Hausdorff Measure

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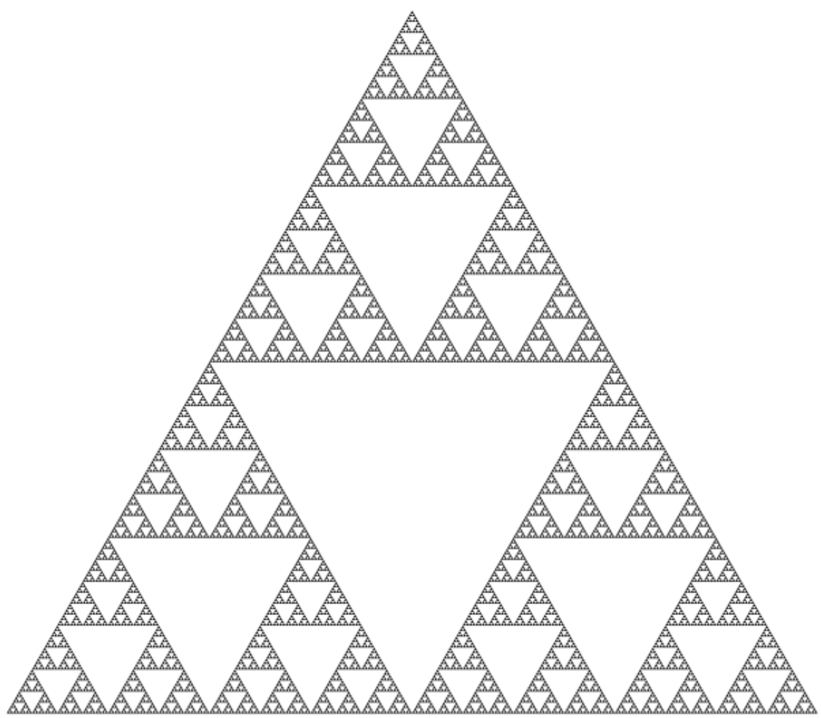


1 Introduction

In this report, we explore the the measurement of arbitrary subsets of the metric space (X, ρ), a topological space X along with its distance function ρ. We introduce Hausdorff Measure as a natural way of assigning sizes to these sets, especially those of smaller "dimension" than X. After an exploration of the salient properties of Hausdorff Measure, we proceed to a definition of Hausdorff dimension, a separate idea of size that allows us a more robust comparison between rough subsets of X.

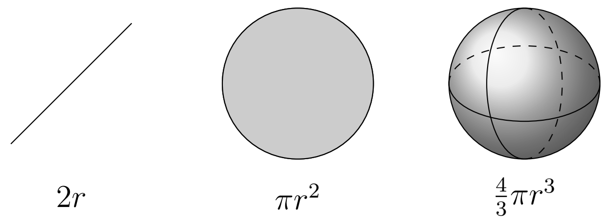
Many of the theorems in this report will be summarized in a proof sketch or shown by visual example. For a more rigorous treatment of the same material, we redirect the reader to Gerald B. Folland's *Real Analysis: Modern techniques and their applications.* Chapter 11 of the 1999 edition served as our primary reference.

2 Hausdorff Measure

2.1 Measuring low-dimensional subsets of X

The need for Hausdorff Measure arises from the need to know the size of lower-dimensional subsets of a metric space. This idea is not as exotic as it may sound. In a high school Geometry course, we learn formulas for objects of various dimension embedded in $\mathbb{R}^3$. In Figure 1 we see the line segment, the circle, and the sphere, each with it's own idea of size. We call these *length, area,* and *volume*, respectively.

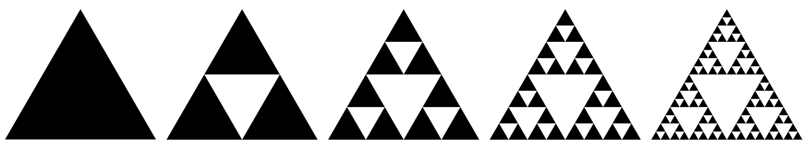
Figure 1: low-dimensional subsets of $\mathbb{R}^3$.



