Haar Measure on LCH Groups

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Abstract

This expository article is an introduction to Haar measure on locally compact Hausdorff (LCH) groups. Haar measure is a translation invariance measure and is widely used in pure mathematics, physics, and even statistics. This article begins with an introduction to topological groups, discusses Haar measure's elementary properties, gives ideas about its construction and uniqueness, and ends with the discussion of the relationship between left and right Haar measures. The major reference is from [1].

1 Topological Groups

The main object we focus on is topological group. The notion of topological group serves as a linkage between topology and algebra.

Definition 1.1 (Topological groups)**.** *A topological group G is a group endowed with a topology such that the group operations $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.*

Normed vector space is an additive topological group. The continuity of addition essentially follows from the triangle inequality of the norm. Also the collection of invertible $n \times n$ real matrices is a multiplicative non-Abelian topological group, with the Euclidean topology on $\mathbb{R}^{n^2}$. Meanwhile, it is clear that all groups with discrete topology are topological groups.

Before proceeding to the properties, we first introduce our notations. Let G be a topological group. For $x, y \in G, A, B \subset G$, we denote

- $e =$ identity element in G,
- $xA = \{xy : y \in A\}$,
- $Ax = \{yx : y \in A\}$,
- $AB = \{xy : x \in A, y \in B\}$,
- $A^{-1} = \{x^{-1} : x \in A\}$.

And we say A is symmetric if $A^{-1} = A$.

We now provide several properties of topological groups.

Proposition 1.1. *Let G be a topological group. Then:*

1. *Topology of G is translation invariant. i.e. U open $\implies$ Ux, xU open $\forall x \in G$.*

2. *For every neighborhood U of e, there is a symmetric neighborhood V of e such that $V \subset U$.*

3. *For every neighborhood U of e, there is a neighborhood V of e such that $VV \subset U$.*

4. *K_1, K_2 compact in $G \implies K_1K_2$ compact in G.*

The above four propositions are all direct results of the continuity of group operations. For part 1, fix $x \in G$, we know $y \mapsto xy$ is a homeomorphism. Hence U open $\iff$ xU open. And similar arguments work for Ux. For part 2, WLOG we assume U is open (otherwise just work on the interior of U). Then U^{-1} is open because $x \mapsto x^{-1}$ is a homeomorphism. Now we can check $V = U \cap U^{-1}$ is a symmetric open neighborhood of e and $V \subset U$. For part 3, by continuity at the identity, U is an open neighborhood of e implies $\exists$ open neighborhood A, B of e such that $AB \subset U$. Then we can check $V = A \cap B$ is exactly what we want. For part 4, note that $K_1 \times K_2$ is compact with respect to the product topology. Then by continuity of $(x, y) \mapsto xy$, we immediately have K_1K_2 is compact.

Meanwhile, we shall note that part 2 and part 3 together implies the following:

Proposition 1.2. *For every neighborhood U of e, there is a symmetric neighborhood V of e such that $VV \subset U$.*

This proposition will be used frequently throughout this article. Now we give several other properties of topological groups.

Proposition 1.3. *Let G be a topological group. Then:*

1. *H is a subgroup of $G \implies \overline{H}$ (the closure of H) is a subgroup of G.*

2. *Every open subgroup of G is also closed.*

For part 1, just recall the closure of H is the set of points which is the limit of a net in H. Then for $x, y \in \overline{H}$, $\exists \{x_\alpha\}_{\alpha \in A}, \{y_\beta\}_{\beta \in B}$ such that $x_\alpha \to x, y_\beta \to y$. Then by the fact that $x_\alpha y_\beta \to xy$ and $x_\alpha^{-1} \to x^{-1}$, we know $xy, x^{-1} \in \overline{H}$. Then noting that $e \in \overline{H}$, we conclude $\overline{H}$ is a subgroup. For part 2, let H be the open subgroup of G. We may write $G \setminus H = \bigcup_{x \notin H} xH$. Hence $G \setminus H$ is open because each xH is open. So we conclude that H is closed.

