of $\chi \in \mathrm{Hom}(\Lambda, T_1)$.

This gives surjectivity; for injectivity if χ_1, χ_2 both give the same line bundle $L \in \mathrm{Pic}(X)$ then we have

$$\chi_1(\lambda) = \chi_2(\lambda) \frac{h(\lambda + v)}{h(v)}$$

for some h (and all λ, v). Since $|\chi_1|, |\chi_2| = 1$ we conclude $|h(\lambda + v)| = |h(v)|$ for all $\lambda \in \Lambda$ and $v \in V$; this lets us conclude h is bounded and Liouville gives that h is constant so $\chi_1 = \chi_2$.

Canonical factors. So now we have our isomorphism of SES's

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\Lambda, T_1) & \longrightarrow & P(\Lambda) & \longrightarrow & NS(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & NS(X) & \longrightarrow & 0 \end{array}.$$

Explicitly, the map $P(\Lambda) \cong \mathrm{Pic}(X)$ is given by mapping a pair (H, χ) to the line bundle L with $c_1(L) = H$ coming from the cocycle

$$a_{H,\chi}(\lambda, v) = \chi(\lambda) \exp\left(\pi H(\lambda, v) + \tfrac{\pi}{2} H(\lambda, \lambda)\right);$$

this cocycle is called the *canonical factor* associated to $L = L(H, \chi)$. Most questions about line bundles on X then boils down to explicit computations with the canonical factors $a_{H,\chi}$.

Basic properties of semi-characters and canonical factors:

1. If χ is a semicharacter and $\lambda \in \Lambda$ then $\chi(n\lambda) = \chi(\lambda)^n$. (Induct on n; formula for semicharacters gives you something with a $H(\lambda, \lambda) = 0$ by alternating...)

2. $a_L(\lambda, v + w) = a_L(\lambda, v) e^{\pi H(w, \lambda)}$.

3. $1/a_L(\lambda, v) = a_L(-\lambda, v) e^{-\pi H(\lambda, \lambda)}$.

Behavior of line bundles under holomorphic maps. Recall: a holomorphic map between complex tori is a composition of a translation and a homomorphism, so to describe pullbacks under a general holomorphic map it's sufficient to describe pullbacks under these special cases.

Lemma 1: Let $t_x : X \to X$ be the translation-by-x map ($t_x(y) = x + y$); then

$$t_x^* L(H, \chi) = L(H, \chi e^{2\pi i \operatorname{Im} H(\widetilde{x}, \cdot)})$$

where $\widetilde{x}$ is a lift of x to the universal cover V.

Lemma 2: If $f : X' \to X$ is a homomorphism (with $X = V/\Lambda$ and $X' = V'/\Lambda'$) then

$$f^* L(H, \chi) = L(f_{\text{an}}^* H, f_{\text{Int}}^* \chi)$$

where $(f^* H)(u, v) = H(f(u), f(v))$, etc.

Corollary (theorem of squares): Suppose $v, w \in L$ and $L \in \operatorname{Pic}(X)$; then

$$t_{v+w}^* L \cong t_v^* L \otimes t_w^* L \otimes L^{-1}.$$

Corollary (theorem of the cube): Let X_1, X_2, X_3 be complex tori and L a line bundle on $X_1 \times X_2 \times X_3$. If L is trivial when restricted to each of the three "faces" $X_1 \times X_2 \times \{0\}$, $X_1 \times \{0\} \times X_3$, and $\{0\} \times X_2 \times X_3$, then L is trivial on $X_1 \times X_2 \times X_3$.

Remark: Theorem of the cube works if X_1, X_2, X_3 are complete varieties, and even more (see Mumford's book on abelian varieties p. 55), you only need 2 of them to be complete + the third to be connected.

Corollary: If $n \in \mathbb{Z}$ and $n_X : X \to X$ is the isogeny $x \mapsto nx$, and $L \in \operatorname{Pic}(X)$, then

$$n_X^* L = L^{(n^2+n)/2} \otimes (-1)^* L^{(n^2-n)/2}$$

where here L^k means the tensor product of k copies of L.

Proof: Take $L = L(H, \chi)$ and consider the RHS $L(H', \chi')$. Looking at the group law on $P(\Lambda)$ we find $H' = \frac{n^2+n}{2} H + (-1)^* \frac{n^2-n}{2} H$ and $\chi' = \chi^{(n^2+n)/2} (-1)^* \chi^{(n^2-n)/2}$. Now, what does $(-1)^*$ do? For H, we have $((-1)^* H)(u, v) = H(-u, -v) = H(u, v)$ by Hermitianness, so

$$H' = \frac{n^2 + n}{2} H + \frac{n^2 - n}{2} H = n^2 H$$

and similarly $(-1)^* \chi = \chi^{-1}$ so $\chi' = \chi^n$. Thus we conclude

$$L^{(n^2+n)/2} \otimes (-1)^* L^{(n^2-n)/2} = L(n^2 H, \chi^n).$$

On the other hand, applying the definition of pullbacks directly to n_X^* we get $n_X^* L = L(n^2 H, \chi^n)$.

Definition: Say a line bundle L is symmetric if $(-1)^* L = L$; then $n_X^* L = L^{n^2}$. Lemma: $L = L(H, \chi)$ is symmetric iff χ maps into $\{\pm 1\}$. Proof: $(-1)^* L(H, \chi) = L(H, \chi^{-1})$.

Dual complex torus. Goals: Any $X = V/\Lambda$ has a dual $\widehat{X}$, with functorial properties; for each $L \in \operatorname{Pic}(X)$ we want to get $\varphi_L : X \to \widehat{X}$. Fancier stuff: look at $X \times \widehat{X}$, the Poincaré bundle, and biextensions.

What's the idea: Appell-Humbert says $\operatorname{Hom}(\Lambda, T_1) \cong \operatorname{Pic}^0(X)$, and we know $\operatorname{Hom}(\Lambda, T_1) \cong (\mathbb{R}/\mathbb{Z})^{2g}$. Question: can we naturally make $\operatorname{Pic}^0(X)$ into a complex torus itself? Answer: yes, and it will be canonically isomorphic to $\widehat{X}$.

So start with $X = V/\Lambda$, and consider $\Omega = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$, the space of $\mathbb{C}$-linear functionals $\ell : V \to \mathbb{C}$; this is the cotangent space to X at 0. Similarly set $\overline{\Omega} = \operatorname{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ to be the space of $\mathbb{C}$-antilinear functionals. In fact we have

$$\overline{\Omega} \cong \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$$

by $\ell \mapsto K = \operatorname{Im} \ell$ and $K \mapsto \ell(v) = -K(iv) + iK(v)$.

It follows that we have a bilinear map $\overline{\Omega} \times V \to \mathbb{R}$ given by $\langle \ell, v \rangle = \operatorname{Im} \ell(v)$. This is $\mathbb{R}$-bilinear and nondegenerate. Then

$$\widehat{\Lambda} = \{\ell \in \overline{\Omega} : \langle \ell, \lambda \rangle \in \mathbb{Z} \forall \lambda \in \Lambda\}$$

is a lattice in $\overline{\Omega}$. Define $\widehat{X} = \overline{\Omega}/\widehat{\Lambda}$; this is the dual torus to X.

Theorem: There is a canonical isomorphism $\widehat{X} \cong \operatorname{Pic}^0(X)$ induced by $\overline{\Omega} \to \operatorname{Hom}(\Lambda, T_1)$ given by $\ell \mapsto e^{2\pi i \operatorname{Im} \ell}$. Proof: Nondegenerate implies the map is surjective, and by definition the kernel is $\overline{\Lambda}$. So $\operatorname{Pic}^0(X)$ is naturally a complex torus.

11 Lecture - 03/08/2016

As usual: $X_i = V_i/\Lambda_i$. Today we want to cover basic functorial properties of $\widehat{X} = \mathrm{Pic}^0(X)$, which we constructed as a complex torus last time.

First thing: If you have a homomorphism $f : X_1 \to X_2$ we want a dual map. First of all, let $f_{an} : V_1 \to V_2$ be the corresponding homomorphism on the covering spaces; then we can define a map $f_{an}^* : \overline{\Omega}_2 \to \overline{\Omega}_1$ which has the property that $f_{an}^*[\overline{\Lambda}_2] \subseteq \overline{\Lambda}_1$. Thus this descends to a homomorphism $\widehat{f} : \widehat{X}_2 \to \widehat{X}_1$. Moreover, under the natural identifications $\widehat{X}_i \cong \mathrm{Pic}^0(X_i)$, this homomorphism $\widehat{f}$ corresponds to the pullback of line bundles $f^* : \mathrm{Pic}^0(X_2) \to \mathrm{Pic}^0(X_1)$. (If we further use the isomorphism $\mathrm{Pic}^0(X_i) \cong \mathrm{Hom}(\Lambda_i, T_1)$ then f^* corresponds to $f_{\mathrm{Int}}^* : \mathrm{Hom}(\Lambda_2, T_1) \to \mathrm{Hom}(\Lambda_1, T_2)$).

If $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are two homomorphisms, we can check that $\widehat{g \circ f} = \widehat{f} \circ \widehat{g}$. We can also check that $\widehat{\mathrm{id}_X} = \mathrm{id}_{\widehat{X}}$. Thus duality $(\hat{\cdot})$ is a contravariant functor. Also note that it's an involution: $\widehat{\widehat{X}} = X$ and $\widehat{\widehat{f}} = f$. Furthermore it's an exact functor: if we have a short exact sequence

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

then the induced sequence

$$0 \to \widehat{X}_3 \to \widehat{X}_2 \to \widehat{X}_1 \to 0$$

is also exact. Then we have a diagram of SES's

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \end{array};$$

since the cokernels are trivial, the snake lemma gives us a short exact sequence of the kernels

$$0 \to \Lambda_1 \to \Lambda_2 \to \Lambda_3 \to 0.$$

Then applying $\mathrm{Hom}(\cdot, T_1)$ to it get a short exact sequence

$$0 \to \mathrm{Hom}(\Lambda_3, T_1) \to \mathrm{Hom}(\Lambda_2, T_1) \to \mathrm{Hom}(\Lambda_1, T_1) \to 0.$$

(To prove that this remains exact, we could either use that Λ_3 is free and thus projective so the SES splits which means $\mathrm{Hom}(\cdot, T_1)$ is exact on it, or alternatively use the long exact sequence in cohomology and use that the Ext groups that show up are trivial again because Λ_3 is projective).

Isogenies and duality. Proposition: if $f : X_1 \to X_2$ is an isogeny, then the dual homomorphism $\widehat{f} : \widehat{X}_2 \to \widehat{X}_1$ is also an isogeny. In fact, $\ker(\widehat{f}) = \mathrm{Hom}(\ker(f), T_1)$, and we have $\deg \widehat{f} = \deg f$.

Proof: Everything follows from the computation $\ker(\widehat{f}) = \mathrm{Hom}(\ker(f), T_1)$ (because then $\ker(\widehat{f})$ is the dual group to $\ker(f)$ so has the same size). But we know $\ker(\widehat{f})$ is the kernel of

$$f_{\mathrm{Int}}^* : \mathrm{Hom}(\Lambda_2, T_1) \to \mathrm{Hom}(\Lambda_1, T_1),$$

which is the subgroup $\mathrm{Hom}(\Lambda_2/f_{\mathrm{Int}}^*[\Lambda_1], T_1)$. But $\Lambda_2/f_{\mathrm{Int}}[\Lambda_1]$ is $\ker(f)$.

Line bundles and duality. Question: Do line bundles descend under isogeny? Proposition: If $f : X_1 \to X_2$ is an isogeny and $L = L(H, \chi)$ is a line bundle in $\mathrm{Pic}(X_1)$, then $L = f^*M$ for some $M \in \mathrm{Pic}(X_2)$ iff $H[f_{\mathrm{an}}^{-1}[\Lambda_2] \times f_{\mathrm{an}}^{-1}[\Lambda_2]] \subseteq \mathbb{Z}$.

Proof: We have an explicit formula for pullbacks, and we can explicitly see that if we start with a line bundle on X_2 and pull back to X_1 we get a $L(H, \chi)$ with this property. The content of the theorem is proving that the containment is sufficient (which is where we actually use the hypothesis of being an isogeny). So suppose $H[f_{\text{an}}^{-1}[\Lambda_2] \times f_{\text{an}}^{-1}[\Lambda_2]] \subseteq \mathbb{Z}$; by definition this means $H_1 = (f_{\text{an}}^{-1})^* H$ is an element of the Néron-Severi group $NS(X_2)$. Then there exists $\widetilde{M} \in \text{Pic}(X_2)$ with $c_1(\widetilde{M}) = H_1$. By the pullback formula we find $c_1(f^*\widetilde{M}) = H$. Then we have two bundles L and $f^*\widetilde{M}$ with $c_1 = H$; can consider $L \otimes (f^*\widetilde{M})^{-1}$ which has trivial first Chern class.

Then, since $\widehat{f} : \text{Pic}^0(X_2) \to \text{Pic}^0(X_1)$ is surjective (because f is an isogeny and thus so is $\widehat{f}$), there exists $N \in \text{Pic}^0(X_2)$ with $f^*N = L \otimes (f^*\widetilde{M})^{-1}$; rearranging we find $L = f^*(\widetilde{M} \otimes N)$ in the Picard group.

Remark: All of this is fairly elementary; tracking through it we could have done everything in general (not using the transcendental theory over $\mathbb{C}$). The next thing is much heavier and really uses the theory over $\mathbb{C}$ we've developed (you can do it over general fields but you need the theorem of the cube in general which is pretty involved).

Line bundles and maps $f : X \to \widehat{X}$. Fix $L \in \text{Pic}(X)$. We write down a map $\varphi_L : X \to \widehat{X} = \text{Pic}^0(X)$ by $\varphi_L(x) = t_x^* L \otimes L^{-1}$ (need to add the L^{-1} so we land in Pic^0 not just Pic). Can check explicitly that x is mapping to the line bundle $L(0, \exp(2\pi i \operatorname{Im} H(v, \cdot)))$, so in particular φ_L really only depends on $H = c_1(L)$. Easy to write this down, but proving that φ_L is a homomorphism requires the theorem of the square to let us conclude

$$t_{x+y}^* L \otimes L^{-1} \cong t_x^* L \otimes t_y^* L \otimes L^{-2}.$$

Moreover, the analytic representative of this map φ_L is really just the map $\varphi_H : V \to \overline{\Omega}$ given by $v \mapsto H(v, \cdot)$ (only depending on $H = c_1(L)$!).

If you tensor two line bundles and look at $\varphi_{L \otimes M}$, it turns out that $\varphi_{L \otimes M} = \varphi_L \oplus \varphi_M$ (either from the explicit formula, or from the fact that Chern classes add). Also, if $f : X \to Y$ is a homomorphism then we have $\varphi_{f^*L} = \widehat{f} \circ \varphi_L \circ f$.

Let $K(L)$ denote the kernel of $\varphi_L : X \to \widehat{X}$. To study this, we set up some more notation:

$$\Lambda(L) = \{v \in V : \operatorname{Im} H(v, \lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\} = \varphi_H^{-1}(\widehat{\Lambda}).$$

Then $K(L) \cong \Lambda(L)/\Lambda$. Obviously $\Lambda(L)$ and thus $K(L)$ only depend on $H = c_1(L)$. Similarly, $K(L \otimes P) \cong K(L)$ for $P \in \text{Pic}^0(X)$. We have $K(L) = X$ iff $L \in \text{Pic}^0(X)$. Also, can check $K(L^n) = n_X^{-1}[K(L)]$ (preimage under the multiplication-by-n map? or inverse isogeny? or both?). If $n \neq 0$ we then have $K(L) = n_X[K(L^n)]$. (For these computations, recall that if $L = L(H, \chi)$ then $L^n = L(nH, \chi^n)$). Explicitly, can check

$$\Lambda(L^n) = \{v \in V : \operatorname{Im} H(nv, \lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\} = \frac{1}{n}\Lambda(L).$$

Definition: H is called nondegenerate if $H(v, w) = 0$ for all v implies $w = 0$, and $H(v, w) = 0$ for all w implies $v = 0$. We say $L \in \text{Pic}(X)$ is nondegenerate if $c_1(L) = H$ is nondegenerate, which is iff $\operatorname{Im} H$ (the associated alternating form) is nondegenerate. Lemma (to be proved next time): L is nondegenerate iff $K(L)$ is finite, and $\deg \varphi_L = \det(\operatorname{Im} H) = [\Lambda(L) : \Lambda]$.

12 Lecture - 03/10/2016

Correction from last time: if $c_1(L_1) = c_1(L_2)$ then $\varphi_{L_1} = \varphi_{L_2}$ and thus $K(L_1) = K(L_2)$ (equality, not just isomorphism). Also remark: "semicharacters" are sometimes called "pseudocharacters".

Poincaré bundle. Let X be a complex torus as usual, and $\widehat{X} = \mathrm{Pic}^0(X)$ be the dual complex torus. A point in $\widehat{X}$ gives a line bundle on X and vice-versa (by duality). Does there exist a *universal* line bundle on $X \times \widehat{X}$ such that the restriction to $X \times \{\widehat{x}\}$ is the line bundle on X corresponding to the point $\widehat{x} \in \widehat{X}$ and the restriction to $\{x\} \times \widehat{X}$ is the line bundle corresponding to x.

Definition: A *Poincaré bundle* for X is a holomorphic line bundle $\mathcal{P}$ on $X \times \widehat{X}$ such that $\mathcal{P}|_{X\times\{L\}} \cong L$ and $\mathcal{P}|_{\{0\}\times\widehat{X}}$ is the trivial line bundle on $\widehat{X}$. (The first condition is the universal property; the second is really just a choice of normalization).

Theorem: There exists a unique Poincaré bundle on $X \times \widehat{X}$ which is uniquely determined up to isomorphism.