Note that while only volume measures something of the same dimension as the host space, $\mathbb{R}^3$, length, and area can still be of interest to us, especially in applications. Sets of higher dimension and sets whick are less smooth are not as easy to measure.

As an example, we will consider the Sierpinski Carpet, a fractal subset of R^2. The Carpet can be constructed iteratively starting with an equilateral triangle, which we shall call T_0. To pass from one iteration to the next, we remove the middle quarter from each triangle in the set. Sierpinski Carpet is the figure T_∞ where the iteration has been applied a countably infinite number of times. Figure 2 shows T_0 through T_4.

Figure 2: The first five iterations of T_k, from left to right.



Let A_i and P_i be defined as the area and perimeter of T_i, respectively, and let each of A_0 and P_0 be finite. It is clear that for each $k \in \mathbb{N}$.

$$A_k = A_0 \left(\frac{3}{4}\right)^k$$
$$P_k = P_0 \left(\frac{3}{2}\right)^k$$

As $k \to \infty$ we see that $A_k \to 0$ and $P_k \to \infty$. This means that the Sierpinski Carpet, T_k has zero area but infinite perimeter. This lack of precision in measurement is frustrating. We need a way to assign a finite number to the size of a Sierpinksi Carpet.

2.2 Constructing Hausdorff Measure

We will proceed now to define the Hausdorff Outer Measure, H^d, of dimension d on a general metric space (X, ρ). Hereafter, we will show that one can restrict Hausdorff Outer Measure to the Borel sets of X to obtain a measure.

Before defining Hausdorff Outer Measure, we will remind the reader of another definition.

Definition. Let $S \subset X$. Then let the *diameter* of S be defined by

$$\operatorname{diam}(S) = \max\{\rho(x, y) : x, y \in S\}.$$

That is to say, the diameter of a set is the distance between the farthest two points in the set.

Definition. Let S be any subset of X, and $\delta > 0$ a real number. We define the Hausdorff Outer Measure of dimension d bounded by δ (written H^d_δ) by:

$$H^d_\delta(S) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \operatorname{diam} U_i < \delta \right\}.$$

where the infimum is taken over all countable covers of S by sets $U_i \subseteq X$ satisfying $\operatorname{diam}(U_i) < \delta$.

If we allow δ to approach zero, the infimum is taken over a decreasing collection of sets, and is therefore increasing. We can conclude that

$$\lim_{\delta \to 0} H^d_\delta(S) = H^d(S)$$

exists, but may be infinite. We call this limit Hausdorff Outer Measure of dimension d. We will explore Hausdorff Outer Measure more rigorously soon, but first, we will provide a few comments.

1. The superscript d corresponds roughly to the dimensionality of the set being measured. That is, H^d is used to measure a d-dimensional subset of the host space. The amount of S contained in a region of diameter r is proportional to r^p. This is measuring the length of a curve with line segments, the area of a shape with circles, or the volume of a manifold with spheres, as demonstrated in Figure 3.

2. Intuitively, the reason we decrease r toward zero to account for the "roughness" of S. Covers made up of sets with large diameters may fail to capture the complex shape of a set. For example, consider $\mathcal{C}$, the Topologist's Sine Wave. $\mathcal{C}$ is a subset of $\mathbb{R}^2$ defined as the union of the vertical line segment from $(0, -1)$ to $(0, 1)$ and points of the form $(x, \pi/\sin(x))$ for $x \in (0, 1]$. Figure 4 shows $\mathcal{C}$.