From now on we concentrate on the case where the topology on G is locally compact Hausdorff (LCH). And by the following proposition we can see the Hausdorff-ness turns out to be not much of a restriction.

Proposition 1.4. *Let G be a topological group. Then:*

1. *If G is T_1 then G is T_2 (Hausdorff).*
2. *If G is not T_1, let H be the closure of $\{e\}$. Then H is a normal subgroup. And if G/H is given the quotient topology, then G/H is a Hausdorff topological group.*

We will not pursue the details of proof here. But it is worthwhile to note that if G is not T_1, we can alternatively work on G/H, which ensures the Hausdorff-ness. i.e. we just "quotient out" the non-separable part.

2 Haar Measure and Its Properties

Before the formal definition of Haar measure, we first introduce several terms. The following definitions have both left and right versions. Without further note we concentrate on the left case.

Definition 2.1 (Left/right translates)**.** *Let $f : G \to \mathbb{R}$. We define $L_y f : G \to \mathbb{R}$ to be the left translation of f through $y \in G$ by $L_y f(x) = f(y^{-1}x)\ \forall x \in G$. Similarly we define the right translate of f through y by $R_y f(x) = f(xy)$.*

Note that we are using y^{-1} on the left and y on the right because we want $L_{yz} = L_y \circ L_z$ and $R_{yz} = R_y \circ R_z$.

Definition 2.2 (Left/right uniformly continuity)**.** *$f : G \to \mathbb{R}$ is called left uniformly continuous if $\forall \epsilon > 0, \exists$ a neighborhood V of e such that $\|L_y f - f\|_u < \epsilon$ for all $y \in V$, where $\|\cdot\|_u$ denotes the uniform norm. We can define right uniform continuity in the same fashion.*

If $G = \mathbb{R}$, then either left or right uniform continuity is equivalent to the classical "uniform continuity", because $\mathbb{R}$ is an additive Abelian topological group.

Recall $C_c(G)$, the space of continuous functions with compact support. This functional space has a very good property, as indicated in the following proposition.

Proposition 2.1. *If $f \in C_c(G)$, then f is both left and right uniformly continuous.*

Proof. We shall consider right uniform continuity (the left case is essentially the same). Let $K = \text{supp}(f), \epsilon > 0$. For each $x \in K$, by continuity of f, $\exists$ a neighborhood U_x of e such that $|f(xy) - f(x)| < \frac{\epsilon}{2}$ for $y \in U_x$. By proposition 1.2, $\exists$ a symmetric neighborhood V_x of e such that $V_x V_x \subset U_x$. Since $\{xV_x\}_{x \in K}$ covers K, by compactness, $\exists x_1, \ldots, x_n$ such that $K \subset \bigcup_{j=1}^n x_j V_{x_j}$. Let $V = \bigcap_{j=1}^n V_{x_j}$. We claim that $|f(xy) - f(x)| < \epsilon$ if $y \in V$, for all $x \in G$, which is exactly the statement of right uniform continuity.

- If $x \in K$, then $x \in x_j V_{x_j}$ for some j. Hence $x_j^{-1} x \in V_{x_j}$ and thus we have $xy = x_j(x_j^{-1}x)y \in x_j V_{x_j} V_{x_j} \subset x_j U_{x_j}$. Also $x \in x_j V_{x_j} \subset x_j U_{x_j}$. Then we have
$$|f(xy) - f(x)| \le |f(xy) - f(x_j)| + |f(x) - f(x_j)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- If $x \notin K$, $f(x) = 0$. Then either $f(xy) = 0$ (if $xy \notin K$) or $xy \in x_j V_{x_j}$ for some j (if $xy \in K$). For the first case, it is clear that $|f(xy) - f(x)| = 0$. And for the second case, we have $xy \in x_j U_{x_j}$ and $x \in x_j V_{x_j} y^{-1} \subset x_j V_{x_j} V_{x_j}^{-1} \subset x_j U_{x_j}$. Then again we have

$$|f(xy) - f(x)| \leq |f(xy) - f(x_j)| + |f(x) - f(x_j)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

From now on we fix G to be a LCH group and $(G, \mathcal{B}_G)$ to be our measurable space, where $\mathcal{B}_G$ is the Borel σ-algebra. We introduce the notion of invariant measure and invariant linear functional.