Proof of existence: Can explicitly construct $\mathcal{P}$. If $X = V/\Lambda$ we've explicitly constructed $\widehat{X} = \overline{\omega}/\widehat{\Lambda}$ so $X \times \widehat{X} \cong (V \times \overline{\Omega})/(\Lambda \times \widehat{\Lambda})$. We need to construct a line bundle $L = L(H, \chi)$ on this complex torus. We define this by letting our Hermitian form $H : (V \times \overline{\Omega}) \times (V \times \overline{\Omega}) \to \mathbb{C}$ by

$$H\big((v_1, \ell_1), (v_2, \ell_2)\big) = \overline{\ell_2(v_1)} + \ell_1(v_2).$$

Easy to see it's non-degenerate. We need to verify that

$$\operatorname{Im} H[\Lambda \times \widehat{\Lambda}, \Lambda \times \widehat{\Lambda}] \subseteq \mathbb{Z},$$

which is true by definition of the lattices. So there exists a line bundle $\mathcal{L} = L(H, \chi)$ for some semicharacters χ; want to pick a specific χ. We define this $\chi : \Lambda \times \widehat{\Lambda} \to T_1$ by

$$\chi(\lambda, \ell_0) = \exp(\pi i \operatorname{Im} \ell_0(\lambda)).$$

Can check this is actually a semicharacter for H.

So now we need to check that $\mathcal{P} = L(H, \chi)$ actually gives us a Poincaré bundle. Proof of claim: consider the associated canonical factor

$$a_{\mathcal{P}}((\lambda, \ell_0), (v, \ell)) = \chi(\lambda, \ell_0) \exp\Big(\pi\big(H((\lambda, \ell_0), (v, \ell)) - \tfrac{1}{2}H((\lambda, \ell_0), (\lambda, \ell_0))\big)\Big).$$

Now we want to check property (1) of being a Poincaré bundle: if we fix $L = L(0, \exp(2\pi i \operatorname{Im} \ell)) \in \widehat{X} \in \mathrm{Pic}^0(X)$ (for some $\ell \in \overline{\Omega}$), then $\mathcal{P}|_{X\times\{L\}}$ corresponds to the 1-cocycle $a_P|_{(\Lambda,0)\times(V,\ell)}$, which when we plug into our formula and get this is $\exp(\pi\ell(\lambda))$. On the other hand, $a_L(\lambda, v) = \exp(2\pi i \operatorname{Im} \ell(\lambda))$; to show that these two functions are the same we need to show that they are in the same cohomology group, i.e. differ by 1-coboundary. But the coboundary

$$\frac{\exp(\pi\overline{\ell(v)})}{\exp(\pi\overline{\ell(v + \lambda)})}$$

works. For property (2), it's straightforward to see $\mathcal{P}|_{\{0\}\times\widehat{X}}$ has $a_{\mathcal{P}}((0, \ell_0), (0, \ell)) = 1$ as its 1-cocycle so it's trivial.

This proves existence. What about uniqueness? It's true, and will follow from the following theorem we're not proving: the "see-saw principle", that if X, Y are compact complex manifolds, and $\mathcal{L}$ is a holomorphic line bundle on $X \times Y$, and $\mathcal{L}|_{X\times\{z\}}$ is trivial for all z in some open set $U \subseteq Y$, and $\mathcal{L}|_{\{x_0\}\times Y}$ is trivial for some $x_0 \in X$, then $\mathcal{L}$ is trivial. (If $\mathcal{P}_1, \mathcal{P}_2$ are two Poincaré bundles then apply it to $\mathcal{P}_1 \otimes \mathcal{P}_2^{-1}$).

Remarks: (1) The Poincaré line bundle is nondegenerate - not too hard to prove. (2) There's a universal property of the Poincaré line bundle (this is difficult to prove though): if T is any normal complex analytic space ("normal" meaning stalks are integrally closed), X a complex torus, and L a line bundle on $X \times T$ such that (1) $L|_{X\times\{t\}} \in \mathrm{Pic}^0(X)$ for all $t \in T$ (or even for one $t \in T$, if T is connected) and (2) $L_{\{0\}\times T}$ is trivial, then there exists a unique holomorphic map $\psi : T \to \widehat{X}$ such that $L \cong (\mathrm{id} \times\psi)^*\mathcal{P}$. The proof uses Zariski's main theorem and a more general seesaw theorem.

A few applications of the Poincaré bundle. Definition: If L_1, L_2 are two line bundles on X, you'd like to say L_1 is analytically equivalent to L_2 (written $L_1 \sim_{\mathrm{an}} L_2$) if there's an analytic way to go from one to the other in a family: there exists a connected complex analytic space T, a line bundle $\mathcal{L}$ on $X \times T$, and have $t_1, t_2 \in T$ with $\mathcal{L}|_{X \times \{t_i\}} \cong L_i$.

Proposition: TFAE for L_1, L_2 line bundles on $X = V/\Lambda$.

1. $L_1 \sim_{\mathrm{an}} L_2$.
2. $L_1 \otimes L_2^{-1} \in \mathrm{Pic}^0(X)$.
3. $\varphi_{L_1} = \varphi_{L_2}$.
4. $c_1(L_1) = c_1(L_2)$.

Corollary: analytic equivalence classes the cosets in $\mathrm{Pic}/\mathrm{Pic}^0$; call the class of H to be $\mathrm{Pic}^H(X)$.

Proof: We've basically done that (2), (3), and (4) are equivalent. For (2) $\implies$ (1), assume $L_1 \otimes L_2^{-1} \in \mathrm{Pic}^0(X)$. Consider projection $p : X \times \widehat{X} \to X$ and take $\mathcal{L} = \mathcal{P} \otimes p^* L_2$. Then this is a line bundle on $X \times \widehat{X}$ (a connected complex analytic space) with $\mathcal{L}|_{X \times \{0\}} \cong L_2$ and $\mathcal{L}|_{X \times \{L_1 \otimes L_2^{-1}\}} \cong L_1$.

For (1) $\implies$ (4), assume $L_1 \sim_{\mathrm{an}} L_2$, so there exists some $\mathcal{L}$ on $X \times T$ with L_1, L_2 as two restrictions. Consider the map $T \to H^2(X, \mathbb{Z})$ given by $t \mapsto c_1(\mathcal{L}|_{X \times \{t\}})$. Since everything is holomorphic/functorial, this is a continuous map from a connected space T to a discrete space $H^2(X, \mathbb{Z})$. So it's constant and thus $c_1(L_1) = c_2(L_2)$.

Lemma: If $L, L' \in \mathrm{Pic}(X)$ are two line bundles on a complex torus, and L is nondegenerate, then $L \sim_{\mathrm{an}} L'$ iff there exists $x \in X$ with $L' \cong t_x^* L$. Proof: $\impliedby$ is always true (can make a family $x \mapsto t_x^* L$). For $\implies$, we just saw $L' \otimes L^{-1} \in \mathrm{Pic}^0(X)$. Since L is nondegenerate, our map $\varphi_L : X \to \mathrm{Pic}^0(X)$ is surjective; by our construction (ultimately following from the theorem of the square) we have $\varphi_L(x) = t_x^* L \otimes L^{-1}$ and thus there exists x with this hitting $L' \otimes L^{-1}$.

Question: Given a homomorphism $f : X \to \widehat{X}$, does there exist L with $f = \varphi_L$? Theorem: If $X = V/\Lambda$ and $f : X \to \widehat{X}$ is a homomorphism with $f_{\mathrm{an}} : V \to \overline{\Omega}$, then TFAE:

1. $f = \varphi_L$ for some $L \in \mathrm{Pic}(X)$.
2. The map $F : V \times V \to \mathbb{C}$ by $(v, w) \mapsto f_{\mathrm{an}}(v)(w)$ is Hermitian.

Proof uses the following lemma: For $M \in \mathrm{Pic}(X)$ and $n \in \mathbb{Z}$, then TFAE:

- $M = L^n$ for some $L \in \mathrm{Pic}(X)$.
- $X[n] \subseteq K(M)$ (the kernel of φ_M).

Biextension of abelian groups. Reason for studying this: want to show the Poincaré bundle $\mathcal{P}$ minus the zero section is a biextension of $X \times \widehat{X}$ by $\mathbb{C}^\times$. Next time we'll define the set $Biext(B \times C, A)$ of bi-extensions, and show that this is naturally a group.

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Biextensions. Recall that a group extension of a group G by an abelian group A is a group E that sits in the middle of an exact sequence

$$0 \to A \to E \to G \to 0.$$

This gives an action of G on A, so we can take group cohomology $H^i(G, A)$. Theorem: Equivalence classes of extensions are in bijection with $H^2(G, A)$.

So what's a biextension? Our running example will be the Poincaré bundle minus the zero section. Let A, B, C be three abelian groups (so $A = C^\times$, $B = X$, and $C = \widehat{X}$ in our Poincaré bundle example). A biextension of $B \times C$ by A is a set G with:

1. A free action of A on G (free action means $gx = x$ for any x implies $g = 1$). [In our example this is just scalars acting on the bundle.]
2. A map $\pi : G \to B \times C$ such that $B \times C \cong G/A$ by π (i.e. the preimage $\pi^{-1}[g]$ is the orbit Ag). [In our example this is the projection from the bundle.]
3. Maps $+_1 : G \times_B G \to G$ and $+_2 : G \times_C G \to G$ (here $G \times_B G$ is the set of (g_1, g_2) with $\pi(g_1)$ and $\pi(g_2)$ having same B-component). [In our example, the sets $G \times_B G$ is basically the union of the restrictions $\mathcal{P}|_{\{x\} \times \widehat{X}}$ and ???]

such that:

1. For all $b \in B$, $G'_b = \pi^{-1}[\{b\} x C]$ is an abelian group is an abelian group under $+_1$, induces a surjective homomorphism $\pi_1 : G'_b \to C$, and we have $A \cong \ker \pi_1$ via the action of A on G'_b.
2. For all $c \in C$, $G'_c = \pi^{-1}[B \times \{c\}]$ is an abelian group is an abelian group under $+_2$, induces a surjective homomorphism $\pi_2 : G'_c \to B$, and we have $A \cong \ker \pi_2$ via the action of A on G'_c.
3. Compatibility relation: with $x, y, u, v \in G$ with $\pi(x) = (b_1, c_1)$, $\pi(y) = (b_1, c_2)$, $\pi(u) = (b_2, c_1)$, and $\pi(v) = (b_2, c_2)$ then we have

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v).$$

   (For the Poincaré bundle this is nontrivial and is known as Lang duality!)

Remark: $Biext(B \times C, A)$ (the equivalence classes of bi-extensions) itself forms a group. In fact these classes can be expressed in terms of "cocycles" modulo "coboundaries". (Reference: Mumford "Bi-extensions of formal groups" - he wanted to construct a generalization of Poincaré bundles in settings for formal groups where they don't exist).

Cohomologies of line bundles on complex tori. Will talk about characteristics of line bundles, theta functions (which give H^0's). Then want to compute all H^i's, and to do this need to talk about vanishing theorems (this involves harmonic forms). Then will talk about Riemann-Roch.

Characteristics. Fix $H \in NS(X)$. Given a "nice" decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$, one can distinguish a line bundle $L_0 \in \mathrm{Pic}^H(X)$. If H is nondegenerate, can write $L = t_c^* L$; call c the characteristic. Along the way, we compute explicitly $K(L) = \ker(\varphi_L)$.

Let $X = V/\Lambda$ be a g-dimensional complex torus. Fix $L \in \mathrm{Pic}(X)$, let $H = c_1(L)$ be our Hermitian form on V, and $E = \mathrm{Im}(H)$ be the corresponding $\mathbb{Z}$-valued alternating form on $\Lambda \cong \mathbb{Z}^{2g}$. This is our setup.

Lemma: There exists a basis $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ for Λ such that the associated matrix of E is of the form

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$$

where D is a diagonal matrix with entries $d_1, \ldots, d_g$ with $d_i | d_{i+1}$ and $d_i \geq 0$. Proof sketch: For any matrix $A : \mathbb{Z}^n \to \mathbb{Z}^n$ you can do row/column operations to get $UAV = D$ diagonal for $U, V \in \mathrm{GL}_n(\mathbb{Z})$ - the Smith normal form. So for our general case suppose we take a matrix for E; we can write it in the form

$$\begin{bmatrix} F & A \\ -A^\top & G \end{bmatrix}$$

with F, G, A all $g \times g$ and both F, G skew-symmetric. By using a Gram-Schmidt-like argument can get $F = G = 0$. Then if we have the Smith normal form $UAV = D$ for A, get

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V^\top \end{bmatrix} \begin{bmatrix} 0 & A \\ -A^\top & 0 \end{bmatrix} \begin{bmatrix} U^\top & 0 \\ 0 & V \end{bmatrix}.$$

Since they correspond to invariant factors, the list $(d_1, \ldots, d_g)$ in this lemma is uniquely determined by E, and thus uniquely by H, and thus uniquely by L. We call it the *type* of E (or H, or L).

Remark: (1) We will see $K(L) = \ker \varphi_L$ is isomorphic to $K_1 \oplus K_2$ with $K_i \cong \bigoplus(\mathbb{Z}/d_i\mathbb{Z})$. (2) If $d_i > 0$ then H, L, or E is nondegenerate.

Definition: A basis $\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ as above (giving rise to the matrix with the type) is called a *canonical* or *symplectic* basis for Λ for L.

Definition: A sublattice Λ_1 of Λ is called *totally isotropic* for E if $E(\lambda, \lambda') = 0$ for all $\lambda, \lambda' \in \Lambda_1$. Abelian varieties people call this "isotropic" but in bilinear forms there's a slightly different definition of that, so we say "totally isotropic" for clarity. (Example: $\Lambda = \langle \lambda_1, \ldots, \lambda_g \rangle$ for our basis above).

Definition: A decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ is called a "decomposition for E (or H, or L)" if Λ_1, Λ_2 are both totally isotropic. We just showed these exist - can take $\Lambda_1 = \langle \lambda_1, \ldots, \lambda_g \rangle$ and $\Lambda_2 = \{\mu_1, \ldots, \mu_g\}$.

Definition: A decomposition $V = V_1 \oplus V_2$ of V (with V_i a g-dimensional $\mathbb{R}$-vector space) is called a "decomposition for E (or H, or L)" if it's induced by a decomposition of Λ, i.e. $(\Lambda \cap V_1) \oplus (\Lambda \cap V_2)$ is a decomposition as in the previous definition. (Warning: You can have a decomposition of V into totally isotropic subspaces without it giving such a decomposition of Λ).

Now: developing some more theory. (It will be used in the case that H is nondegenerate, but doesn't need that hypothesis yet). Let $H \in NS(X)$, and let $V = V_1 \oplus V_2$ be a decomposition for H (or for $L \in \mathrm{Pic}^H$). Define $\chi_0 : V \to T_1$ by

$$\chi_0(v) = \exp\left(\pi i \operatorname{Im} H(v_1, v_2)\right) = \exp\left(\pi i E(v_1, v_2)\right)$$

where $v = v_1 + v_2$ for $v_i \in V_i$.

Lemma: if $v = v_1 + v_2$ and $w = w_1 + w_2$ (under our decomposition) then

$$\chi_0(v + w) = \chi_0(v)\chi_0(w)\exp(\pi i E(v, w))\exp(-2\pi i E(v_2, w_1)).$$

So $\chi_0|_\Lambda$ is a semicharacter for H. (Should not say χ_0 is a semicharacter on all of V because we have this weird last term - but it's trivial on Λ because E is integer-valued on Λ).

Definition: The character $L_0 = L(H, \chi_0)$ (for this semicharacter χ_0 we just constructed) is the *distinguished element* of $\mathrm{Pic}^H(X$ with respect to the decomposition $V = V_1 \oplus V_2$. This will be the basepoint for our analytic families.

Lemma: Let $V = V_1 \oplus V_2$ be a decomposition for $H \in NS(X)$. Assume H is nondegenerate. Then:
(a)
(b) For all $L \in \mathrm{Pic}^H$, there exists a $c \in V$ uniquely determined up to translation by $\Lambda(L)$ such that $L = t^*_{c+\Lambda} L_0$.

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Finishing lemma from last time: Lemma: Let $V = V_1 \oplus V_2$ be a decomposition for $H \in NS(X)$. Assume H is nondegenerate. Then:
(a) The distinguished line bundle $L_0 = L(H, \chi_0)$ is the unique element in $\mathrm{Pic}^H(X)$ whose semicharacter is trivial on $\Lambda_1 = V_1 \cap \Lambda$ and $\Lambda_2 = V_2 \cap \Lambda$.
(b) For all $L \in \mathrm{Pic}^H$, there exists a $c \in V$ uniquely determined up to translation by $\Lambda(L)$ such that $L = t^*_{c+\Lambda} L_0$.

Proof: Part (a) essentially follows from the definition; won't use it so won't say much about it. Part (b): Have seen that $L \cong t^*_{\bar{c}} L_0$ for some $\bar{c} = c + \Lambda$ in $X = V/\Lambda$. Note: $\bar{c}$ is well-defined up to $\Lambda(L)/\Lambda = \ker \varphi_L : X \to \widehat{X}$ (which is the map $x \mapsto t^*_x L \otimes L^{-1}$, so certainly $t^*_{\bar{c}_1} L_0 = t^*_{\bar{c}_2} L_0$ iff $\bar{c}_1 - \bar{c}_2$ is in this kernel).

Definition: c as above is called the *characteristic* of L WRT the decomposition $V_1 \oplus V_2$ for L. (Only defined for nondegenerate L).