Figure 3: Covers of 1-, 2-, and 3-dimensional subsets of $\mathbb{R}^3$

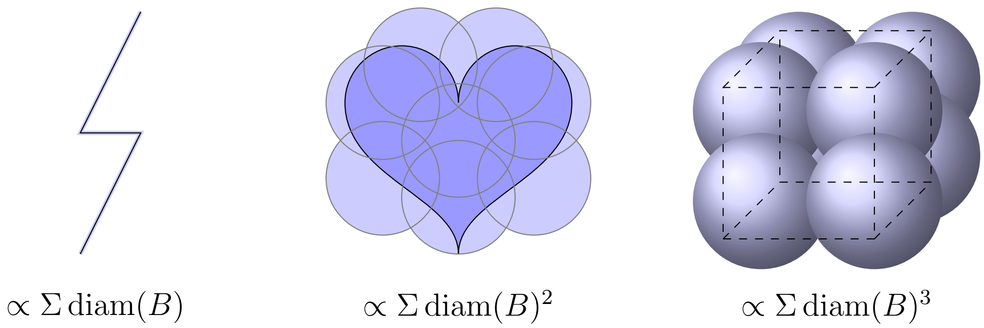
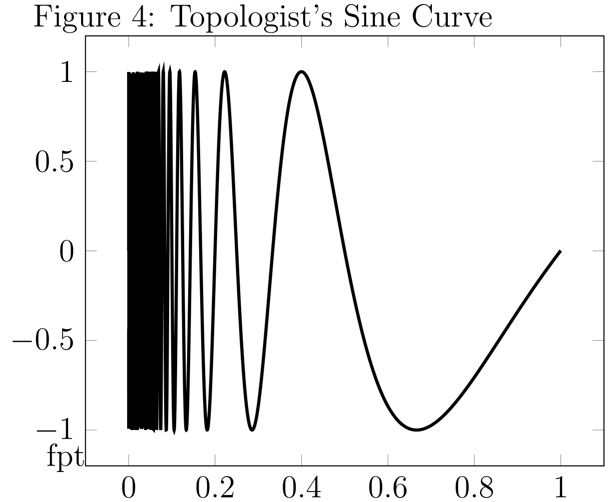


Figure 4: Topologist's Sine Curve



If we do not restrict the Hausdorff Measure by some finite, δ we can simply cover $\mathcal{C}$ by itself and show that

$$H^1_\infty(\mathcal{C}) \leq \text{diam}(\mathcal{C}) \leq \sqrt{(5)}.$$

But as $\delta \to 0$, sets in the cover grow small enough so that they tend toward a trace of the Curve's infinite length, giving us the anticipated result that

$$H^1(\mathcal{C}) = \infty$$

3. H^d is invariant under isometries of X. The idea here is that because isometries preserve length, we can "translate" the sets in a cover of

> X to form a new cover of a "translated" copy of X, and the summed diameters of the new cover will be the same as the summed diameters of the original cover. We enclose the word "translate" in scare-quotes to warn that isometries are not limited to translations. Nevertheless, the picture remains the same.

Recall that we have only declared H^d to be an outer measure, we will now move toward its restriction to a measure. This is a simple consequence of a property of H^d we define now.

Definition. μ^*, an outer measure, is a *metric outer measure* if and only if

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) \text{ when } \rho(A, B) > 0.$$

Where $\rho(A, B)$ is the minimum value of $\rho(x, y)$ for $x \in A, y \in B$.

Theorem. H^d is a metric outer measure.

Proof sketch. The key here is to remember that H^d is defined as a limit H^d_δ for decreasing δ. Then let A and B be two sets in X such that $\rho(A, B) = \epsilon > 0$. For any $\delta < \epsilon/3$ we see that the covers of A and B which define $H^d_\delta(A)$ and $H^d_\delta(B)$ are disjoint, and therefore do not "double count." ▲

Knowing that H^d is a metric outer measure, we apply the following theorem (given without proof).

Theorem. If μ^* is a metric outer measure on X then the Borel subsets of X are μ^*-measurable.