Definition 2.3 (Left/right invariant measure)**.** *A Borel measure μ on G is called left invariant if $\mu(xE) = \mu(E)$ $\forall x \in G, E \in \mathcal{B}_G$. Respectively we can define the right invariant Borel measure.*

Definition 2.4 (Left/right invariant linear functional)**.** *A linear functional I on $C_c(G)$ is called left invariant if $I(L_y f) = I(f)$ $\forall f \in C_c(G), y \in G$. Respectively we can define the right invariant linear functional.*

Given our construction so far, it is easy to state the definition of Haar measure.

Definition 2.5 (Left/right Haar measure)**.** *A left Haar measure on G is a nonzero left invariant Radon measure on G. Respectively we can define a right Haar measure.*

We shall recall the definition of the Radon measure: it is finite on compact sets, outer regular for Borel sets and inner regular for open sets. Immediately we can see that the Lebesgue measure on $\mathbb{R}^n$ is a left and right Haar measure. Also the counting measure on any group with discrete topology is a left and right Haar measure, because the cardinality of subset of a group is translation invariant.

Now we let $C_c^+ = \{f \in C_c(G) : f \geq 0, \|f\|_u > 0\}$. And we state several properties of the Haar measure.

Proposition 2.2.

1. *A Radon measure μ on G is a left Haar measure iff the measure $\widetilde{\mu}$ defined by $\widetilde{\mu}(E) = \mu(E^{-1})$ $\forall E \in \mathcal{B}_G$ is a right Haar measure.*

2. *A nonzero Radon measure μ on G is a left Haar measure iff $\int f d\mu = \int L_y f d\mu$ $\forall f \in C_c^+, y \in G$.*

3. *If μ is a left Haar measure on G, then $\mu(U) > 0$ for every nonempty open $U \in \mathcal{B}_G$ and $\int f d\mu > 0$ $\forall f \in C_c^+$.*

4. *If μ is a left Haar measure on G, then $\mu(G) < \infty$ iff G is compact.*

Part 1 tells us an elementary relationship between left and right Haar measures. As a result, if we construct a left Haar measure, we automatically get a right Haar measure for free. Part 2 tells us an alternative characterization of the Haar measure: when the measure is already Radon, in order to specify a Haar measure, we just need to specify its invariant behavior on integration for functions in C_c^+. Therefore, if we want to construct a Haar measure, we just need to construct a invariant positive linear functional on $C_c(G)$ and induce the Riesz representation theorem.

Proof. For part 1, by symmetry it suffices to show one direction. Assume μ is a left Haar measure. We show $\widetilde{\mu}$ is a right Haar measure. We first check the right invariance. For $A \in \mathcal{B}_G, x \in G$, we have $\widetilde{\mu}(Ax) = \mu(x^{-1}A^{-1}) = \mu(A^{-1}) = \widetilde{\mu}(A)$. Also, since $x \mapsto x^{-1}$ is a homeomorphism, K compact iff K^{-1} compact. Thus $\widetilde{\mu}(K) = \mu(K^{-1}) < \infty$. Then we check outer regularity. Note that $\mathcal{C} := \{S \subset G : S^{-1} \in \mathcal{B}_G\}$ is a σ-algebra. Meanwhile, we know U open iff U^{-1} open. Hence $\mathcal{C} \supset \{U : U\text{open}\}$, which implies $\mathcal{C} \supset \mathcal{B}_G$. Thus $A \in \mathcal{B}_G$ iff $A^{-1} \in \mathcal{B}_G$. Now assume $A \in \mathcal{B}_G$, by outer regularity, we have $\widetilde{\mu}(A) = \mu(A^{-1}) = \inf\{\mu(U) : A^{-1} \subset U, U\text{open}\}$. With some computations we have $\widetilde{\mu}(A) = \inf\{\mu(U^{-1}) : A \subset U, U\text{open}\}$, which means $\widetilde{\mu}$ is outer regular for Borel sets. The arguments for inner regularity is essentially the same.