Extended canonical factors. Recall that we had our canonical factor (1-cocycle) associated to L: $a_L : \Lambda \times V \to \mathbb{C}^\times$ defined by

$$a_L(\lambda, v) = \chi(\lambda) \exp\big(\pi H(v, \lambda) + \tfrac{\pi}{2} H(\lambda, \lambda)\big).$$

Using the notion of characteristic we can extend this canonical factor to a function $V \times V \to \mathbb{C}^\times$. Namely, define

$$a_L(u, v) = \chi_0(u) \exp(2\pi i E(c, u)) \cdot \exp\big(\pi H(v, u) + \tfrac{\pi}{2} H(u, u)\big).$$

By part (b), χ agrees with $\chi_0(u) \exp(2\pi i E(c, u))$ on Λ, so this is actually an extension! Let χ denote the extended map $u \mapsto \chi_0(u) \exp(2\pi i E(c, u))$.

Easy properties: If $u = u_1 + u_2$, $v = v_1 + v_2$, and w are elements of $V = V_1 \oplus V_2$, then

- $a_L(u, v + w) = a_L(u, v) \exp(\pi H(w, u))$.
- $a_L(u + v, w) = a_L(u, v + w) a_L(v, w) \exp(2\pi i E(u_1, u_2))$.
- $a_L(u, v)^{-1} = a_L(-u, v) \chi_0(u)^{-2} \exp(-\pi H(u, u))$.
- If $L' = t^*_w L$ then $a_{L'}(u, v) = a_L(u, v) \exp(2\pi i E(w, u))$.

Lemma: if $L \in \mathrm{Pic}(X)$ is nondegenerate with $\Lambda = \Lambda_1 \oplus \Lambda_2$ a decomposition for L and $V = V_1 \oplus V_2$ the induced decomposition, then:
(a) $\Lambda(L) = \Lambda(L)_1 \oplus \Lambda(L)_2$ where $\Lambda(L)_i = V_i \cap \Lambda(L)$.
(b) $K(L) = K_1 \oplus K_2$ for $K_i = \Lambda(L)_i / \Lambda_i$.
(c) $K_i = \bigoplus_{j=1}^g \mathbb{Z}/d_j\mathbb{Z}$ where $(d_1, \ldots, d_g)$ is the type of L.
Definition: Type $(1, 1, \ldots, 1)$ is called *principal*; these are the ones with trivial $K(L)$.

Proof: everything follows from (a). Clearly $\Lambda(L) \supseteq \Lambda(L_1) \oplus \Lambda(L_2)$. For the converse, first note that $\Lambda(L_i) \supseteq \Lambda_i$ by construction so

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \subseteq \Lambda(L)_1 \oplus \Lambda(L)_2.$$

Now, let $v \in \Lambda(L)$ decompose as $v_1 + v_2$; need to show $v_1 \in \Lambda(L)$ But by definition this means we're asking that $E(v_1, \lambda) \in \mathbb{Z}$ for all $\lambda \in \Lambda$. But if $\lambda = \lambda_1 + \lambda_2$ then

$$E(v_1, \lambda) = E(v_1, \lambda_1 + \lambda_2) = E(v_1, \lambda_2) = E(v_1 + v_2, \lambda_2) = E(v, \lambda_2) \in \mathbb{Z}$$

by definition of $v \in \Lambda(L)$ (using that E is trivial on $V_1 \times V_1$ and $V_2 \times V_2$).

Remark: If H is nondegenerate can set $\Lambda' = \Lambda(L)_1 \oplus \Lambda_2$ and $\Lambda'' = \Lambda_1 \oplus \Lambda(L)_2$; these sit between Λ and $\Lambda(L)$. Then can define complex tori $X' = V/\Lambda' = X/K_1$ and $X'' = V/\Lambda'' = X/K_2$; then $p_i : X \to X/K_i$ is an isogeny. Note $a_L : V \times V \to \mathbb{C}^\times$ restricted to $\Lambda' \times V$ and $\Lambda'' \times V$ give 1-cocycles and thus line bundles $M_1 \to X_1$ and $M_2 \to X_2$. Moreover, M_i is the descent of L with respect to the isogeny p_i, and the characteristic of M_i is the same c. As we vary c within $\Lambda(L)$ we obtain all descents of L via p_i. (Later on we'll talk about the relevance of this).

Theta functions. First goal: if H is positive definiite, we want an explicit basis (of "theta functions") for $H^0(X, L)$, where $H = c_1(L)$. As usual let $X = V/\Lambda$ and $\pi : V \to X$ the covering map. Recall:

$$H^0(X, L) \cong H^0(V, \pi^* L) \cong H^0(V, \mathbb{C} \times V)^\Lambda.$$

A theta function will be a holomorphic function $\vartheta : V \to \mathbb{C}$ such that $\vartheta(v + \lambda) = f(\lambda, v)\vartheta(v)$ where $f \in Z^1(\Lambda, H^0(V, \mathcal{O}_V^\times))$ is a factor of automorphy for L. (Remember that changing f by a coboundary gives isomorphic complex vector spaceS).

Appell-Humbert gave us a canonical factor $a_L = a_{H,\chi}$. However, we want to get a "better" factor, the *classical factor of automorphy*, which makes H^0 very explicit. This will be possible in the case that H is positive definite.

So let $L = L(H, \chi) \in \mathrm{Pic}(X)$, with H positive-definite (so $H(x, x) > 0$ for all $x \neq 0$). Say L is *positive* or *positive-definite*. As usual, let $E = \operatorname{Im} H$ and $V = V_1 \oplus V_2$.

Lemma: V_2 generates V as a $\mathbb{C}$-vector space. More precisely, $V = V_2 \oplus iV_2$. (This really needs H positive definite).

Proof: Want to show that $U = V_2 \cap iV_2$ is trivial; then we get the direct sum decomposition we want by dimension considerations. Since E is trivial on V_2, $E(v, w) = 0$ for $v, w \in U$. For any $v, w \in U$ we then have

$$H(v, w) = E(iv, w) + iE(v, w) = 0$$

because if $v, w \in U$ then $iv \in U$ too. But for the case $v = w$, positive-definiteness means $H(v, v) > 0$ if $v \neq 0$; so we can only have $v = 0$ in U.

Note: $E = \operatorname{Im} H$ is zero on V_2 means H is symmetric (real) on V_2. Let $B : V \times V \to \mathbb{C}$ be the $\mathbb{C}$-bilinear extension of $H|_{V_2 \times V_2}$ (by the lemma). Finally, get that $H - B : V \times V \to \mathbb{C}$ is Hermitian with the properties:
(a) $(H - b)(v, w)$ is 0 if $w \in V_2$, and is $2iE(v, w)$ if $v \in V_2$.
(b) $\mathrm{Re}(H - B)$ is positive-definite on V_1.

Now can define the classical factor of automorphy $e_L : \Lambda \times V \to \mathbb{C}^\times$ given by

$$e_L(\lambda, v) = \chi(\lambda) \exp\left(\pi(H - B)(v, \lambda) + \tfrac{\pi}{2}(H - B)(\lambda, \lambda)\right).$$

Easy to see this is equivalent to the canonical factor a_l; in fact

$$e_L(\lambda, v) = a_L(\lambda, v) \left(\frac{\exp(\frac{\pi}{2} B(v, v))}{\exp(\frac{\pi}{2} B(v + \lambda, v + \lambda))} \right).$$

Then a *classical theta function* is a theta function with respect to the 1-cocycle e_L.

Remarks: (1) For L of type $(1, \ldots, 1)$, the corresponding classical theta functions were studied by Riemann. (2) We'll see $e(\lambda, \lambda_2) = 1$ which gives us periodicity and thus lets us do Fourier analysis on our theta functions.

Definition: If L is of type $(d_1, \ldots, d_g)$, and E has skew-symmetric form

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

define its Pfaffian $Pf(L) = Pf(E) = d_1 \cdots d_g$. (In general: If A is skew-symmetric then $\det(A)$ is always formally a square, and the square root is the Pfaffian). Note in our case, $Pf(L) = \sqrt{|K(L)|}$. A theorem we'll be aiming for is that in the positive-definite case, $h_0 = \dim H^0(X, L)$ is equal to the Pfaffian $Pf(L) = d_1 \cdots d_g$.

15 Lecture - 03/22/2016

Recall: extended our canonical factor a_L to a function $a_L : V \times V \to \mathbb{C}$. Had a lemma (which we'll call "the lemma") telling us a bunch of properties of a_L; will use that repeatedly.

Main theorem we want: If L is positive-definite then $h^0(X, L) = d_1 \cdots d_g$. Have several steps to put into this, but eventually will write down an explicit basis of theta functions. (Note that this says the cohomological behavior is entirely determined by $c_1(L) = H$ - something very special to this case).

Assume $L \in \mathrm{Pic}^H(X)$. Let $V = V_1 \oplus V_2$ be a decomposition for L, and let L_0 be the basepoint we chose (which had zero as its characteristic).

Lemma 1: $H^0(X, L) \cong H^0(X, L_0)$. (So right off the bat, seeing that H^0 doesn't change as we vary in analytic families). Proof: Present the H^0 with classical theta functions; we claim that the map $H^0(X, L) \to H^0(X, L_0)$ taking a theta function $\vartheta \in H^0(X, L)$ to $\widetilde{\vartheta}$ given by

$$\widetilde{\vartheta}(v) = \exp(\pi(HB)(v, c))\vartheta(v - c)$$

where c is the characteristic of L.

The only thing that really has to be shown is that $\widetilde{\vartheta}$ is a theta function for the 1-cocycle e_{L_0}. This is an easy computation using that $L = t_c^* L_0$ means

$$e_L(\lambda, v) = e_{L_0}(\lambda.v) \exp(2\pi i E(c, \lambda));$$

using this plus the lemma gives that $\widetilde{\vartheta}$ satisfies the right functional equation.

Lemma 2: We have $h^0(X, L_0) \leq Pf(E) = d_1 \cdots d_g$. Proof: Let $\vartheta \in H^0(X, L_0)$ be a classical theta function, i.e.

$$\vartheta(v + \lambda) = e_{L_0}(\lambda, v)\vartheta(v)$$

for all $\lambda \in \Lambda$ and $v \in V$. But if $\lambda_2 \in \Lambda_2$ then

$$e_{L_0}(\lambda_2, v) = \chi_0(\lambda_2) \exp\left(\pi(H - B)(v, \lambda_2) + \tfrac{1}{2}(H - B)(\lambda_2, \lambda_2)\right);$$

the lemma from last time told us that $H - B$ is trivial if the second entry is in λ_2 so the entire exp goes away. Also χ_0 is explicitly defined and we get

$$\chi_0(\lambda_2) = \exp(\pi i E(0, \lambda_2)) = 0.$$

So we conclude that our functional equation tells us $\vartheta(v + \lambda_2) = \vartheta(v)$ for all $\lambda_2 \in \Lambda_2$. This means we have a Fourier series, of the form

$$\vartheta(v) = \sum_{\lambda \in \Lambda(L)_1} \alpha_\lambda \exp(\pi(H - B)(v, \lambda)).$$

(One way to get this is by properties of $H - B$ discussed last time. Alternatively there exist suitable coordinates $v = (v_1, \ldots, v_g)^\top$ for V and $\lambda = (\lambda_1, \ldots, \lambda_g)^\top$ for Λ such that we can write $\exp(\pi(H-B)(v, \lambda)) = \exp(-2\pi i v^\top \lambda)$).

Now, want to say we only have a few possible choices for what the α_λ's can be, restricted by the functional equation. For instance if $\lambda_1 \in \Lambda_1$ then

$$\vartheta(v + \lambda_1) = e_{L_0}(\lambda_1, v)\vartheta(v);$$

plugging in our Fourier expansion and comparing coefficients gives

$$\alpha_{\lambda - \lambda_1} = \alpha_\lambda e_{L_0}(\lambda_1, 0)^{-1} \exp(\pi(H - B)(\lambda_1, \lambda))$$

for any $\lambda \in \Lambda(L_1)$ and $\lambda_1 \in \Lambda_1$. So if we fix values α_λ for one representative of each coset of $\Lambda(L)_1/\Lambda_1$ we've determined all possible λ; there are $d_1 \cdots d_g$ such cosets, and thus the space of all ϑ's is at most $d_1 \cdots d_g$-dimensional.

Remark: Let $X = V/\Lambda$, fix $H \in NS(X)$ positive-definite of type $(d_1, \ldots, d_g)$, and take a symplectic basis $\{\lambda_1, \ldots, \lambda_g; \mu_1, \ldots \mu_g\}$ for Λ for $\mathrm{Im}(H - E)$. Recall this means that with respect to this basis, E is of the form

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$$

for D the diagonal thing with entries d_i, and $V = V_1 \oplus V_2$ for $V_1 = \langle \lambda_i \rangle$ and $V_2 = \langle \mu_i \rangle$.

Define $e_i = \frac{1}{d_i}\mu_i$. Last time we saw $\{e_i\}$ was a $\mathbb{C}$-basis of V. Consider the period matrix with respect to $\{e_i\}$ for V and $\{\lambda_i, \mu_i\}$ for Λ; this is $\Pi = [Z|D]$ for D what we already had and $Z \in M_g(\mathbb{C})$. Can then check we have: $Z = Z^\top$ (actually symmetric, not Hermitian!) and satisfies $\mathrm{Im}(Z) > 0$, i.e. Z is in Siegel's upper half space. (This leads to moduli interpretations and things). Moreover, with respect to the $e_1, \ldots, e_g$ basis for V we have:

- The matrix for H with respect to this basis is $\mathrm{Im}(Z)^{-1}$.
- $B(v, w) = v^\top (\mathrm{Im}\, Z)^{-1} w$.
- $(H - B)(v, w) = -2iv^\top w_1$ for $w = w_1 + w_2 \in V_1 \oplus V_2$.

Next: want to give an explicit basis for $H^0(X, L)$. Suppose $L \in \mathrm{Pic}(X)$ with characteristic of L equal to c with respect to a decomposition $V_1 \oplus V_2$. Start by defining an explicit theta function

$$\vartheta^c(v) = \exp\big(-\pi H(v, c) + \tfrac{1}{2}H(c, c) + \tfrac{1}{2}B(v + c, v + c)\big) \sum_{\lambda \in \Lambda_1} \exp\big(\pi(H - B)(v + c, \lambda) - \tfrac{1}{2}(H - B)(\lambda, \lambda)\big).$$

Theorem: ϑ^c is a canonical theta function for $L = t_c^* L_0$. (Straightforward part of the theorem: showing this satisfies the functional equation; maybe harder is showing that it's holomorphic). Proof sketch: (1) ϑ^c is holomorphic on V. First of all, need to show that $f(v) = \sum_{\lambda \in \Lambda_1} |\cdots|$ converges uniformly on every compact subset of V_1. To do this fix $\|\cdot\| : V \to \mathbb{R}$ with $\|\Lambda\| \subseteq \mathbb{Z}$. Since $Re(H - B)$ is positive definite on V_1 and Λ_1 is discrete, there exists $R > 0$ such that $|\exp(-\frac{\pi}{2}(H - B)(\lambda, \lambda))| \leq \exp(-R\|\lambda\|^2)$. Also for all $r > 0$ there exists $R' > 0$ such that if $\|v\| < r$ then $|\exp(\pi(H - B)(v + c, \lambda))| \leq \exp(R'\|\lambda\|)$. So we have

$$f(v) \leq \sum_{\lambda \in \Lambda_1} \exp(R'\|\lambda\| - R\|\lambda\|^2)$$

and this converges. So get uniform convergence; as a sum of holomorphic thing that converges uniformly on compact subsets get that the sum is holomorphic.

(2) Automorphy (for the canonical factor a_L, not the classical one e_L): We need $\vartheta^c(v + \lambda) = a_L(\lambda, v)\vartheta(v)$. For $c = 0$ this boils down to "the lemma". For arbitrary c, use the relation

$$\vartheta^c(v) = \exp\big(-\pi H(v, c) + \tfrac{1}{2}H(c, c)\big)\vartheta^0(v + c).$$

Remark: The proof actually shows ϑ^c is a canonical theta function for M_2 (descent of L to $V/(\Lambda_1 \oplus \Lambda(L)_2)$).

So far we've constructed only one canonical theta function for L, i.e. one element of $H^0(X, L)$. We can get more as follows. For $\overline{\omega} \in K(L)$ define

$$\vartheta^c_{\overline{\omega}}(v) = a_L(\omega, v)^{-1}\vartheta^c(v + \omega).$$

Easy to check this is independent of the choice of lift ω.

Lemma: For all $\overline{\omega}$, $\vartheta^c_{\overline{\omega}}$ is a canonical theta function for L. Proof: use "the lemma".

Theorem: $\{\vartheta^c_{\overline{\omega}} : \overline{\omega} \in K(L)_1\}$ is a basis for the space of canonical theta functions for the $\mathbb{C}$-vector space $H^0(X, L)$. Proof: Just need to show it's linearly independent by our dimension counting. Will do this next time; will go back to Fourier analysis.

16 Lecture - 03/24/2016

From last time: the matrix of H was actually $\mathrm{Im}(Z)^{-1}$.

Theorem (where we left off from last time): If $L = L(H,\chi)$ is a positive definite line bundle on X, and c is the characteristic of L with respect to the decomposition $V_1 \oplus V_2$ for L, then the set of canonical theta functions $\{\vartheta^c_{\overline{\omega}} : \overline{\omega} \in K(L)_1\}$ is a basis for the $\mathbb{C}$-vector space $H^0(X,L)$. Corollary: $h^0(L) = d_1 \cdots d_g$.

Proof of theorem: (1) We already know $h^0(X,L) \leq d_1 \cdots d_g$, so what remains to show is that these theta functions $\vartheta^c_{\overline{\omega}}$ are linearly independent.

(2) The expression we gave last time generalizes to

$$\vartheta^c_{\overline{\omega}}(v) = \exp\big(-\pi H(v,c) - \tfrac{\pi}{2}H(c,c)\big)\vartheta^0_{\overline{\omega}}(v+c),$$

so it's sufficient to prove linear independence in the case $c = 0$.