A proof of this theorem is available in Folland's book. The relevant result is that H^d is indeed a measure when restricted to the Borel sets of X. Now we have fully defined Hausdorff Measure on the Borel sets of an arbitrary metric space.

2.3 Hausdorff Measure and Lebesgue Measure

In this section, we restrict our attention to a special case of Hausdorff Measure, specifically H^n on the metric space defined by $\mathbb{R}^n$ with Euclidean metric. We will discover that H^n differs from Lebesgue measure only by a constant factor.

Let Q_n be any finite n-dimensional cube embedded in R_n. We claim that

$$0 < H^n(Q_n) < +\infty.$$

This fact is the direct consequence of a recurrence relation we will explore in section 3.3 of this report. For now, the fact that $H_n(Q_n)$ has a finite, nonzero value gives us two important lemmas.

(i) H^n is not the zero measure.

(ii) H^n is finite on compact sets.

(i) is trivial. (ii) can be shown by constructing a cube around any compact set in $\mathbb{R}^n$ and invoking monotonicity. We need a lemma to move forward.

Lemma. Every measure on the Borel subsets of R^n which is finite on compact sets is regular, and therefore Radon.

Theorem. Let λ^n denote n-dimensional Lebesgue Measure. There exists a $\gamma_n > 0$ such that

$$\lambda^n = \gamma_n H^n$$

Proof. By the last lemma and by (i), we know that H^n is a non-zero Radon measure on $\mathbb{R}^n$. Additionally, recall from our "remarks" after the definition of Hausdorff Measure that it is invariant under the isometries of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a locally compact Hausdorff space, this is enough to conclude that H^n is also a Haar measure on $\mathbb{R}^n$. We know that Lesbesgue Measure is a Haar measure and that all Haar Measures on a space are equivalent up to a scaling factor. Thus we are done. ▲

You may wonder about the value of γ_n. It can be shown that it is equal to the n-volume of an n-dimensional ball of radius 1 in the usual norm. That is

$$\gamma_n = \frac{\pi^{2/n}}{2^n \Gamma(n/2 + 1)}$$

where Γ is Euler's Gamma Function.

3 Hausdorff Dimension

3.1 Defining the Hausdorff dimension of a set

Theorem. If $H^p(A) < \infty$, then $H^q(A) = 0$ for all $q > p$. Also, if $H^p(A) > 0$, then $H^q(A) = \infty$ for all $q < p$.

Proof. Let $H^p(A) < \infty$ and $q < p$. Then,

$$\begin{aligned}
H^q(A) &= \lim_{\delta\to 0} H^q_\delta(A) \\
&= \lim_{\delta\to 0} \inf \left\{ \sum_{\alpha\in I} (\text{diam } U_\alpha)^q : \bigcup_{\alpha\in I} U_\alpha \supseteq A, \text{diam } U_\alpha < \delta \right\} \\
&\leq \lim_{\delta\to 0} \inf \left\{ \sum_{\alpha\in I} (\text{diam } U_\alpha)^p \delta^{q-p} : \bigcup_{\alpha\in I} U_\alpha \supseteq A, \text{diam } U_\alpha < \delta \right\} \\
&= \lim_{\delta\to 0} \delta^{q-p} \inf \left\{ \sum_{\alpha\in I} (\text{diam } U_\alpha)^p : \bigcup_{\alpha\in I} U_\alpha \supseteq A, \text{diam } U_\alpha < \delta \right\} && \text{factor out constant} \\
&= 0 && \text{let } \delta \to 0
\end{aligned}$$