For part 2, we first show the "only if" part. Note that for a simple function φ, $\int L_y\varphi d\mu = \int \varphi d\mu$. For $f \in C_c^+$, f can be approximated by a sequence of simple functions. Then the result easily follows by using monotone convergence theorem. For the "if" part, we need to show the left invariance of μ. If we can show $\mu(U) = \mu(xU)$ for open U, then using outer regularity, for $A \in \mathcal{B}_G$ we have $\mu(xA) = \inf\{\mu(U) : U \supset xA, U\text{open}\} = \inf\{\mu(xU) : xU \supset yA, yU\text{open}\} = \inf\{\mu(U) : U \supset A, U\text{open}\} = \mu(A)$. Hence it suffices to show $\int \chi_U d\mu = \int L_y\chi_U d\mu$. But unfortunately $\chi_U \notin C_c^+$ all the time. However we do have a way to approximate characteristic functions using continuous functions: Urysohn's lemma. The lemma states: if K is compact in G and U is an open neighborhood of K, then $\exists f \in C_c(G)$ such that $\chi_K \leq f \leq \chi_U$. By the existence of such f, along with inner regularity, it is easy to see that $\mu(U) = \sup\{\mu(K) : K \subset U, K\text{compact}\} = \sup\{\int f d\mu : f \in C_c^+, \|f\|_u \leq 1, \text{supp}(f) \subset U\}$. Using this alternative characterization of inner regularity, after some computations we can derive $\mu(U) = \mu(yU)$, which completes the proof for part 2.

For part 3, since $\mu \neq 0$, we have $\mu(G) > 0$. Since G is open in G, by inner regularity we have $\mu(G) = \sup\{\mu(K) : K \subset G, K\text{compact}\} > 0$. Hence $\exists K$ compact in G such that $\mu(K) > 0$. Now for a nonempty open U, $\{xU\}_{x\in G}$ is an open cover of K. So we get a subcover $\{x_iU\}_{i=1}^n$ of K. Then $\mu(\bigcup_{i=1}^n x_iU) \geq \mu(K) > 0$, which gives $\mu(U) > 0$. Now we let $U = \{x \in G : f(x) > \|f\|_u/2\}$. It is clear that U is open by continuity of f and the definition of $\|\cdot\|_u$. Then $\int f d\mu \geq \int_U \|f\|_u/2 d\mu = \frac{1}{2}\|f\|_u\mu(U) > 0$.

For part 4, the "if" part is true by the definition of the Haar measure. So we show the "only if" part. Assume G is not compact. Let K be a compact neighborhood of e (such K exists since G is LCH). Then by proposition 1.2, $\exists$ a symmetric open neighborhood U of e such that $UU \subset K$. Note that there is no finite number of translates of K that covers G (if there is one, let $\mathcal{O}$ be any open cover of G, $\mathcal{O}$ covers x_iK, so there is a finite subcover of x_iK. Take the union of these finite subcovers over $x_1, \ldots, x_n$, we get a finite subcover of G, which means G is compact). Now we construct a net $\{x_n\}_{n\in\mathbb{N}} \subset G$ by choosing x_n such that $x_n \notin \bigcup_{j<n} x_jK$. We claim that $x_iU \bigcap x_jU = \varnothing$ for $i \neq j$. If the claim holds, then $\mu(G) \geq \mu(\bigcup_{n\in\mathbb{N}} x_nU) = \sum_{n\in\mathbb{N}} \mu(x_nU) = \sum_{n\in\mathbb{N}} \mu(U) = \infty$ because $\mu(U) > 0$ by part 3. The proof for the claim is straightforward: suppose $x_iU \bigcap x_jU \neq \varnothing$ for $i < j$, then $\exists u, v \in U$ such that $x_iu = x_jv$. So $x_j = x_iuv^{-1} \in x_iUU^{-1} = x_iUU \subset x_iK$, which contradicts our construction of $\{x_n\}_{n\in\mathbb{N}}$. □

3 Construction of Haar Measure

We now construct the Haar measure on the LCH topological group G. Note that by proposition 2.2 part 1, it suffices to construct the left Haar measure, as given a left Haar measure μ on G, the measure $\widetilde{\mu}$ defined by $\widetilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure. We start by giving an intuition as to how one might measure the Borel subsets of G.