(3) We still have no tools to deal with these $\vartheta^0_{\overline{\omega}}$'s, though! To get a handle on them we want to translate from canonical to classical theta functions. Remember the canonical and classical factors are related by

$$e_L(\lambda,v) = a_L(\lambda,v)\frac{\exp(\pi B(v,v))}{\exp(\pi B(v+\lambda,v+\lambda))}.$$

We see that

$$\theta^c_{\overline{\omega}}(v) = \exp(-\tfrac{\pi}{2}B(v,v))\vartheta^c_{\overline{\omega}}(v)$$

is a classical theta function. So it's sufficient to show that $\{\theta^c_{\overline{\omega}} : \overline{\omega} \in K(L)_1\}$ is linearly independent. (And by (2) it's sufficient to do $c = 0$).

(4) We can now work with Fourier analysis to do the proof. Let $\omega_1, \ldots, \omega_N \in \Lambda(L)_1$ be representatives for the cosets in $K(L)_1$. We'll show these are linearly independent. But note

$$\theta^0_{\omega_i}(v) = \exp(-\tfrac{\pi}{2}B(v.v))\vartheta^0_{\omega_i}(v),$$

and it's easy to expand this out as

$$\theta^0_{\omega_i}(v) = \sum_{\lambda \in \Lambda_1 - \omega_i} \exp\big(-\tfrac{\pi}{2}(H-B)(\lambda,\lambda)\big)\exp\big(\pi(H-B)(v,\lambda)\big).$$

So $\theta^0_{\omega_i}$ has a Fourier expansion with nonzero coefficients only in the coset $-\omega_i + \Lambda_1$. Since these cosets are all disjoint, the functions are evidently linearly independent.

Corollary: If $L = L(H,\chi)$ and $L' = L(H,\chi')$, and c, c' are the characteristics of L and L', and we define $\tau : V \to \mathbb{C}^\times$ by

$$\tau(v) = \exp\big(\pi i \,\mathrm{Im}\, H(c',c) - \pi H(v, c-c') - \tfrac{\pi}{2}H(c'-c, c'-c)\big).$$

Then $H^0(x,L) \to H^0(X,L)$ given by $\vartheta \mapsto \tau t^*_{c'-c}\vartheta$ is an isomorphism, and is diagonal WRT the basis given above. (Proof: A simple computation).

Positive semi-definite line bundles. Recall that if L is degenerate, then $K(L) = \ker(\varphi_L) = \Lambda(L)/\Lambda$ is not finite, for

$$\Lambda(L) = \{v \in V : \mathrm{Im}\, H(v,\lambda) \in \mathbb{Z} \forall \lambda\}.$$

Let $\Lambda(L)_0$ be the connected component of $\Lambda(L)$ containing 0; this is a subspace of V; it's then given by

$$\Lambda(L)_0 = \{v \in V : H(v,x) = 0 \forall x \in V\}$$

(using that $\varphi^{an}_L(v) = H(v,\cdot)$). Let $K(L)_0 = \Lambda(L)_0/(\Lambda(L)_0 \cap \Lambda)$; this is a subtorus of X. The idea is to take our possibly-bad line bundle L on X and descend it to a quotient torus where we fix the problems.

So define $\overline{X} = X/K(L_0) = \overline{V}/\overline{\Lambda}$ where $\overline{V} = V/\Lambda(L)_0$ and $\overline{\Lambda} = \Lambda/(\Lambda(L)_0 \cap \Lambda)$, and let $p : X \to \overline{X}$ be the natural quotient map. Lemma: There exists $\overline{L}$ on $\overline{X}$ with $L \cong p^*\overline{L}$ (i.e. L descends to $\overline{X}$) iff

$L|_{K(L)_0} = 0$. (This is the obvious condition we want, and the proof is straightforward). Moreover, if $\overline{L}$ exists then $h^0(X, L) = h^0(\overline{L}, \overline{X})$ and $\overline{L}$ is nondegenerate. From the definite case we recover:

Theorem: If $L = L(H, \chi)$ is positive semi-definite on X, then $h^0(X, L)$ is equal to the Pfaffian of the reduced E if $L|_{K(L)_0}$ is trivial, and 0 otherwise. This Pfaffian is then $\prod_{d_i \neq 0} d_i$ (which is by definition 1 if all d_i's are zero).

Now onto deeper things. Let $L \in \mathrm{Pic}(X)$ on $X = V/\Lambda$ a g-dimensional complex torus. Suppose H has r positive eigenvalues and s negative eigenvalues (with $r + s \leq g$). Then:

Theorem (Mumford, Kempf, Deligne, etc.): We have:
(a) $H^q(X, L) = 0$ if $q < s$ or $q > g - r$. (Note that in the positive definite case this limits us to $q = 0$, the case we dealt with).
(b) $H^q(X, L)$ are either all trivial or all nontrivial for $s \leq q \leq g - r$.
(c) $h^q(X, L) = \binom{g-s-r}{q-s} h^s(X, L)$ for $s \leq q \leq g - r$.
(d) We have $h^s(X, L)$ is $Pfr(E)$ if $L|_{K(L)_0}$ is trivial and 0 otherwise.

Corollary: (a) If L is nondegenerate then $r + s = g$ so cohomology is nontrivial only for $q = s = g - r$; we denote this number the *index* of L, $i(L)$.
(b) If $h^s \neq 0$ then $h^q \neq 0$ for $s \leq q \leq g - r$.
(c) If L is positive-definite (so $r = g$ and $s = 0$ then only H^0 is nontrivial).
(d) If L is positive semi-definite then $r < g$ and $s = 0$, and $H^0, \dots, H^{g-r}$ are all nontrivial if $L|_{K(L)_0}$ is trivial.

Corollary (Analytic Riemann-Roch theorem for complex tori): Define

$$\chi(X, L) = \sum_{i=s}^{g-r} (-1)^i h^i(X, L),$$

where $c_1(L)$ has s negative eigenvalues and r positive ones. Then $\chi(X, L) = (-1)^s Pf(E)$.

Proof: If $L|_{K(L)_0}$ is nontrivial, the LHS is a sum of zeros and the RHS has some $d_i = 0$ so we get $0 = 0$. If $L|_{K(L)_0}$ is trivial then

$$\chi(X, L) = \sum_{q=s}^{g-r} (-1)^q \binom{g-r-s}{q-s} Pfr(E).$$

Now the sum we have can be written

$$\sum_{q=s}^{g-r} (-1)^q \binom{g-r-s}{q-s} = (-1)^s \sum_{i=0}^{N} (-1)^i \binom{N}{i}$$

for $N = g - r - s$; this sum is 1 if $N = 0$ and 0 if $N > 0$. So we get $\chi(X, L)$ is $(-1)^s Pfr(E) = (-1)^s Pf(E)$ if $r + s = g$, and is 0 if $r + s < g$, and in the latter case some d_i is zero so $Pf(E) = 0$.

Remarks: (1) $\deg(\varphi_L) = \det(E) = Pf(E)^2 = \chi(X, L)^2$. (2) It's useful (e.g. to get an algebraic/geometric statement) to use $1/g! \cdot L^g$ instead of $(-1)^s Pf(E)$, where L^g is the self-intersection of the divisor L (the g-fold cup product, viewing them as things in cohomology). (3) Given $f : X' \to X$ and $L \in \mathrm{Pic}(X)$, we have $\chi(X', f^*L) = \deg(f)\chi(X, L)$. (This is a shadow of Grothendieck-Riemann-Roch). (4) Exercise: For the Poincaré bundle $\mathcal{P}$ on $X \times \widehat{X}$, we have $h^q(X \times \widehat{X}, \mathcal{P})$ is 1 if $q = g$ and 0 otherwise. (Hint: we've given $c_1(\mathcal{P})$ explicitly).

Vanishing theorem of Mumford and Kempf. Recall we have an exact sequence of sheaves (by $\overline{\partial}$-Poincaré):

$$0 \to \Omega^p_{hol} \hookrightarrow \Omega^{p,0} \to \Omega^{p,1} \to \cdots .$$

For $p = 0$ this gives

$$0 \to \mathcal{O}_X \hookrightarrow \Omega^{0,0} \to \Omega^{0,1} \to \cdots .$$

Since L is locally free we also get an exact sequence of sheaves from tensoring:

$$0 \to L \to \Omega^{0,0}(L) \to \Omega^{0,1}(L) \to \cdots ,$$

where $\Omega^{0,q}(L) = \Omega^{0,q} \otimes_{\mathcal{O}_X} L$.

17 Lecture - 04/05/2016

Today: the vanishing theorem of Mumford-Kempf. Need to study harmonic L-valued forms. Recall we had a Dolbeault resolution of sheaves

$$0 \to \Omega^p_{hol} \hookrightarrow \Omega^{p,0} \to \Omega^{p,1} \to \cdots$$

with the maps given by $\overline{\partial}$ (exactness was the $\overline{\partial}$ Poincaré lemma). Then we can tensor the $p = 0$ one with the locally free sheaf L (in our case, a line bundle) and obtain another exact sequence

$$0 \to L \to \Omega^{0,0}(L) \to \Omega^{0,1}(L) \to \cdots$$

where $\Omega^{p,q}(L) = \Omega^{p,q} \otimes L$ is the "sheaf of smooth forms of type (p, q) with values in L" and the connecting maps are $\overline{\partial} = \overline{\partial} \otimes \mathrm{id}$.

Apply the functor $\Gamma(X, -) = H^0(X, -)$ to get the complex

$$0 \to H^0(X, L) \to \Gamma(X, \Omega^{0,0}(L)) \to \Gamma(X, \Omega^{0,1}(L)) \to \cdots$$

Then take the cohomology of this complex to get $H^{0,q}_{\overline{\partial}}(X, L)$ (a variant of Dolbeault cohomology with values in L). Then there's a Dolbeault theorem in this setting: we have $H^q(X, L) \cong H^{0,q}_{\overline{\partial}}(X, L)$. Moreover there's a Hodge theorem stating that $H^{0,q}_{\overline{\partial}}(X, L)$ is isomorphic to a space of harmonic forms $\mathcal{H}^q(L)$. This is a vector subspace $\ker \Delta$ of $\Gamma(X, \Omega^{0,q}(L))$, i.e. the "harmonic forms with values in L with respect to Δ".

What is this Δ? Remember we took $\Delta = \overline{\partial \delta} + \overline{\delta \partial}$, but we then need to define $\overline{\delta}$, which was supposed to be the adjoint to $\overline{\partial}$. More specifically it should be the adjoint with respect to a "global inner product" on $\Gamma(X, \Omega^{0,q}(L))$. So how do we get this?

Recall previously that a metric ds^2 on X gave us a $(1, 1)$-form ω and then a (n, n)-form (volume form) $dv = \frac{1}{g!} \bigwedge^g \omega$. Then defined our inner product by

$$(\varphi, \psi) = \sum \int \psi_{IJ} \varphi_{IJ} dv.$$

But this is not enough for what we're doing now: need to fix some "metric data" on L.

Definition: A Hermitian metric on a line bundle is a positive definite form on each fiber L_x, depending smoothly on $x \in X$. In other words, we take a $h \in \Gamma_{sm}(X, (L \otimes \overline{L})^v)$ such that $h_p(\eta, \overline{\xi}) = \overline{h_p(\xi, \overline{\eta})}$ and $h_p(\xi, \overline{\xi}) > 0$ if $\xi \neq 0$, for any $p \in X$ and any $\xi, \eta \in L_p$. (Remark: Can compute Chern classes once you've fixed a metric. For a Hermitian line bundle (L, h), you get a curvature $(1, 1)$-form we can call $c_1(L, h)$, and its cohomology class is the Chern class $c_1(L)$. So can get the Chern class explicitly by choosing a metric! And lots of other stuff - Poincaré-Lelong.)

Constructions for a line bundle $L = L(H, \chi)$ on a complex torus X. In the $(0, 0)$ case: Note $\Gamma(X, \Omega^{0,0}(L))$ is the set of smooth functions $f : V \to \mathbb{C}$ with $f(x + \lambda) = a_L(\lambda, v)f(v)$ (these are "smooth" theta functions). For $f, g \in \Gamma(X, \Omega^{0,0}(L))$ define a new function $h = \langle f, g\rangle$ on V by

$$h(v) = f(v)\overline{g(v)} \exp(-\pi H(v, v)).$$

Note $h(v + \lambda) = h(v)$ by explicit computation, so h gives a C^∞ element of $\Gamma(X, \Omega^{0,0})$, i.e. a smooth global section. Thus we have a Hermitian form

$$\langle\ ,\ \rangle : \Gamma(X, \Omega^{0,0}(L)) \times \Gamma(X, \Omega^{0,0}(L)) \to \Gamma(X, \Omega^{0,0}).$$

This determines a Hermitian metric of the form we wanted above.

Now, a work with a Kahler metric ds^2 on $V/\Lambda = X$. Fix a basis $e_1, \ldots, e_g$ of V such that $H = c_1(L)$ is diagonalizable with respect to this basis. (Note Hermitian matrices always are diagonalizable). If $h_1, \ldots, h_g$ are the diagonal entries of the matrix, let $v_1, \ldots, v_g$ be the coordinate functionals corresponding to the v_i's. This means by definition we have $H(v, w) = \sum h_i v_i(v) v_i(w)$ which we denote $\sum h_i v_i w_i$.

Fix $k_1, \ldots, k_g$ positive real numbers (will pick them precisely later). Then

$$ds^2 = \sum_{i=1}^{g} k_i dv_i \otimes d\overline{v}_i$$

defines a Kähler metric (an "easy" one, for which it's easy to prove the Hodge theorem). Then the associated $(1,1)$-form

$$\omega = -\frac{1}{2}\operatorname{Im}(ds^2) = \frac{i}{2}\sum_{j=1}^{g} k_j dv_j \wedge d\overline{v}_j$$

is closed, and gives us a volume form

$$dv = \frac{1}{g!}\int^g \omega = \left(\frac{i}{2}\right)^g \prod_{j=1}^{g} k_i \bigwedge_{j=1}^{g} dv_j \wedge d\overline{v}_j.$$

Definition: We define an inner product on $\Gamma(X, \Omega^{0,0}(L))$ by

$$(f, g) = \int_X \langle f, g\rangle(v) dv.$$

This gives the $(0,0)$ case of the inner product we needed. For the $(0,q)$ case, recall we have $\omega = \sum_{|I|=q} \varphi_I d\overline{v}_I$ and $\omega' = \sum_{|I|=q} \psi_I d\overline{v}_I$ for $\omega, \omega' \in \Gamma(X, \Omega^{0,q}(L))$, with $\varphi_I \in \Gamma(X, \Omega^{0,0}(L))$. Then define

$$(\omega, \omega') = \sum_{|I|=q} \mathbf{k}^{-I}(\varphi_I, \psi_I)$$

where $\mathbf{k}^{-I} = \prod_{i\in I} k_i^{-1}$. So this is the obvious thing except with a scalar for each I coming from our parameters k_i above.

Notation: Let $\partial_i = \partial/\partial v_i$ and $\overline{d}_i = \partial/\partial\overline{v}_i$. Recall $\Gamma(X, \Omega^{0,0}(L))$ consists of $f : V \to \mathbb{C}$ with $f(v+\lambda) = a_L(\lambda, v) f(v)$. Since a_L is holomorphic $\overline{\partial}_i a_L = 0$ so $\overline{\partial}_i$ is a linear operator on $\Gamma(X, \Omega^{0,0}(L))$. So there exists $\overline{\partial} : \Gamma(X, \Omega^{0,q}(L)) \to \Gamma(X, \Omega^{0,q+1}(L))$ given by

$$\overline{\partial}(\psi d\overline{v}_I) = \sum_i (\overline{\partial}_i \varphi) d\overline{v}_i \wedge d\overline{v}_I.$$

This is the map on the complex we had. Define $\overline{\delta}_i$ to be the adjoint of $\overline{\partial}_i$ and $\overline{\delta}$ the adjoint of $\overline{\partial}$, with respect to the $(\ ,\ )$ we just defined.

Explicitly computing these adjoints: Lemma: Let $\varphi \in \Gamma(X, \Omega^{0,0}(L))$. Then (a) $\overline{\delta}_i\varphi = -\partial_i\varphi + \pi h_i \overline{v}_i \varphi$ (recall h_i are the diagonal entries of H, and $\overline{v}_i$ are the coordinates). (b) If J is the index set $j_1 < j_2 < \cdots < j_{q+1}$ then

$$\overline{\delta}(\varphi dv_J) = \sum_{i=1}^{q+1} (-1)^{i-1} \frac{1}{k_{j_i}} (\overline{\delta}_{j_i}\varphi) d\overline{v}_{J\setminus\{j_i\}}.$$

Proof: (b) follows from (a). For (a), need to show that this actually works, i.e. that for all φ and ψ we have

$$(\overline{\partial}_i\varphi, \psi) = (\varphi, -\partial_i\varphi + \pi h_i \overline{v}_i \varphi).$$

Equivalently want to show the difference of these is zero; and the difference is the integral of

$$(*) = \langle \overline{\partial}_i \varphi, \psi\rangle - \varphi, -\partial_i\varphi + \pi h_i \overline{v}_i \varphi.$$

But we can explicitly compute that $(*)$ is $\overline{\partial}_i\langle\varphi, \psi\rangle$, and then its integral with respect to the volume form is 0 because it's a closed form on a boundaryless space (by Stokes' theorem).