Now let $H^p(A) > 0$ and $q > p$. Then,

$$\begin{aligned}
H^q(A) &= \lim_{\delta\to 0} H^q_\delta(A) \\
&= \lim_{\delta\to 0} \inf \left\{ \sum_{\alpha\in I} (\text{diam } U_\alpha)^q : \bigcup_{\alpha\in I} U_\alpha \supseteq A, \text{diam } U_\alpha < \delta \right\} \\
&\geq \lim_{\delta\to 0} \inf \left\{ \sum_{\alpha\in I} (\text{diam } U_\alpha)^p \delta^{q-p} : \bigcup_{\alpha\in I} U_\alpha \supseteq A, \text{diam } U_\alpha < \delta \right\} && \text{since } q-p<0 \\
&= \lim_{\delta\to 0} \frac{1}{\delta^{p-q}} \inf \left\{ \sum_{\alpha\in I} (\text{diam } U_\alpha)^p : \bigcup_{\alpha\in I} U_\alpha \supseteq A, \text{diam } U_\alpha < \delta \right\} && \text{factor out constant} \\
&= +\infty && \text{let } \delta \to 0
\end{aligned}$$

▲

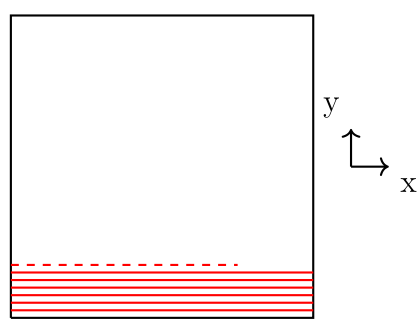
Definition. We then see that if $H^p(A)$ is a finite nonzero value, then for all $k \in [0, +\infty)$ such that $k \neq p$, $H^k(A)$ is either infinite or zero. Then, we define the Hausdorff Dimension of set A, $\dim_H(A) = p$ if and only if $H^p(A)$ is a finite, nonzero number.

Remark. If the set A has Hausdorff dimension $\dim_H(A) = k$, then

$$H^p(A) = \begin{cases} 0 & \text{if } k < p \\ \text{finite, nonzero number} & \text{if } k = p \\ +\infty & \text{if } k > p \end{cases}$$

To illustrate, let's consider a situation where $k > p$. For concreteness, we'll measure the length ($p = 1$) of a square ($k = 2$). First consider using a ruler to measure an edge of the square in the x direction. Let's say the side length is some value, $L > 0$ Then move that ruler an infintesimal distance in the y direction and record the length, which must still be $L > 0$. By gradually moving the ruler continuously over this extra y dimension, the length of the square becomes the sum of an uncountable number of L's. This brings the length to $L + L + L + L + L + L ... = +\infty$. More formally, this is shown by integrating this uncountable set of L's with respect to the counting measure. We retrieve from this that the square has an infinite length.

$H^1(A) = +\infty$



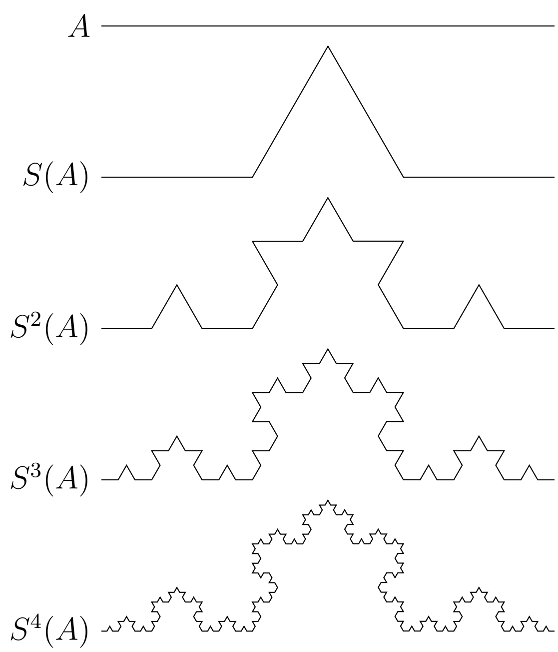
Now let's set $k < p$. For concreteness, we'll measure the area ($p = 2$) of a line ($k = 1$). Area can be regarded as the product of length and width. If we lay our line of length L in the x direction, then the line in the y direction can only occupy a single point, giving the line a width of zero in the y direction.