Lemma: For $\Delta = \overline{\partial\delta} + \overline{\delta\partial}$ and $\varphi d\overline{v}_I \in \Gamma(X, \Omega^{0,q}(L))$ for $I = (i_1 < \cdots < i_q)$ and $\varphi \in \Gamma(X, \Omega^{0,0}(L))$, we have

$$\Delta(\varphi d\overline{v}_I) = \sum_{i=1}^{g} \frac{1}{k_i} \delta_i \overline{\partial}_i \varphi d\overline{v}_I + \pi \sum_{j=1}^{q} \frac{1}{k_{i_j}} h_{i_j} \varphi d\overline{v}_I.$$

18 Lecture - 04/07/2016

Last time: stated a lemma that

$$\Delta(\varphi d\overline{v}_I) = \sum_{i=1}^{g} \frac{1}{k_i}(\overline{\delta}_i\overline{\partial}_i\varphi)d\overline{v}_I + \pi\sum_{j=1}^{q}\frac{1}{k_{i_j}}h_{i_j}\varphi d\overline{v}_I.$$

Here Δ was the operator we defined, $\varphi d\overline{v}_I$ was an element of $\Gamma(X, \Omega^{0,q}(L))$ with $I = (i_1 < \cdots < i_q)$ an index set and $\varphi \in \Gamma(X, \Omega^{0,0}(L))$. Also the $k_i > 0$'s were such that $ds^2 = \sum k_i dv_i \otimes d\overline{v}_i$ is our metric, and the h_i's were the diagonal entries for H, which is diagonal with respect to the coordinates $v_i, \overline{v}_i$ we chose. Upshot is we get this explicit formula for the Laplacian.

First remark: This formula tells us that the Laplacian doesn't "mix" the indices; it acts on the subvector space of "monomials" $A_I = \{\varphi_I d\overline{v}_I\}$ for each fixed I. So, $\mathcal{H}^q(L) = \ker(\Delta)$ has a canonical decomposition

$$\mathcal{H}^q(L) = \bigoplus_{|I|=q} \ker(\Delta|_{A_I}) = \bigoplus_{|I|=q} \mathcal{H}^q_I(L).$$

Now, this was all set up for an arbitrary choice of k_i's and h_i's; what happens if we want to be clever about our choices? Let's fix "nice" ones satisfying:

1. After permuting and scaling the fixed basis $\{e_1, \ldots, e_g\}$ we used to diagonalize H, we can assume that $h_1 = \cdots = h_r = 1$, $h_{r+1} = \cdots = h_{r+s} = -1$, and $h_{r+s+1} = \cdots = h_g = 0$ (where r is the number of positive eigenvalues and s the number of negative ones).

2. Set $k_i = 1/(s+1)$ for $i \le r$ and $k_i = 1$ for $i > r$.

Key proposition: Given a multiindex I, let $R_I = |I \cap \{1, \ldots, r\}|$ and $S_I = |I \cap \{r+1, \ldots, r+s\}|$. With k_i's and h_i's as above, for any $\varphi d\overline{v}_I \in \Gamma(X, \Omega^{0,q}(L))$ we have

$$\big(\Delta(\varphi_I d\overline{v}_I), \varphi_I d\overline{v}_I\big) \ge \pi\big((s+1)R_I - S_I\big)(\varphi d\overline{v}_I, \varphi d\overline{v}_I).$$

Remark: We then can use this to prove vanishing theorems, i.e. that there's no harmonic forms; since if $\varphi_I d\overline{v}_I$ is harmonic then the pairing on the LHS is zero and if we can arrange $(s+1)R_I - S_I > 0$ then this forces the pairing on the RHS to be zero and thus $\varphi d\overline{v}_I = 0$.

Proof: Plug in the formula for $\Delta(\varphi d\overline{v}_I)$ from our lemma, and get

$$\big(\Delta(\varphi_I d\overline{v}_I), \varphi_I d\overline{v}_I\big) = \sum_{i=1}^{g}\frac{1}{k_i}\big(\overline{\delta}_i\overline{\partial}_i\varphi_I d\overline{v}_I, \varphi_I d\overline{v}_I\big) + \pi\left(\sum_{j=1}^{q}\frac{h_{i_j}}{k_{i_j}}\right)(\varphi d\overline{v}_I, \varphi d\overline{v}_I).$$

Now the first sum is

$$\sum_{i=1}^{g}\frac{1}{k_i}\big(\overline{\delta}_i\overline{\partial}_i\varphi_I d\overline{v}_I, \varphi_I d\overline{v}_I\big) = \sum_{i=1}^{g}\frac{1}{k_i}\big(\overline{\partial}_i\varphi_I d\overline{v}_I, \overline{\partial}_i\varphi_I d\overline{v}_I\big) \ge 0,$$

and by the choices of h_i and k_i we have we get $\sum_{j=1}^{q} h_{i_j}/k_{i_j} = R_I(s+1) - S_I$. So we get

$$\big(\Delta(\varphi_I d\overline{v}_I), \varphi_I d\overline{v}_I\big) \ge 0 + \pi\big((s+1)R_I - S_I\big)(\varphi d\overline{v}_I, \varphi d\overline{v}_I).$$

Corollary: $\mathcal{H}^q_I(L) = 0$ if $R_I > 0$. Proof: By assumption $R_I \ge 0$ and by definition $S_I \le s$, so $(s+1)R_I - S_I \ge 1 > 0$. So by the lemma, if $\varphi_I d\overline{v}_I$ is a harmonic form we get

$$0 = \big(\Delta(\varphi_I d\overline{v}_I), \varphi_I d\overline{v}_I\big) \ge \pi\big((s+1)R_I - S_I\big)(\varphi d\overline{v}_I, \varphi d\overline{v}_I)$$

means $0 \ge (\varphi d\overline{v}_I, \varphi d\overline{v}_I)$ and thus $\varphi d\overline{v}_I = 0$.

Theorem (Mumford-Kempf; Deligne): If $X = V/\Lambda$ is a complex torus, and $H = c_1(L)$ has r positive and s negative eigenvalues, then $H^q(X, L) = 0$ if $q > g - r$ or $q < s$.

Proof: if $q > g - r$ then for any I with $|I| = q$ we must have $I \cap \{1, \dots, r\} \neq \emptyset$ and thus $R_I > 0$. So by the corollary above, $\mathcal{H}^q_I(L) = 0$, and then $\mathcal{H}^q(L) = \bigoplus_I \mathcal{H}^q_I(L) = 0$. By the Hodge theorem, $H^q(X, L) \cong \mathcal{H}^q(L) = 0$ too.

For the $q < s$ case we'll use Serre duality. For any (smooth) compact complex manifold of dimension g, and L is a holomorphic line (or vector) bundle over X, and K is the canonical bundle (a particular line bundle), then there is a canonical isomorphism

$$H^q(X, L) \cong H^{g-q}(X, K \otimes L^\vee)^*.$$

(Correct statement is there's a nondegenerate bilinear form between these things which gives this duality). Remark: Grothendieck has a more general version for non-smooth things and coherent duality.

So specializing this to X a complex torus: $K = \Omega^g$ is isomorphic to $\mathcal{O}_X$ and $L^\vee \cong L^{-1}$, so Serre duality gives $H^q(X, L) \cong H^{g-q}(X, L^{-1})^*$. Since $c_1(L^{-1}) = -c_1(L)$, the number of positive eigenvalues of L^{-1} is s, the number of negative ones for L. So our first case tells us that $H^{g-q}(X, L^{-1})$ is zero for $g - q > g - s$, i.e. $q < s$. Thus by duality conclude $H^q(X, L) = 0$ for $q < s$.

Next part: What if $s \leq q \leq g - r$, the range we haven't considered yet? First of all we'll see they all come down to the case of $q = s$:

Theorem: if $s \leq q \leq g - r$ then $h^q(X, L) = \binom{g-r-s}{q-s} h^s(X, L)$. In fact, $H^q(X, L) \cong \bigoplus H^s(X, L)$ for a direct sum over $\binom{g-r-s}{q-s}$ indices.

Proof: Recall we had $\mathcal{H}^q(L) = \bigoplus_{|I|=q} \mathcal{H}^q_I(L)$. Claim that in fact we have

$$\mathcal{H}^q(L) = \bigoplus_{\substack{|I|=q \\ R_I=0, S_I=s}} \mathcal{H}^q_I(L).$$

We already proved that $\mathcal{H}^q_I(L) = 0$ if $R_I \neq 0$ so that restriction is obvious. So need to prove that $\mathcal{H}^q_I(L) = 0$ if $R_I = 0$ and $S_I < s$. From our lemma earlier we have $\Delta(\varphi d\overline{v}_I) = \psi d\overline{v}_I$ for

$$\psi = \sum_{j=1}^{g} \frac{1}{k_j} \overline{\delta}_j \overline{\partial}_j \varphi - \pi \sum_{j=1}^{q} \frac{h_{i_j}}{k_{i_j}} \varphi,$$

and the latter term is $\pi S_I \varphi$. Then if $J = I \cap \{r+1, \dots, r+s\}$, we have $R_J = 0$, $S_J = S_I$, and by our formula for Δ we get $\Delta(\varphi d\overline{v}_J) = 0$ iff $\Delta(\varphi d\overline{v}_I) = 0$. Thus we have an isomorphism $\mathcal{H}^q_I(L) \cong \mathcal{H}^{S_I}_J(L)$. But $S_I < s$ puts us in the range of vanishing for $H^{S_I}(X, L)$ and thus $\mathcal{H}^{S_I}_J(L) = 0$ too.

So we've proven our claim, that

$$\mathcal{H}^q(L) = \bigoplus_{\substack{|I|=q \\ R_I=0, S_I=s}} \mathcal{H}^q_I(L).$$

We now want to prove in fact that every $\mathcal{H}^q_I(L)$ in this sum is isomorphic to $\mathcal{H}^s(L)$. This follows by the argument above, since $\mathcal{H}^q_I(L) \cong \mathcal{H}^s_J(L)$ for $J = \{r+1, \dots, r+s\}$ and $\mathcal{H}^s_J(L) = \mathcal{H}^s(L) \cong H^s(X, L)$ by the claim for $q = s$. Now, how many summands are there? It's the number of subsets of q elements of $\{1, \dots, g\}$ that contain nothing in $\{1, \dots, r\}$ and everything in $\{r+1, \dots, r+s\}$; such I's correspond to choices of $q - s$ elements in $\{r+s+1, \dots, g\}$, i.e. $\binom{g-r-s}{q-s}$.

So we're reduced to looking at $H^s(X, L)$. What happens here? If $s = 0$ we've already analyze the positive-definite (which we found a basis in terms of theta functions) and positive-semidefinite case (which is either zero or descends to a positive-definite one on a quotient) so we're done. If $s > 0$ we also reduce to the $s = 0$ case by finding a complex torus X' and a positive-semidefinite line bundle L' on it such that $H^s(X, L) \cong H^0(X', L')$. This is "Wirtinger's trick"; the idea is to change the complex structure.

Recall $\Lambda(L)_0$, the connected component of $\Lambda(L)$, is the radical of H. We have fixed a basis $\{e_1, \ldots, e_g\}$ such that H has r entries of 1, s entries of -1, and $q - r - s$ entries of 0; then $V_0 = \Lambda(L)_0$ is spanned by $e_{r+s+1}, \ldots, e_g$. Define V_+ as the span of $e_1, \ldots, e_r$ and V_- as the span of $e_{r+1}, \ldots, e_{r+s}$. So $\mathbb{C}^g \cong V = V_+ \oplus V_- \oplus V_0$.

Now, let $\mathbb{R}^{2g} \cong W$ be the underlying $\mathbb{R}$-vector space. Let j be the complex structure on w corresponding to V, i.e. the $\mathbb{R}$-linear map $j : W \to W$ given by $v \mapsto iv$. We have $W = W_+ \oplus W_- \oplus W_0$ as before, and now define a new complex structure j' by $j'(w) = j(w)$ if $w \in W_+ \oplus W_0$ and $j'(w) = -j(w)$ if $w \in W_-$. Then let V' be the induced complex vector space by j'. Then $\Lambda \subseteq V'$ is still obviously a lattice, and we take $X' = V'/\Lambda$. So this is a complex torus (which is equal to X as a set but with a different complex structure).

Now, how do we define L? A line bundle on X' is given by a Hermitian form and a character. Recall if $H = c_1(L)$ we took $E = \operatorname{Im} H$ and

$$H(v, w) = E(iv, w) + iE(v, w) = E(j(v), w) + iE(v, w);$$

so define a new Hermitian form

$$H'(v, w) = E(j'(v), w) + iE(v, w).$$

Don't change the character, so $L' = L(H', \chi)$.

Theorem: $H^s(X, L) \cong H^0(X', L')$ with L' positive-semidefinite on X. To explicitly give this isomorphism, let $f : W \to \mathbb{C}^\times$ be the map $w \mapsto \exp(\pi H'(w_-, w_+))$ (using the decomposition $W = W_+ \oplus W_- \oplus W_0$).

Lemma: Consider $\{\varphi d\overline{v}_j : \varphi \in \Gamma(X, \Omega^{0,0}(L))\} \hookrightarrow \Gamma(X', \Omega^{0,0}(L'))$. Then the map $\varphi d\overline{v}_j \mapsto \varphi f$ induces the above isomorphism.

19 Lecture - 04/12/2016

Today: Start moving from complex tori in general to the special case of abelian varieties. Let X be a g-dimensional complex torus (or really, a g-dimensional complex variety) and $L_1, \dots, L_g \in \text{Pic}(X)$. Will define the *intersection number* of these line bundles as

$$(L_1, \dots, L_g) = \int_X c_1(L_1) \wedge \cdots \wedge c_1(L_g).$$

Remark: Remember that $c_1(L) \in H^2(X, \mathbb{Z})$ sits inside $H^2(X, \mathbb{C})$, and with respect to the Hodge decomposition it sits inside $H^{1,1}$. Thus $c_1(L) \in H^2(X, \mathbb{Z}) \cap H^{1,1}$. So the wedge product in our integral is a (g, g)-form which we think about as a volume form, and thus we get a number in $\mathbb{C}$.

Why do we call this an intersection number? Note $c_1(L) \in H^2(X, \mathbb{Z})$ is Poincaré dual to $\{D\} \in H_{2g-2}(X, \mathbb{Z})$ for any divisor D associated to L. (Remark: It's not entirely obvious it makes sense that D gives a class in homology; need to check it's triangulizable, which was done by Hironaka). So this can be thought of as an intersection of associated divisors in the sense of topology.

In particular, for $L \in \text{Pic}(X)$ we can define a self-intersection of L as $(L^g) = (L, \dots, L) = \int_X \bigwedge^g c_1(L)$. Theorem: $(1/g!)(L^g) = (-1)^s Pf(L)$. Since the RHS is the RHS of the Riemann-Roch theorem we proved, conclude that the Euler characteristic is given by $\chi(X, L) = (1/g!)(L^g)$. This statement is "algebraic Riemann-Roch" or "geometric Riemann-Roch". So we need to prove this theorem.

Lemma: Let L be of type $(d_1, \dots, d_g)$. Let $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ be a symplectic basis for $E = \text{Im}(H)$, and $\{x_1, \dots, x_g, y_1, \dots, y_g\}$ the corresponding coordinate functions. Then

$$c_1(L) = -\sum_{j=1}^{g} d_j dx_j \wedge dy_j.$$

Proof: Follow the definitions.

Remark 1: If L is degenerate and thus there exists $d_i = 0$, conclude $\bigwedge^g c_1(L) = 0$. But in this case $Pf(L) = 0$ too so the theorem we're trying to prove is true as it says $0 = 0$.

Remark 2: So we're reduced to assuming L is nondegenerate. In this case $s = g - r$ is the index of L, and $\chi(X, L) = h^s(X, L)$. At this point it's straightforward to see the theorem is true up to sign. So to determine the sign we need:

Lemma: We have

$$\int_X \bigwedge_{j=1}^{1} (dx_j \wedge dy_j) = (-1)^{g+s}.$$

(In other words this gives us explicitly the orientation associated to the symplectic basis).

Proof: We've seen that $\{\mu_1, \dots, \mu_g\}$ is a $\mathbb{C}$-basis for V; let $v_1, \dots, v_g$ the corresponding coordinates. Know that

$$\left(\frac{i}{2}\right)^g \int (dv_j \wedge d\overline{v}_j)$$

is the natural positive orientation. Let Π be the period matrix with respect to $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ our $\mathbb{Z}$-basis for Λ and $\mu_1, \dots, \mu_g$ the $\mathbb{C}$-basis for V. Thus $\Pi = [Z|I]$ for some $g \times g$ matrix Z. A simple computation is then

$$\left(\frac{i}{2}\right)^g \int (dv_j \wedge d\overline{v}_j) = (-1)^g \det(\text{Im}\, Z) \bigwedge_{j=1}^{g} dx_j \wedge dy_j.$$

Then it's sufficient to show that $(-1)^s \det(\text{Im}\, Z) > 0$.

So, let $Y \in M_{g \times g}(\mathbb{C})$ be the matrix of H with respect to the $\mathbb{C}$-basis $\{\mu_1, \dots \mu_g\}$. Then the matrix of H with respect to the $\mathbb{R}$-basis $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ is

$$\Pi^{\tilde{r}} Y \overline{\Pi} = \begin{bmatrix} Z^{\tilde{r}} \\ I \end{bmatrix} Y \begin{bmatrix} \overline{Z} & I \end{bmatrix} = \begin{bmatrix} Z^\top Y \overline{Z} & Z^\top Y \\ Y \overline{Z} & Y \end{bmatrix}.$$

The matrix of $E = \operatorname{Im} H$ with respect to $\lambda_1, \ldots, \lambda_g$ is then the imaginary part of this matrix, but it's also what we set up before, i.e.

$$\operatorname{Im} \begin{bmatrix} Z^\top Y \overline{Z} & Z^\top Y \\ Y\overline{Z} & Y \end{bmatrix} = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}.$$

So in particular Y is real, so we get $D = (\operatorname{Im} Z^\top)Y$. Since $\det(D) > 0$ by construction, find that the signs of $\det \operatorname{Im} Z^\top$ and $\det Y$ are the same; but the sign of $\det Y$ is $(-1)^s$ by construction. So we've shown $(-1)^s \det(\operatorname{Im} Z) > 0$, as desired.

Now, note that

$$\bigwedge^g c_1(L) = \bigwedge^g \left(-\sum_{j=1}^g d_j (dx_j \wedge dy_j) \right) = (-1)^g g! (d_1 \cdots d_g) \bigwedge^g (dx_j \wedge dy_j).$$

Integrating this over X and using the previous lemma gives $(L^g) = (-1)^s g! Pf(L)$, which is what we wanted. So we've proven algebraic/geometric Riemann-Roch.

Corollary: If $f : X' \to X$ is a surjective homomorphism of complex tori and $L \in \operatorname{Pic}(X)$, then

$$\chi(X', f^* L) = (\deg f) \chi(X, L).$$

Proof: if f is not an isogeny, we've seen $f^* L$ is degenerate so the equality is $0 = 0$. If f is an isogeny, so $\deg f = [\Lambda : f_{\mathrm{Int}}(\Lambda')]$ and a change-of-variables computation gives

$$\int_{X'} \bigwedge c_1(f^* L) = (\deg f) \int_X \bigwedge c_1(L).$$

Abelian varieties. A complex torus $X = V/\Lambda$ is called an abelian variety if it admits a positive-definite line bundle, i.e. there's some $H \in NS(X)$ which is positive-definite. Remark: we'll see this agrees with our initial definition of complex tori that are "algebraic" or "algebraizable". (Hint: If $X \hookrightarrow \mathbb{P}^N$ then there's $\mathcal{O}(1)$ on $\mathbb{P}^N$ we can pull back to $f^* \mathcal{O}(1)$ which must be positive-definite. The converse will be that this is sufficient: if we have a positive-definite line bundle on X it will give an embedding).

Definition: A *polarization* on X is a choice of a positive-definite L, or its first Chern class $c_1(L)$. The *type* of a polarization is the type of L or $c_1(L)$, i.e. the list $(d_1, \ldots, d_g)$. A polarization is *principal* if it's of type $(1, \ldots, 1)$. For a pair (X, L), or just (X, H), is a *polarized abelian variety*, i.e. a PAV. A homomorphism of PAVs is $f : (Y, M) \to (X, L)$ such that $f : Y \to X$ is a homomorphism of complex tori such that $c_1(f^* L) = f^* c_1(L) = c_1(M)$. (Note that $f^* c_1(L)$ and $c_1(f^* L)$ are pullbacks in two different senses, we need a theorem that they are equal).

Remark: a homomorphism of PAVs must have $f^* L$ and M analytically equivalent. Positive-definiteness forces f to have finite kernel (otherwise $f^* L$ would be degenerate). Conversely if $f : Y \to X$ has finite kernel then $f^* L \in \operatorname{Pic}(Y)$ is positive-definite. This means if X is an abelian variety so is Y, and if X is polarized we can give Y the *induced polarization.*

Corollary: Complex subtori of abelian varieties are abelian varieties. Corollary: If X is an abelian variety and Y is a complex torus isogenous to it, then Y is an abelian variety. Corollary: If X is an abelian variety then the dual $\widehat{X}$ is an abelian variety.

20 Lecture - 04/14/2016

Recall: An abelian variety is a complex torus X such that there exists a $H \in NS(X)$ which is positive definite. The H is a polarization for X, and $(d_1, \dots, d_g)$ is its type ("principal polarization" if type is $(1, \dots, 1)$). Said that if $f : X \to Y$ is a map with finite kernel with Y an abelian variety then so is X. In particular, X is an abelian variety iff $\widehat{X}$ is.

Lemma: Every polarization is induced by a principal polarization by an isogeny. (In other words, if X is a polarized abelian variety, there's an isogeny of polarized abelian varieties $f : X \to Y$ with Y principally polarized. Remember this means f^* pulls back the polarization of Y to one of X).

Proof: Let (X, L) be a polarized abelian variety of type $(d_1, \dots, d_g)$. Recall we have an isogeny $P_1 : X \to X_1$ with $X_1 = V/(\Lambda(L)_1 \oplus \Lambda_2)$, and we have a line bundle $M_1 \in \operatorname{Pic}(X_1)$ pulling back to L. Clear M_1 is positive definite, and claim that the type of M_1 is $(1, \dots, 1)$. (So the remark constructing this X_1 and M_1 was giving us a principally polarized thing in the isogeny class of (X, L)).

Proof that M_1 is principal: By how pullback works and Riemann-Roch we have

$$d_1 \cdots d_g = \chi(L) = \deg(p_1)\chi(M) = d_1 \cdots d_g \chi(M_1)$$

so $\chi(M_1) = 1$ and thus the Pfaffian is 1.

Example: If $X = \mathbb{C}/\Lambda$ is an elliptic curve, then $\Lambda \cong \mathbb{Z}^2$, say $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. By rotating lattices we may assume $\omega_1 = 1$ and $\omega_2 = \tau$ is in the upper half-plane. Then we have a Hermitian form $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $H(v, w) = v\overline{w}/\operatorname{Im}(\omega_1\overline{\omega}_2)$, which is scaled so it's integer-valued on Λ. This is obviously positive-definite. So $\mathbb{C}/\Lambda$ is an abelian variety. (This is very rare for any $\mathbb{C}^n/\Lambda$ for $n > 1$).

Recall the theorem of the square: for $\overline{v}, \overline{w} \in X$, we have

$$t^*_{\overline{v}+\overline{w}}L \cong t^*_{\overline{v}}L \otimes t^*_{\overline{w}}L \otimes L^{-1}$$

which in particular gives us $t^*_x L \otimes t^*_{-x} L \cong L^2$. Also get

$$t^*_{\overline{v}} \otimes t^*_{\overline{w}} \otimes t^*_{-\overline{v}-\overline{w}} L \cong L^3.$$

This generalizes:

Lemma: Let $\overline{v}_1, \dots, \overline{v}_n \in X$ with $\sum \overline{v}_i = 0$. Then

$$\bigotimes_{i=1}^n t^*_{\overline{v}_i} L \cong L^n.$$

Proof: if $L = L(H, \chi)$ then the LHS is

$$L(nH, \prod \chi e^{2\pi i \operatorname{Im} H(\overline{v}, -)}) = L(nH, \chi^n) = L^n$$

because $\sum \overline{v}_i = 0$.

Remark: (1) I'll give an interpretation of this in terms of divisors. (2) One can use this to give an explicit basis for $H^0(X, L^n)$ in terms of a basis for $H^0(X, L)$. (If L is of type $(1, \dots, 1)$ then $H^0(X, L)$ is 1-dimensional and easy to deal with, but doing $H^0(X, L^n)$ directly from our theory of line bundles is a pain).

Riemann relations. Question: What does it mean in terms of the period matrix Π for X to be an abelian variety? Let $X = V/\Lambda$ with V having a $\mathbb{C}$-basis $\{e_1, \dots, e_g\}$ and Λ having a $\mathbb{Z}$-basis $\{\lambda_1, \dots, \lambda_{2g}\}$, which gives its period matrix.

Theorem (Riemann relations): X is an abelian variety iff there exists a nondegenerate antisymmetric matrix $A \in M_{2g}(\mathbb{Z})$ such that:

1. $\Pi A^{-1} \Pi^\top = 0$
2. $i\Pi A^\top \overline{\Pi}^\top$ is a positive-definite matrix.

(Remark: condition A having integer entries gives us integrality condition and the matrix for E; then condition (1) is about Hermitianness and (2) is about positive-definiteness).

Proof sketch: Start with a complex torus, and let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be an arbitrary nondegenerate alternating form on Λ (in $NS(X)$). Let A be its matrix with respect to the λ_i, so $a_{ij} = E(\lambda_1, \lambda_j)$. Define $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ by $H(u, v) = E(iu, v) + iE(u, v)$. Then define

$$J = \begin{bmatrix} \Pi \\ \overline{\pi} \end{bmatrix}^{-1} \begin{bmatrix} iI_g & 0 \\ 0 & -iI_g \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{\pi} \end{bmatrix}.$$

Then this matrix J satisfies $i\Pi = \Pi J$; so we can rephrase our linear algebra about multiplying by i using J. Note also $E(\Pi x, \Pi y) = x^\top Ay$.

Claim 1: A is Hermitian iff (1) holds. Proof: we've seen H is Hermitian iff $E(iu, iv) = E(u, v)$, so by our comments before we have H is Hermitian iff $J^\top AJ = A$. Claim 2: If (1) holds, then the matrix of H with respect to $\{e_1, \ldots, e_g\}$ is $2i(\Pi A^{-1}\overline{\Pi})^{-1}$. (Won't prove these claims; they're messy computations).

Remark: Assuming $\{\lambda_i\}$ is symplectic, so

$$A = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} \qquad\qquad \Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}$$

then the relations can be stated as $\Pi_2 D^{-1}\Pi_1^\top - \Pi_1 D^{-1}\Pi_2^\top = 0$ and $i(\Pi_2 D^{-1}\overline{\Pi}_1^\top - \Pi_1 D^{-1}\overline{\Pi}_2^\top)$ is positive-definite.

Divisors vs. Line bundles. Let $(X, \mathcal{O}_X)$ be a complex g-dimensional manifold with a holomorphic structure sheaf (not necessarily compact). Everything has an algebraic analogue as we'll remark (see Chapters 4-6 of Hartshorne, though that makes things complicated).

Let $\mathcal{K}_X$ be the sheaf of rings of meromorphic functions; so we start with a presheaf with $\mathcal{K}_X(U)$ being the set of meromorphic functions on U, and sheafify. Algebraically, $K_X(U) = \mathcal{O}_X(U)$ localized the multiplicative system of non-zero divisors. If X is "integral" then $K_X(U)$ is the fraction field of $\mathcal{O}_X(U)$.

We have an exact sequence of sheaves

$$0 \to \mathcal{O}_X^\times \hookrightarrow \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 0.$$

The associated long exact sequence in cohomology gives

$$H^0(X, \mathcal{K}_X^\times) \to H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times).$$

Now $H^1(X, \mathcal{O}_X^\times)$ is the Picard group. We'll take $H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ as the sheaf of "Cartier divisors". Next time we'll describe this (and the connecting maps) explicitly, and see the connection to "Weil divisors" in the nicest settings.

21 Lecture - 04/19/2016

Let $(X, \mathcal{O}_X)$ be a complex g-dimensional manifold. Let $\mathcal{K}_X$ be the sheaf of meromorphic functions on X, and $K_X^\times$ the sheaf (of multiplicative groups) of invertible elements. Get an exact sequence

$$0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 0$$

which induces a LES and in particular an exact triple

$$H^0(X, K_X^\times) \to H^0(X, K_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times).$$

Recall $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$; define $CDiv(X) = H^0(X, K_X^\times/\mathcal{O}_X^\times)$ ("Cartier divisors"). Want to study these things explicitly. (Remark: In the algebraic setting we define $\mathcal{K}_X(U)$ to be $\mathcal{O}_X(U)$ localized by the multiplicative system of non-zerodivisors; if X is integral this is the fraction field. But this presheaf isn't necessarily a sheaf, so we need to sheafify).

If $j_* : H^0(X, K_X^\times) \to H^0(X, K_X^\times/\mathcal{O}_X^\times)$ is the induced map from the exact triple above, define $CPrin(X) = j_*[H^0(X, K_X^\times)]$ as the "principal Cartier divisors". Exactness of the sequence tells us that we have an embedding $CDiv(X)/CPrin(X) \hookrightarrow \mathrm{Pic}(X)$ induced via the boundary map $\delta : H^0(X, K_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times)$. Remark: This will not be an isomorphism in general, because the next term in the long exact sequence may not be zero. Important remark: If X is a submanifold of $\mathbb{P}^n$ (algebraically: if X is projective) then $H^1(X, \mathcal{K}_X^\times) = 0$ and thus δ induces an isomorphism $CDiv(X)/CPrin(X) \cong \mathrm{Pic}(X)$.

Now want to make this all completely explicit. First of all we recall that $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles over X, and we had an isomorphism of this with $H^1(X, \mathcal{O}_X^\times)$. In particular if L is a line bundle, there exists an open cover $\{U_\alpha\}$ of X on which there's trivializations $\varphi_\alpha : L(U_\alpha) \cong U_\alpha \times \mathbb{C}$. Then there are associated transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ for any pair of indices, in particular such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ maps $(z, w) \mapsto (z, g_{\alpha\beta}(z))$. This gives us a collection of holomorphic functions $g_{\alpha\beta}$ satisfying $g_{\alpha\beta} g_{\beta\alpha} = 1$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. So $\{g_{\alpha\beta}\}$ is a Čech 1-cocycle. Moreover, changing φ_α to $\varphi'_\alpha = f_\alpha \varphi_\alpha$ changes the transition functions to $g'_{\alpha\beta} = (f_\alpha/f_\beta) g_{\alpha\beta}$ which is a 1-coboundary. So L gives a well-defined Čech cohomology class, independent of the choices of trivializations. Conversely, any 1-cocycle $\{g_{\alpha\beta}\}$ gives a line bundle $(\coprod U_\alpha \times \mathbb{C})/\sim$ where the equivalence relation is determined by the $g_{\alpha\beta}$.

Cartier divisors, explicitly. So now let's go back to $CDiv(X) = H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ (note this is very general - makes sense for any scheme, or even any locally ringed space). A global section of $K_X^\times/\mathcal{O}_X^\times$ can be presented by an open cover $\{U_\alpha\}$ of X and a collection of meromorphic functions f_α on U_α (not locally identically zero), with the gluing condition that $f_\alpha/f_\beta \in \mathcal{O}_X^\times(U_\alpha \cap U_\beta)$ for all α, β. (Remark: Since we're working modulo holomorphic functions, replacing f_α by a multiple by any nonvanishing holomorphic function gives the same section thing). Notation: $D = (\{U_\alpha\}, \{f_\alpha\})$.

What's a principal Cartier divisor? Well, something coming from $H^0(X, \mathcal{K}_X^\times)$. In the notation of the previous paragraph, it should be something coming from a single global meromorphic function f. In other words, we should have a presentation $D = (\{U_\alpha\}, \{f|_{U_\alpha}\})$ for a global meromorphic function f.

Definition: Say two divisors D_1, D_2 are linearly equivalent, denoted $D_1 \sim D_2$, if the divisor $D_1 - D_2$ is principal. Here, to define $D_1 - D_2$ we take presentations with the same open cover (which we can get from the general case by taking a common refinement) $D_1 = (\{U_\alpha\}, \{f_\alpha\})$ and $D_2 = (\{U_\alpha\}, \{g_\alpha\})$ then $D_1 - D_2 = (\{U_\alpha\}, \{f_\alpha/g_\alpha\})$. The shift from multiplicative notation to additive notation is because of how we'll be thinking about things.

Weil Divisors. In nice situations (e.g. for complex manifolds, or for Noetherian integral separated scheme that's regular in codimension 1; this latter condition means that if $\dim \mathcal{O}_{X,\mathfrak{p}} = 1$ then $\mathcal{O}_{X,\mathfrak{p}}$ is a regular ring, i.e. $\mathfrak{m}/\mathfrak{m}^2$ has dimension 1).

Let Y be a codimension 1 analytic submanifold of X. Basic fact: Y is an "analytic hypersurface": there's an open neighborhood U of any $p \in Y$ such that $U \cap Y$ is the zero set of some holomorphic function g. (The analogue in the language of schemes needs all of the conditions listed). Now Y may not be smooth since the holomorphic function may not be irreducible; but split it up as a union of irreducible components. (Formally, remove the singular points, split into connected components, and take their closures). Write $Y = Y_1 \cup \cdots \cup Y_m$ with Y_i irreducible; say Y itself is irreducible if $m = 1$. These are the building blocks of Weil divisors.

Definition: A Weil divisor D on X is a "locally finite" formal linear combination of such things: $D = \sum_i a_i Y_i$ for $a_i \in \mathbb{Z}$ and Y_i irreducible analytic codimension-1 submanifolds (hypersurfaces). The Y_i's are called "prime Weil divisors". Algebraically, always ask for a finite sum, and the prime Weil divisors are closed integral codimension 1 subschemes. (In the analytic situation "locally finite" means that around any point, there's an open set intersecting only finitely many of the Y_i's. For a compact manifold X this forces the whole sum to be finite).

So define $WDiv(X)$ as the group of all Weil divisors. Two questions now: (1) What are principal Weil divisors? (2) How do Weil divisors compare to Cartier divisors?

For the first question: Let Y be a prime Weil divisor. Given a point $p \in Y$, and given a holomorphic function f around this point, say the order of f along Y at p is an integer n if we have $f = g^n h$ in the local ring $\mathcal{O}_{X,Y}$ for g a local equation for Y (a uniformizer in $\mathcal{O}_{X,Y}$) and h a unit in this local ring. So we're remembering the multiplicity of f "along Y". Or equivalently (?), the largest n such that we can write $f = g^n h$ in $\mathcal{O}_{X,p}$ for any $h \in \mathcal{O}_{X,p}$. Fact: this order is locally constant for $p \in Y$ (hence globally since Y is irreducible and thus connected). So define $\mathrm{ord}_Y(f) = n$. (Algebraically things are easier - let η be the generic point of Y and look at local ring $\mathcal{O}_\eta$, which is a DVR by our assumptions. Define the order as the valuation of f for that discrete valuation). Then if f is meromorphic, locally it's g/h, so define $\mathrm{ord}_Y f = \mathrm{ord}_Y g - \mathrm{ord}_Y h$.

Now can define principal Weil divisors. Given any f (global meromorphic function, but can do even more general things), define

$$\mathrm{div}(f) = \sum \mathrm{ord}_Y(f) Y$$

with the summation over all Y. Fact: this is locally finite, so is indeed a Weil divisor. Note if $\mathrm{ord}_Y(f) > 0$ then f has a zero along Y, and if $\mathrm{ord}_Y(f) < 0$ then f has a pole along Y. The set of all $\mathrm{div}(f)$ is the set of principal Weil divisors.