$H^1(A) = 0$

It follows from the remark that there exists a unique Hausdorff Dimension for any subset of (X, ρ).

3.2 Non-integer Hausdorff Dimension

To visualize the necessity of a noninteger Hausdorff dimension, let us consider the Koch curve, K. Which is generated as the limit of the following transformations:



Since this curve is composed of lines, it necessarily has zero area, thus $H^2(K) = 0$, so $\dim_H(K) < 2$. Now we'll consider the length of K. Let's suppose that the initial line from above has a length of one. The result of one transformation is four lines each of length strictly less than one. For the sake of argument, let's say that first transformation yields four lines, each of length $1/3$. This gives us a total length of $4/3$. Repeating this procedure gives us 4 lines of length $(1/3)/3 = 1/9$ for each of the 4 lines, giving us a total length of $16/9 = (4/3)^2$. K is achieved by applying this transformation an arbitrarily large number of times, therefore we can regard the total length of K as

$$H^1(K) = \lim_{n\to\infty} (4/3)^n = +\infty$$

Therefore $dim_H(K) > 1$. It follows that

$$1 < dim_H(K) < 2$$

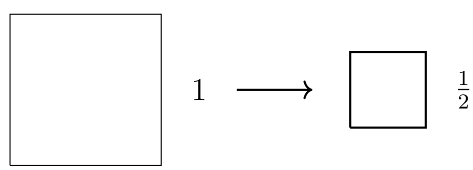
Therefore, K's Hausdorff dimension is necessarily some noninteger value. In order to analyze this result, we will first develop some terminology.

3.3 Calculating $\dim_H(\cdot)$ for self-similar objects

Definition. A set, A is called self-similar if there exists some $A' \subset A$ such that A' is similar to A. In other words, there exists a subset of the self-similar set A such that dilation of that subset gives A exactly.

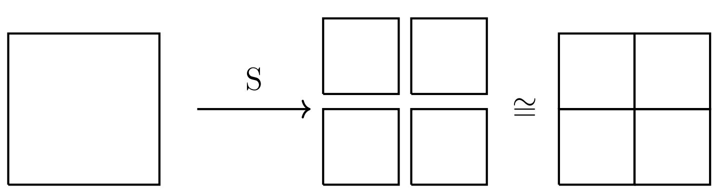
Definition. A similitude is a map $S : \mathbb{R}^n \to \mathbb{R}^n$ of the form $S(x) = r\mathcal{O}(x)+b$ such that $r > 0$, $b \in \mathbb{R}^n$, and $\mathcal{O}$ is an element of the n^{th} orthogonal group, $\mathcal{O} \in O(n)$. r is often called the scaling factor.

Example. The following is an example of a similitude on a square which has a scaling factor of $r = \frac{\text{new}}{\text{old}} = \frac{1/2}{1} = 1/2$



Definition. A similitude family is a collection of similatudes (S_α), $\alpha \in I$. We will be interested in similitude families that leave the Hausdorff p measure invarient.

Example. To demonstrate a similitude family that leaves the Hausdorff p measure invariant, we will consider the following partitioning of a square, A. By associating the Hausdorff 2 measure with the Lebesgue 2 measure, we



can show that this similitude family conserves area.

$$H^2(A) \approx \int_A d^2\lambda = \int_0^1 \int_0^1 dx_1 dx_2 = 1$$

$$\sum_{i=1}^4 H^2(S_i(A)) \approx \sum_{i=1}^4 \int_{S_i(A)} d^2\lambda = \sum_{i=1}^4 \int_0^{1/2} \int_0^{1/2} dx_1 dx_2 = \sum_{i=1}^4 \frac{1}{2*2} = 1$$

Now define $A = \cup_{\alpha \in I} S_\alpha(A) \equiv S(A)$. Since this similitude family $(S_\alpha) \cong S$, leaves the area of a square invariant we have for any $i \in \mathbb{N}$,