Cartier divisors vs. Weil divisors. For complex manifolds (or for integral, Noetherian, separated, regular in codimension 1, and locally factorial (all local rings are UFDs)) there exists an isomorphism $CDiv(X) \cong WDiv(X)$ which respects principal divisors. Explicit isomorphism: Suppose $(\{U_\alpha\}, \{f_\alpha\})$ is a Cartier divisor. Since $f_\alpha / f_\beta \in \mathcal{O}_X^\times(U_\alpha \cap U_\beta)$, for all prime divisors Y we have $\mathrm{ord}_Y(f_\alpha) = \mathrm{ord}_Y(f_\beta)$. So we can associate a Weil divisor

$$D' = \sum_Y \mathrm{ord}_Y(f_\alpha) Y$$

where, for a given Y, we choose some U_α intersecting it nontrivially. In fact this map from $CDiv(X) \to WDiv(X)$ always exists, and always takes principal divisors to principal divisors.

The other direction is more complicated. Given a Weil divisor $D = \sum a_i Y_i$, fix $\{U_\alpha\}$ such that each Y_i has a local defining function $g_{i,\alpha} \in \mathcal{O}(U_\alpha)$. Let $f_\alpha = \prod_i g_{i\alpha}^{a_i} \in K_X(U_\alpha)$. Then take $D' = (\{U_\alpha\}, \{f_\alpha\}) \in CDiv(X)$.

22 Lecture - 04/21/2016

Last time: Set up a correspondence between Cartier divisors and Weil divisors, namely $(\{U_\alpha\}, \{f_\alpha\})$ goes to $\sum_Y \mathrm{ord}_Y(f_\alpha)Y$ and conversely $\sum a_i Y_i$ goes to $(\{U_\alpha\}, \{f_\alpha\})$ where each Y_i has a local defining function $g_{i\alpha}$ on U_α, and $f_\alpha = \prod_i g_{i,\alpha}^{a_i}$.

Lemma: These maps are homomorphisms and are inverses of each other, respecting $WPrin$ and $CPrin$. So from now on in nice situations (i.e. the ones we're working in) we will just talk about $\mathrm{Div}(X)$, which we'll take to be the Cartier divisors, but identify with Weil divisors. Does take some practice to go back and forth between the two languages, though.

Also recall the boundary map in cohomology gives $\delta : CDiv(X)/CPrin(X) \hookrightarrow \mathrm{Pic}(X)$. Now want to give this explicitly; i.e. describe how to go from a divisor (in either of our languages) to a line bundle (described explicitly as a cocycle).

Important remark: If X is a Riemann surface, prime divisors are points, so $\mathrm{Div}(X)$ is huge. The same is true for projective complex manifolds - lots of divisors there too (coming from hypersurfaces). But in general this is not true, $\mathrm{Div}(X)$ might even be empty. A converse is true - if there are "enough" divisors then X is projective (Kodaira embedding theorem).

Divisors versus line bundles. We have a boundary map $\delta : \mathrm{Div}(X) \to \mathrm{Pic}(X)$, which we'll denote $D \mapsto \mathcal{L}(D)$ or $\mathcal{O}_X(D)$. Explicitly this map will be given by mapping a Cartier divisor $(\{U_\alpha\}, \{f_\alpha\})$ to the Čech 1-cocycle for $\{U_\alpha\}$ given by $g_{\alpha\beta} = f_\alpha/f_\beta \in \mathcal{O}_X^\times(U_\alpha \cap U_\beta)$. Note that by definition f_α, f_β are *meromorphic* functions so it's not totally immediate that this is actually a 1-cocycle. Then composing with our map from Weil divisors, have an explicit way to get from a Weil divisor $D = \sum a_i Y_i$ to a line bundle $\mathcal{L}(D)$. Easy to check the following:

- $\{g_{\alpha\beta}\}$ is a Čech 1-cocycle.
- Changing the f_α's (for the same D) gives the 1-coboundary condition, so the map is well-defined.
- This map is a homomorphism, and actually agrees with the connecting homomorphism δ in homology.
- If D is principal then $g_{\alpha\beta} = 1$ and thus $\mathcal{L}(D) \cong \mathcal{O}_X$ is trivial.
- Conversely, if $\mathcal{L}(D)$ is trivial, then $g_{\alpha\beta} = h_\alpha/h_\beta$ for $h_\alpha \in \mathcal{O}_X^\times(U_\alpha)$ (this is saying it's a coboundary) so if it comes from $(\{U_\alpha\}, \{f_\alpha\})$ then we also have $g_{\alpha\beta} = f_\alpha/f_\beta$, so $f_\alpha h_\alpha^{-1} = f_\beta h_\beta^{-1}$. Thus $f = f_\alpha h_\alpha$ gives a global meromorphic function which realizes D as a principal divisor.

Definition: We say $D \sim D'$ (linear equivalence) if $D - D' = \mathrm{div}(f)$ for some $f \in H^0(X, \mathcal{K}_X^\times)$; thus $D \sim D'$ iff $\mathcal{L}(D) \cong \mathcal{L}(D')$. Remark: The assignment $D \mapsto \mathcal{L}(D)$ is functorial. (Note that given $f : X' \to X$ we already have a definition of a pullback of line bundles; we also have a pullback $f^* : \mathrm{Div}(X) \to \mathrm{Div}(X')$ by saying that if $D = (\{U_\alpha\}, \{f_\alpha\})$ then $f^*D = (\{f^{-1}[U_\alpha]\}, \{f_\alpha \circ f\})$. At least, this makes sense if $f[X']$ does not end up as a subset of D viewed as a Weil divisor). Functoriality means that $f^*\mathcal{L}(D) \cong \mathcal{L}(f^*D)$.

In terms of Weil divisors: Want to send $\sum a_i Y_i$ to $\sum a_i f^*(Y_i)$. What is the Weil divisor $f^*(Y)$ for a prime divisor Y? Will lie over the hypersurface $f^{-1}[Y]$, but perhaps with multiplicity (the degree of f matters).

Divisors of sections. For $f \in H^0(X, \mathcal{K}_X^\times)$, a global meromorphic function, we defined $\mathrm{div}(f)$. More generally we want a notion of $\mathrm{div}(s)$ where s is a global meromorphic section of an arbitrary line bundle $\mathcal{L}$, i.e. $s \in H^0(X, L \otimes_{\mathcal{O}_X} \mathcal{K}_X)$. Why do we want to define these? Will have a theorem that $L = \mathcal{L}(\mathrm{div}(s))$ in this situation. Corollary: L will lie in the image of $\delta : \mathrm{Div}(X) \to \mathrm{Pic}(X)$ iff L has a nonzero global meromorphic section.

So let L be a line bundle on X. As before, fix $\{U_\alpha\}$ with trivializations $\varphi : L(U_\alpha) \cong U_\alpha \times \mathbb{C}$, and transition functions $g_{\alpha\beta} \in \mathcal{O}_X^\times(U_\alpha \circ U_\beta)$. Note: A holomorphic (meromorphic) section of L over an open set U is given by a collection of functions $s_\alpha \in \mathcal{O}_X(U \cap U_\alpha)$ (or in $\mathcal{K}_x(U \cap U_\alpha)$, respectively) satisfying $s_\alpha = g_{\alpha\beta}s_\beta$ on $U \cap U_\alpha \cap U_\beta$. Note 2: If s, s' are global meromorphic sections of L then s/s' is a global meromorphic function on our manifold (trivial; $s_\alpha/s'_\alpha = g_{\alpha\beta}s_\beta/g_{\alpha\beta}s'_\beta$ so s/s' agrees on overlaps). One consequence: if $D = \mathrm{div}(s)$ and $D' = \mathrm{div}(s')$ we'll have $D - D' = \mathrm{div}(s/s')$ is a principal divisor, so $D \sim D'$ and thus $\mathcal{L}(D) = \mathcal{L}(D')$.

The definition: If s is a meromorphic section of L given by $\{s_\alpha\}$, then $s_\alpha/s_\beta = g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$ so $\mathrm{ord}_Y(s_\alpha) = \mathrm{ord}_Y(s_\beta)$ and thus it's well-defined to write

$$\mathrm{div}(s) = \sum_Y \mathrm{ord}(s_\alpha) Y$$

where again for each Y we take α such that $U_\alpha \cap Y \neq 0$. Example: If $D = (\{U_\alpha\}, \{f_\alpha\}) \in CDiv(X)$. Then $\{f_\alpha\}$ give a global section s for $\mathcal{L}(D)$, and $\mathrm{div}(s)$ as written is the Weil divisor associated to D. Following the definitions, the theorem and corollary we mentioned before are immediate.

Remark: A Weil divisor $D = \sum a_i Y_i$ is called *effective* if all of the coefficients satisfy $a_i \geq 0$. Equivalently, the Cartier divisor $(\{U_\alpha\}, \{f_\alpha\})$ is effective if f_α actually lies in $\mathcal{O}_X(U_\alpha)$ rather than just $\mathcal{K}_X(U_\alpha)$. Also equivalently, if $D = \mathrm{div}(s)$ for s a global holomorphic section.

Another interpretation of global holomorphic sections and effective divisors. Let $D = \sum a_i Y_i \in \mathrm{Div}(X)$ be a divisor. Define

$$R(D) = \{f \in H^0(X, \mathcal{K}_X^\times) : \mathrm{div}(f) + D \geq 0\}.$$

Why did Riemann define this? Basically it's giving a set of meromorphic function with controlled singularities. Since you can't have any nontrivial holomorphic functions, this is a good way to relax the situation and get some nontrivial things, but not too many. Can ask what the dimension of this is. (So if $f \in R(D)$, then f is holomorphic on $X \setminus \bigcup Y_i$, and moreover $\mathrm{ord}_{Y_i}(f) \geq -a_i$).

Also define the *complete linear system* of D as $|D| = \{E \in \mathrm{Div}(X) : E \sim D, E \geq 0\}$, i.e. the set of all effective divisors equivalent to D. (Notation: if L is a line bundle, let $|L|$ denote $|D|$ if D is any divisor with $\mathcal{L}(D) = L$).

Lemma: Let $D = \mathrm{div}(s_0)$ for a global meromorphic section s_0 of $L = \mathcal{L}(D)$. Then we have a natural map $R(D) \to H^0(X, L)$ by $f \mapsto fs_0$, which is an isomorphism. Proof: For well-definedness need to see that if $f \in R(D)$ then fs_0 actually lands in $H^0(X, L)$; but it's clearly a meromorphic section, and is actually holomorphic because $\mathrm{div}(s) = \mathrm{div}(f) + \mathrm{div}(s_0) = \mathrm{div}(f) + D \geq 0$. Then this is clearly an injective homomorphism. For surjectivity, if s is a global holomorphic section then we already checked that $f_s = s/s_0$ is a global meromorphic function, with $\mathrm{div}(f_s) = \mathrm{div}(s) - \mathrm{div}(s_0)$. But $\mathrm{div}(s) \geq 0$ as s is holomorphic, and $\mathrm{div}(s_0) = D$, so get $\mathrm{div}(f_s) \geq -D$ and thus $f_s \in R(D)$ maps to s.

Now: $R(D)$ vs $|D|$. Let $E \in |D|$. Then there exists $f \in R(D)$ with $E = D + \mathrm{div}(f)$. (By definition $E \sim D$ means there is f with $E - D = \mathrm{div}(f)$, and moreover that means $\mathrm{div}(f) + D = E$ is effective so f is in $R(D)$ by definition). If X is compact, any two such f are related by a multiplicative constant; if $D + \mathrm{div}(f) = D + \mathrm{div}(h)$ then $\mathrm{div}(f/h) = 0$ and thus compactness tells us f/h is a constant.

Corollary: $|D|$ is the projective space $\mathbb{P}(R(D)) \cong \mathbb{P}(H^0(X, L))$ associated to the vector space $R(D)$.

23 Lecture - 04/26/2016

Recall from last time: for a divisor D we defined

$$R(D) = \{f \in H^0(X, \mathcal{K}_X^\times) : \operatorname{div}(f) + D \geq 0\} \cong H^0(X, \mathcal{L}(D)),$$

and also the "complete linear system"

$$|D| = \{E \geq 0 : E \sim D\}.$$

If X is compact we said $|D|$ is isomorphic to the projectivization $\mathbb{P}(R(D)) \cong \mathbb{P}(H^0(X, L))$.

If this is a "complete" linear system, what's a linear system in general? Definition: A linear system is a family of effective divisors $\mathcal{E} = \{D_\lambda\}_{\lambda \in I}$ corresponding to a linear subspace of $\mathbb{P}(H^0(X, L))$ for some line bundle L (so i.e. $I \cong \mathbb{P}^n$). More explicitly: fix a subspace $V \subseteq H^0(X, L)$ and take $\mathcal{E} = \mathbb{P}(V)$. The dimension or rank of a linear system $\mathcal{E}$ is the dimension of the corresponding projective space, so if $\mathcal{E} = \mathbb{P}(V)$ then the dimension is $\dim V - 1$.

A "pencil" is a linear system of dimension 1. (A "net" is one of dimension 2 and a "web" is one of dimension 3, but these are less widely used). Definition: Let $\mathcal{E} = \{D_\lambda\} \subseteq |D|$ be a linear system. The "base locus" of $\mathcal{E}$, denoted $\bigcap_{\lambda \in I} D_\lambda$, is the set

$$\{F \in \operatorname{Div}(X) : \forall \lambda, D_\lambda - F \geq 0\}.$$

A "fixed component" of $\mathcal{E}$ is a divisor F in the base locus, i.e. such that $D_\lambda - F \geq 0$ for all λ.

Bertini's Theorem: If D is a "generic" element of a linear system $\mathcal{E}$, then D is smooth away from the base locus.

Maps to $\mathbb{P}^N$. Let X be a compact complex manifold. Then we have a one-to-one correspondence between:

- Nondegenerate maps $f : X \to \mathbb{P}^N$, modulo projective transformations.
- Pairs (L, W) of a line bundle L and a subspace $W \subseteq H^0(X, L)$ such that the linear system $\mathcal{E} = \mathbb{P}(W)$ is base-point-free.

Definition: A linear system $\mathcal{E}$ is basepoint-free if there is no $p \in X$ such that for all $s \in W$ we have $s(p) = 0$.

How do we get this correspondence? First look at maps to $\mathbb{P}^N$ given by a base-point-free divisor $\mathcal{E} = \mathbb{P}(W)$. Version 1: can define $f : X \to \mathbb{P}(W^*)$ by $p \mapsto \{s \in W : s(p) = 0\}$. Version 2: $f : X \to \mathbb{P}(W^*)$ by $p \mapsto \{D \in \mathcal{E} : p \in D\}$. Version 3 (explicit): let $s_0, \dots, s_n$ be a basis for W, and take $f : X \to \mathbb{P}^n$ be given by $p \mapsto [s_0(p) : s_1(p) : \cdots : s_n(p)]$.

So, our map α from nondegenerate maps to pairs (L, W): Given $f : X \to \mathbb{P}^n$ let $W = f^*(H^0(\mathbb{P}^n, \mathcal{L}(H)))$ for H any hyperplane. Then $W \subseteq H^0(X, L)$ for $L = f^*(\mathcal{L}(H))$. Easy to see it's base-point-free. Remark: choice of $\{s_0, \dots, s_n\}$ corresponds to a choice of coordinates $\{x_0, \dots, x_n\}$ for $\mathbb{P}^n$.

Remark: Can define degree of $f : X \to \mathbb{P}^n$ as $\int_X \bigwedge^n c_1(L) = (L^n) = (D^n)$ (self-intersections).

Divisors and maps to $\mathbb{P}^n$ for abelian varieties. As usual, let $X = V/\Lambda$ be an abelian variety of dimension g, and let $L \in \operatorname{Pic}(X)$ be a polarization (i.e. a choice of positive-definite line bundle). Want to look at how to get maps $X \to \mathbb{P}^n$. From the above, these come from maps $p \mapsto [s_0(p) : \cdots : s_n(p)]$ for a linear system. But how do we see when this is actually well-defined everywhere, i.e. the linear system is base-point-free.

Given our polarization L, define a meromorphic map $\psi_L : X \dashrightarrow \mathbb{P}^N$ (which is a rational map) as $x \mapsto [\sigma_0(x) : \cdots : \sigma_N(x)]$ where $\sigma_0, \dots, \sigma_N$ span $H^0(X, L)$. This is well-defined for any x where there's some σ_i with $\sigma_i(x)$ nonzero.

But we can do better than just looking at "arbitrary" σ_i's - we did a lot of work constructing theta functions for $H^0(X, L)$. So, fixing a factor of automorphy we can identify $H^0(X, L)$ with a collection of theta functions, and fixing a basis $\vartheta_0, \dots, \vartheta_N$, and write

$$\psi_L(\overline{v}) = [\vartheta_0(v) : \cdots : \vartheta_N(v)]$$

where $\overline{v} = v + \Lambda$. Big question: When is ψ_L an embedding?

Goal: Theorem (Lefschetz): If the type of L is $(d_1, \dots, d_g)$ with $d_i \geq 3$ for every i, then ψ_L is an embedding.

Example/exercise: If L is (positive-definite) of type $(d_1, \dots, d_g)$, then $L^{\otimes k}$, first see that

$$Pf(L^{\otimes k}) = \chi(L^{\otimes k}) = \dim H^0(X, \mathcal{L}^k) = N + 1,$$

where N is the dimension of our projective space. On the other hand, $Pf(L^{oxk}) = Pf(kc_1(L)) = k^g Pf(L)$. If L is principal, this means $N = k^g - 1$. Since our goal is to show that L^3 works, we will be showing that any abelian variety of dimension g embeds into projective space with dimension $3^g - 1$. (Note for $g = 1$ this tells us every elliptic curve embeds into $\mathbb{P}^2$ - which is what we get the Weierstrass equation).

So suppose X is a principally polarized abelian variety (e.g. elliptic curves, or more generally Jacobians). How do we explicitly write this down? Need to write down theta functions for L^k. If L is principally polarized then $\dim H^0(X, L) = 1$, i.e. it has a canonical theta function θ (unique up to scaling) generating it. This is called Riemann's theta function. Exercise: Give a basis $\vartheta_0, \dots, \vartheta_N$ for $L^{\otimes k}$ in terms of θ. Hint: Recall $L^k \cong \bigotimes^k t^*_{\overline{v}_i} L$ for $\sum \overline{v}_i = 0$. So take appropriate shifts of θ (by, say, $(\frac{1}{K}\Lambda/\Lambda)$) and appropriate products and get a basis.

24 Lecture - 04/28/2016

Fixing things from last time: If $D = \sum a_i Y_i$ is a divisor, write $\operatorname{supp}(D) = \bigcup Y_i$. Then the base locus of a set $\{D_\lambda\}$ is defined as the intersection $\bigcap_\lambda \operatorname{supp}(D_\lambda)$. Then $\{D_\lambda\}$ is base-point-free iff the base locus is empty. A divisor F with support contained in the base locus is called a "fixed component" (?).

Picking up from where we left off last time, we said that if X is an abelian variety and L is a positive-definite line bundle and $\vartheta_0, \dots, \vartheta_N$ are a basis of theta-functions for $H^0(X, L)$, then we have a rational map $X \dashrightarrow \mathbb{P}^N$ given by $\overline{p} \mapsto [\vartheta_0(p) : \cdots : \vartheta_N(p)]$. Want to study when this is defined everywhere.

Basic properties of divisors on abelian varieties. Lemma: Let $\overline{v}_1, \dots, \overline{v}_n \in X$ be points on our complex torus with $\sum \overline{v}_i = 0$. If $D \in |L|$ is an effective divisor. Then $\sum t^*_{\overline{v}_i} D \sim nD$. Proof: $\bigotimes_{i=1}^n t^*_{\overline{v}_i} L \cong L^{\otimes n}$ via the theorem of the square.

Remark: $t^*_x D$ is the divisor "$D - x$"; in Weil divisor language if $D = \sum a_i Y_i$ then $D - x = \sum a_i(Y_i - x)$. In Cartier divisor language, if $D = (\{U_\alpha\}, \{f_\alpha\})$ then $D - x$ is $(\{U_\alpha - x\}, \{f_\alpha(\cdot + x)\})$. Remark 2: we have $y \in \operatorname{supp}(t^*_x D)$ iff $x + y \in \operatorname{supp}(D)$ iff $x \in \operatorname{supp}(t^*_y D)$.

Proposition: If X is a complex torus and L is positive-definite of type $(d_1, \dots, d_g)$, with $d_1 \geq 2$ (and thus $d_i \geq 2$ for all d_i because $d_1 | d_2 | \cdots | d_g$), then ψ_L is holomorphic (i.e. $|L|$ is base-point-free). Example: If L is principal then L^2 is base-point-free.

Remark: Why the bound 2? Will come from the theorem of the square! Remark 2: Recall that $L \cong L_1^n$ for some L_1 iff $X[n] \subseteq K(L)$. It follows that $L = L_1^{d_1}$ always (so in our proposition L is a square).

Proof: Let $x \in X$. Need to show that there exists a divisor $E \in |L|$ not containing x. Let $L = M^{d_1}$ by the remark, for some M positive-definite. So there exists $D \in |M|$ (note $\dim R(D) = \dim H^0(X, M) \geq 1$ means $|D| \neq \emptyset$). Consider $t^*_x D = D - x$. Choose $x_1, \dots, x_{d_1 - 1} \notin t^*_x D$, and take $x_{d_1} = -x_1 - \cdots - x_{d_1 - 1}$. Moreover can choose $x_1, \dots, x_{d_1-1}$ such that $x_{d_1} \notin t^*_x D$ by continuity (if x_{d_i} is in $t^*_x D$ by chance, can move a small distance to get outside of it in some direction because it's codimension 1; and can move any x_i by a small distance and stay outside $t^*_x D$ because the complement is open).

So: have $x_1, \dots, x_{d_1}$ summing to zero and not in $t^*_x D$. By remark before, $x \notin t^*_{x_i} D$. Thus $x \notin \sum t^*_{x_i} D$. By the theorem of the square, $\sum t^*_{x_i} D \sim d_1 D \in |L|$. So we have a divisor $\sum t^*_{x_i} D$ with x not in its support, which is in $|L|$; this is what we needed.

Definition: $D = \sum a_i Y_i$ is called "reduced" if all nonzero a_i's are equal to 1.

Lemma: Let $L \in \operatorname{Pic}(X)$ for X a complex torus, with $|L| \neq \emptyset$. (Recall this means either L is positive-definite, or L is positive-semidefinite with $L|_{K(L)_0}$ trivial). A "general member" of $|L|$ is reduced.

Proof: Assume $D = nE + F \in |L|$ for $E > 0$ and $F \geq 0$ and $n \geq 2$. Want to resolve this and say that $D \sim E' + F$ for E' reduced and $\operatorname{supp}(E')$ disjoint from $\operatorname{supp}(F)$. (Then iterate this and get rid of the finitely many coefficients ≥ 2 in D). But $nE \sim \sum_{i=1}^n t^*_{x_i} E$ for any $(x_1, \dots, x_n)$ with $\sum x_i = 0$; the set of "bad" tuples (such that $E' = \sum t^*_{x_i} E$ does *not* work) is Zariski closed of codimension ≥ 1. So pick something in the complement, and we're done.

Lemma: If $L \in \operatorname{Pic}(X)$ is positive-definite then there exists an open dense $U \subseteq |L| \cong \mathbb{P}^N$ such that if $D \in U$ then we have $t^*_x D \cong D$ only for $x = 0$. (Proof is kind of long and nothing new, so omitted).

Remark: If D is effective, define $H(D) = \{x \in X : t^*_x D = D\}$. Above lemma says for "almost all" D we have $H(D) = \{0\}$. We also have $H(D)$ is Zariski closed. Moreover if $L = \mathcal{L}(D) = \mathcal{O}_X(L)$, TFAE:

1. L is positive-definite (ample, in algebraic geometry language).
2. $K(L)$ is finite (a finite flat group scheme, in general).
3. $H(D)$ is finite.

The proof of the above statement uses $K(L)$; if there exists $x \neq 0$ then $x \in K(L)$ so $G = \langle x \rangle$ is finite so have a complex torus $X \to X/G$ of degree $|G| \geq 2$, look at divisors on both and compute Euler characteristics...

Decomposition of polarized abelian varieties. Let (X, L) be a polarized abelian variety. Want to decompose it to "irreducible polarized abelian varieties." One use of this: studying the maps $X \to \mathbb{P}^N$. If $d_1 = 2$, sometimes then the map ψ_L is an embedding. If it's not, there's an obvious obstruction, and irreducibles ψ_L factor through the Kummer variety $X/\langle -1 \rangle$.

Now, given $L = L(D)$, have that $|L|$ has fixed component $F_1 + \cdots + F_r$ (with $F_i \neq F_j$ for $i \neq j$ because generic elements are reduced). Then take $M = L(D - F_1 - \cdots - F_r)$, so M has no fixed components, and $|L| = |M| + F_1 + \cdots + F_r$ (with $|M|$ the "moving part"). Take $N_i = \mathcal{L}(F_i)$. Then $h^0(X, M) > 1$ and $h^0(X, N_i) = 1$. Have $M|_{K(M)_0}$ and $N_i|_{K(N_i)_0}$ are trivial, and thus if we define $p_M : X \to X_M = X/K(M)_0$ and $p_{N_u} : X \to X_{N_i} = X/K(N_i)_0$ we can descend M and N_i to $\overline{M}$ on X_M and $\overline{N}_i$ on X_{N_i} respectively. So we get $(X_M, \overline{M})$ and $(X_{N_i}, \overline{N}_i)$.

Theorem: The homomorphism

$$p = p_M \times p_{N_1} \times \cdots \times p_{N_r} : X \to X_M \times X_{N_1} \times \cdots \times X_{N_r}$$

induces an isomorphism of polarized abelian varieties from (X, L) to

$$(X_M \times X_{N_1} \times \cdots \times X_{N_r}, p_M^* \overline{M} \otimes p_{N_1}^* \overline{N}_1 \otimes \cdots \otimes p_{N_r}^* \overline{N}_r).$$

Informally say this is $(X_M, \overline{M}) \times (X_{N_1}, \overline{N}_1) \times \cdots \times (X_{N_r}, \overline{N}_r)$. Then can study (X, L) by way of studying $(X_M, \overline{M})$ (which has no fixed components) and one-dimensional things.

25 Lecture - 05/10/2016

Today: Want to prove the Lefschetz theorem on when our maps $X \to \mathbb{P}^N$ are embeddings. To do this need to prove injectivity, and to do that we need to analyze the tangent space. Will work with the *Gauss map*. The version we need is as follows:

Let (X, L) be a polarized abelian variety of dimension g. Let $D \in |L|$ be a reduced divisor (so all coefficients are 1) - proved one exists (and in fact a general member of $|L|$ is reduced). Let D_{sm} be the smooth locus of $supp(D)$, which is $g-1$-dimensional. For any point $\overline{\omega} \in D_{sm}$ we have the tangent space $T_{D,\overline{\omega}}$ (the tangent space to D_{sm} at $\overline{\omega}$), a $g-1$-dimensional $\mathbb{C}$-vector space. Let $W_{\overline{\omega}}$ be the translation of $T_{D,\overline{\omega}}$ (via $t_{-\overline{\omega}}$ to the origin); this is a $g-1$-dimensional subspace of the g-dimensional tangent space $T_{X,0} = V$ at the origin. The (abstract) Gauss map $G : D_{Sm} \to \mathbb{P}(V^*)$ is given by $\overline{\omega} \mapsto W_{\overline{\omega}}^{\perp}$. Can generalize this abstract setup.

Explicitly, for abelian varieties: Identify $H^0(X, L)$ with a space of theta functions (which requires fixing a factor of automorphy). Then let $\pi : V \to X = V/\Lambda$ the usual covering map. Given a divisor $D \in |L|$ on X, can pull back to a divisor π^*D on X; this is $\text{div}(\vartheta)$ for some theta function ϑ (note $\text{div}(\vartheta)$ is periodic with respect to λ). Let $\{v_1, \dots, v_g\}$ be the coordinate functions with respect to some basis, and let $\overline{\omega} = \omega + \Lambda$. Then

$$T_{D,\overline{\omega}} = \left\{ (v_1, \dots, v_g) : \sum_{i=1}^{g} \frac{\partial \vartheta}{\partial v_i}(\omega)(v_i - \omega_i) = 0 \right\}.$$

The dual to this in $T^*_{X,0}$ is the span of the vector $(\frac{\partial \vartheta}{\partial v_i}(\omega)(\omega))$. So the Gauss map may be identified with $D_{sm} \to \mathbb{P}^{g-1}$ given by

$$\overline{\omega} \mapsto \left[\frac{\partial \vartheta}{\partial v_1}(\omega) : \dots : \frac{\partial \vartheta}{\partial v_g}(\omega) \right].$$

Proposition: Let L be positive definite and $D \in |L|$ be reduced. Then the image of the Gauss map $G : D_{sm} \to \mathbb{P}(V^*) \cong \mathbb{P}^{g-1}$ is not contained in a hyperplane.

Proof: Assume not; i.e. assume there exists some nonzero $t \in V$ such that contained in all tangent spaces. Choose a basis for V such that $t = (1, 0, \dots, 0)$. Fix a canonical theta function, so $a_L = a_{L(H,\chi)}$. What it means for t to be in all tangent spaces means $(d\vartheta/dv_1)(\omega) = 0$ for all $\omega \in V$ such that $\vartheta(\omega) = 0$. (We know this on the smooth locus, then by continuity know on the full set).

Let $f = \frac{1}{\vartheta}\frac{\partial \vartheta}{\partial v_1}$. Since D is reduced, f is holomorphic on V. From $\vartheta(v+\lambda) = a_{L(H,\chi)}\vartheta(v)$ we obtain $f(v+\lambda) = f(v) + \pi H(t, \lambda)$. Since df is Λ-periodic it's a pullback of a holomorphic differential on X, i.e. in IF^1. Thus $df = \sum \alpha dv_i$ with $\alpha_i \in \mathbb{C}$ (i.e. the coefficients are constant functions!).

Then $f = \sum \alpha_i v_i + C$ by integration. By our formula for $f(v+\lambda)$ before, we conclude $f(v) = \pi H(t, v) + C$. Now we're done; we know f is holomorphic, but from our formula it's also anti-holomorphic (because H is hermitian), which forces f to be identically zero and thus $t = 0$. Contradiction!

Projective embeddings and the Lefschetz theorem. Let (X, L) be a polarized abelian variety of type $(d_1, \dots, d_g)$; associated to L we can write down the associated map $\psi_L : X \to \mathbb{P}^N$. Saw that if $d_1 \geq 2$ then L is base-point-free and thus ψ_L is holomorphic.

Theorem (Lefschetz): If $d_i \geq 3$ then ψ_L is an embedding.

Proof: Recall $\psi_L : X \to \mathbb{P}(W^*)$ is given by $P \mapsto \{D \in |W| : P \in D\}$. Need to show: (1) ψ_L is injective, and (2) for all $x \in X$, $d\varphi_{L,x}$ is injective.

(1) Assume $\psi_L(y_1) = \psi_L(y_2)$ for some $y_1, y_2 \in X$. This means for every $D \in |L|$ we have $y_1 \in D$ iff $y_2 \in D$. Now, since $X[d_1] \subseteq K(L) = (\bigoplus \mathbb{Z}/d_i\mathbb{Z})^2$, we know $L = M^{d_1}$ for some M positive-definite. For a general member $D_M \in |M|$ we know D_M is reduced, and $t_x^*(D_M) = D_M$ only for $x = 0$.

Now, pick any point $x_1 \in t_{y_1}^* D_M$. Pick $x_2, \dots, x_{d_1} \in X$ with $\sum x_i = 0$ and $y_2 \notin t_{x_i}^* D_M$ for $i = 2, \dots, d_1$ (can do this because we have at least two elements to move around). Now look at

$$E = \sum_{i=1}^{d_1} t_{x_i}^* D_M;$$

the theorem of the square tells us this is in the linear system $|M^{d_1}| = |L|$. Since $y_1 \in t_{x_1}^* D_M$ we have $y_1 \in E$, and thus by assumption we have $y_2 \in E$. But $y_2 \notin t_{x_i}^* D_M$ for $2 \le i \le d_1$ by assumption, and this forces $y_2 \in t_{x_1}^* D_M$, and thus $x_1 \in t_{y_2}^* D_M$.

So: we've shown that for any $x_1 \in t_{y_1}^* D_M$ we have $x_1 \in t_{y_2}^* D_M$. The same statement holds with y_1 and y_2 switched, so $t_{y_1}^* D_M$ and $t_{y_2}^* D_M$ are equal as sets. Since D_M is a reduced divisor, this means they're actually equal as divisors. Since D_M was chosen so that no nontrivial translates are the same, this forces $y_1 = y_2$.

(2) Assume not: t is tangent to D at x for all $D \in |L|$ with $x \in D$. Pick $D_M \in |M|$ generic (so reduced). Pick $x_1 \in t_x^* D_M$ and choose (as above) $x_2, \ldots, x_{d_1} \in X$ with $\sum x_i = 0$ and $x \notin t_{x_i}^* D_M$. Then $x \in t_{x_1}^* D_M$ so $x \in E' = \sum t_{x_i}^* D_M \in |L|$. By assumption t is tangent to D at x, and thus t is tangent to $t_{x_1}^* D_M$ at x. This is true for any choice of x_1; thus t is tangent to D_M at all points $u \in D_M$ (for any $u \in D_M$, $0 \in t_u^* D_M$, so $x \in t_{u-x}^* D_M$ and take $x_1 = u - x$...) This says the image of the Gauss map is contained in a hyperplane, contradiction! This finishes the proof.

Definition: A line bundle on a scheme is called *very ample* if ψ_L is an embedding. A line bundle is called *ample* if some power L^n is very ample.

Proposition: If X is a complex torus and $L \in \mathrm{Pic}(X)$, TFAE:

1. L is ample.
2. L is positive-definite.
3. $H^0(X, L) \neq 0$ and $K(L)$ is finite.
4. $H^0(X, L) \neq 0$ and $(L^g) > 0$.

Corollary: If $L \in \mathrm{Pic}(X)$ is ample then L^3 is very ample.

Theorem: If X is a complex torus, TFAE:

1. X has a positive-definite line bundle.
2. X admits the structure of a projective algebraic variety.

Proof: Chow's theorem (GAGA) says that if Y is a complete algebraic variety (i.e. $\mathbb{P}^N$) and $\mathcal{Z} \subseteq Y^{an}$ is a closed analytic subset of Y^{an} (i.e. $\mathcal{Z} = \mathrm{img}\,\psi_L$), then there exists an algebraic subvariety Z of Y with $Z^{an} = \mathcal{Z}$.

When is L^2 actually an embedding? Recall we said that we had an embedding

$$(X, L) \cong (X_M, \overline{M}) \times (X_{N_1}, \overline{N}_1) \times \cdots \times (X_{N_r}, \overline{N}_r).$$

It suffices to understand $(X_M, \overline{M})$ and $(X_{N_i}, \overline{N}_i)$. Two cases:

1. Case where $L = M$ has no fixed component: can prove this always gives an embedding.
2. Case of $L = N_i$ having an irreducible principal polarization. Here we define the Kummer variety $K_X = X/\langle(-1)_X\rangle$; this is an algebraic variety with 2^{2g} singular points. Then $\psi_{L^2} : X \to \mathbb{P}^{2g-1}$ factors through $X \to K_X$, and if L is symmetric the induced map is an embedding.