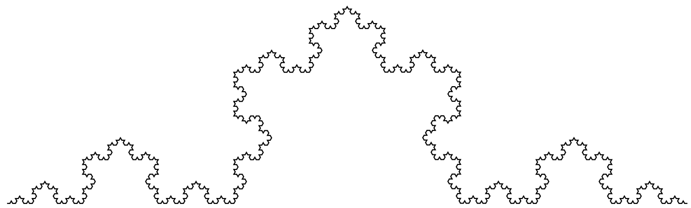
$$A = S(A) = S^2(A) = \cdots = S^i(A) = S^{i+1}(A) = \cdots$$

Now let's calculate the Hausdorff dimension of the square, A. Consider taking the general Hausdorff p measure of A after having been transformed by S i times.

$$\begin{aligned} H^p(S^i(A)) &= H^p(S^{i+1}(A)) \\ 4^i(1/2)^{ip} &= 4^{i+1}(1/2)^{(i+1)p} \\ 1 &= 4(1/2)^p \\ 1/4 &= (1/2)^p \\ \log(1/4) &= p\log(1/2) \\ p &= \log(1/2)/\log(1/4) \\ p &= 2\log(1/2)/\log(1/2) \\ p &= 2 \end{aligned}$$

Therefore, the square A has Hausdorff dimension 2, which agrees with the square classically having dimension 2.

Example. We now turn to calculating the Hausdorff dimension of the Koch Curve, K. In order to construct K, we first consider a line A. We then repeatedly apply the transformation S of the similitude family described by replacing every line of length x with four lines each of length $x/3$ (this is depicted in Section 3). We then describe the object $S^i(A)$, which is that transformation applied to A i times. For visualization purposes, we display the case of $i = 5$.

$S^5(A)$ 

By defining K as $K = \lim_{i\to\infty} S^i(A) \equiv S^\infty(A)$, we can readily observe that

$$S(K) = S\Big(\lim_{i\to\infty} S^i(A)\Big) = \lim_{i\to\infty} S^{i+1}(A) = \lim_{j\to\infty} S^j(A) = K$$

So since S leaves K invariant, it must necessarily leave its p-dimensional Hausdorff measure invariant.

$$H^p(S(K)) = H^p(K)$$

This statement is true for any p, but we aren't interested in just any p, we're interested in the unique p such that $p = \dim_H(K)$. Accordingly, suppose $p = \dim_H(K)$. In this case, $H^p(K) = H^p(\lim_{i\to\infty} S^i(A))$ necessarily converges on a finite nonzero value.

Since only countable sums of nonnegative values are finite, the smallest cover of K can be then expressed as a countable sum under the H^p measure.

And as discussed previously, $H^1(S^{i+1}(A)) > H^1(S^i(A))$, therefore $H^p(S^{i+1}(A)) > H^p(S^i(A))$ for all i, we have that $H^p(S^i(A))$ forms a monotonically increasing sequence with finite limit, $H^p(K)$. We may then apply the monotone convergence theorem to bring the limit out of the countable sum, retrieving,

$$H^p(\lim_{i\to\infty} S^i(A)) = \lim_{i\to\infty} H^p(S^i(A))$$

The calculation is then straightforward

$$\begin{aligned}\lim_{i\to\infty} H^p(S^i(A)) &= H^p(\lim_{i\to\infty} S^i(A)) \\ &= H^p(K) \\ &= H^p(S(K)) \\ &= H^p(\lim_{i\to\infty} S^{i+1}(A)) \\ &= \lim_{i\to\infty} H^p(S^{i+1}(A))\end{aligned}$$

And since both sides converge to a finite value,

$$\lim_{i\to\infty} \left[H^p(S^{i+1}(A)) - H^p(S^i(A))\right] = 0$$

The diameter of a line is its length, and the length of a line in $S^i(A)$ will be $(1/3)^i$. Also, the number of lines in $S^i(A)$ is 4^i. Since the minimum cover of $S^i(A)$ is itself, which is the collection of those 4^i lines, we have that

$H^p(S^i(A)) = 4^i(1/3)^{ip}$. We may then simplify

$$\lim_{i\to\infty} 4^{i+1}(1/3)^{(i+1)p} - 4^i(1/3)^{ip} = 0$$

$$\lim_{i\to\infty} 4^i(1/3)^{ip}\big(4(1/3)^p - 1\big) = 0$$

$$\big(4(1/3)^p - 1\big) \lim_{i\to\infty} 4^i(1/3)^{ip} = 0$$

$$\big(4(1/3)^p - 1\big) H^p(K) = 0$$

By assumption, $0 < H^p(K) < +\infty$. Therefore,

$$4(1/3)^p - 1 = 0$$

Solving for p, we retrieve the Hausdorff dimension of the Koch curve,

$$p = \frac{\log 4}{\log(1/3)} \approx 1.26...$$

Which is in fact a noninteger value. It is easily seen that this is consistent with the previous example

$$1 < 1.26... = \dim_H(K) < 2$$

We would now like to generalize the previous example to show that we can analytically calculate the Hausdorff dimension for any compact object, A, with self-similarity described by a similitude family that creates m copies of itself and has a scaling factor of r. We will also suppose that we are working within a metric space with a continuous, locally finite metric.

Theorem. If m and r are as defined above, then

$$1 = mr^{\dim_H(A)}$$

Proof. With minor adjustments, the proof here is essentially the same as for the Koch curve example above. Suppose that we want to calculate the Hausdorff dimension of a compact object, K, with self similarity described by some similatude family of transformations, S, where S produces m disjoint copies of itself and scales itself by a scaling factor, r. Suppose for nontriviality that $m > 1$. If $r \geq 1$, then K would tile an unbounded region of its metric space. Since K is compact, $r < 1$ necessarily. Letting K be generated as the limit of applications of S on some set, for example, A, we have as before

$$K = \lim_{i\to\infty} S^i(A) \equiv S^\infty(A)$$

and

$$S(K) = S\Big(\lim_{i\to\infty} S^i(A)\Big) = \lim_{i\to\infty} S^{i+1}(A) = \lim_{j\to\infty} S^j(A) = K$$

Now suppose that we have chosen a p such that $H^p(K) \neq +\infty$. we may then write

$$0 = H^p(K) - H^p(K)$$

Supposing we're working in a metric space with a continuous metric, we can as before take the limit out of the Hausdorff p measure, and we have

$$\begin{aligned}
0 &= H^p(K) - H^p(K) \\
&= H^p(S(K)) - H^p(K) \\
&= H^p(\lim_{i\to\infty} S^{i+1}(A)) - H^p(\lim_{i\to\infty} S^i(A)) \\
&= \lim_{i\to\infty} H^p(S^{i+1}(A)) - \lim_{i\to\infty} H^p(S^i(A)) \\
&= \lim_{i\to\infty} m^{i+1}r^{p(i+1)} - \lim_{i\to\infty} m^i r^{pi} \\
&= \lim_{i\to\infty} \Big[m^{i+1}r^{p(i+1)} - m^i r^{pi}\Big]
\end{aligned}$$

Since we assume that the limit of the series exists by the assumption that $H^p(K) < +\infty$. Then by factoring, we recover,

$$0 = (mr^p - 1) \lim_{i \to \infty} m^i r^{ip}$$

Then one of the two factors must be zero. Supposing that p is the Hausdorff dimension of K, we have that

$$0 < H^p(K) = \lim_{i \to \infty} m^i r^{ip}$$

Then,

$$mr^p - 1 = 0$$

Which is the desired result, solving for p, we have

$$\dim_H(K) = p = \frac{\log m}{\log 1/r} = \log_{1/r} m$$