3.1 Haar Covering Number

Let $E \in \mathcal{B}_G$ and let $V \subset G$ be open and non-empty. Then we define the Haar covering number of E with respect to V - $(E : V)$ - by "the smallest number of left translates of V that cover E":

$$(E : V) = \inf\{|A| : E \subset \bigcup_{x\in A} xV\}$$

Intuitively, $(E : V)$ is a way of 'measuring' E using V as a unit of measurement. We can normalize this measurement by picking a set E_0 to have 'measure' 1 by considering $\frac{(E:V)}{(E_0:V)}$. This is in a sense the 'size' of E when E_0 is said to have size 1. This estimate is not very precise, but it gets more precise the 'smaller' we make V (this is tongue and cheek - we don't have a measure yet so we don't know what it means for V to be small).

Notice, $\frac{(E:V)}{(E_0:V)}$ is clearly left-invariant with respect to E. The intuition is that is a V gets smaller and smaller, $\frac{(E:V)}{(E_0:V)}$ approaches a measure. As this measure would be left invariant, this suggests a construction of a left Haar measure by defining $\mu(E) = \lim_{V\to\emptyset} \frac{(E:V)}{(E_0:V)}$. However, this argument is fairly complicated and unwieldy. It is easier to approach this problem from the perspective of integrals of functions rather than measures of sets, and then apply the Riesz Representation Theorem to retrieve the measure.

Thus for $f, \phi \in C_c^+(G)$, we define the Haar covering number of f with respect to g to be:

$$(f : \phi) = \inf\{\sum_{i=1}^{n} c_n : f \leq \sum_{i=1}^{n} c_i L_{x_i}\phi \text{ for some } n \in \mathbb{N} \text{ and } x_1, ..., x_n \in G\}$$

This definition makes sense - $(f:\phi)<\infty$ - as the set $\{x:\phi(x)>\frac{1}{2}||\phi||_u\}$ is an open and non-empty set. Therefore, as supp(f) is compact and the collection of left translates of $\{x:\phi(x)>\frac{1}{2}||\phi||_u\}$ covers supp(f), finitely many left translates of $\{x:\phi(x)>\frac{1}{2}||\phi||_u\}$ cover supp(f). Then, we have for some $x_1,...,x_n$:

$$f\leq\frac{2||f||_u}{||\phi||_u}\sum_{i=1}^{n}L_{x_i}\phi$$

Thus $(f:\phi)\leq\frac{2n||f||_u}{||\phi_u||}<\infty$. We also have $(f:\phi)\geq\frac{||f||_u}{||\phi||_u}$ as if $f\leq\sum_{i=1}^{n}c_iL_{x_i}\phi$, then we have:

$$||f||_u\leq\Big|\Big|\sum_{i=1}^{n}c_iL_{x_i}\phi\Big|\Big|_u\leq\sum_{i=1}^{n}c_i||L_{x_i}\phi||_u=||\phi||_u\sum_{i=1}^{n}c_i$$

Thus, $\frac{||f||_u}{||\phi||_u}\leq\sum_{i=1}^{n}c_i$, thus $\frac{||f||_u}{||\phi||_u}\leq(f:\phi)$. Now we prove several facts about the Haar Covering Number:

3.2 Properties of the Haar Covering Number

Let $f,g,\phi\in C_c^+(G)$. Then:

a. $(f:\phi)=(L_x(f):\phi)=(f:L_x(\phi))$ for all $x\in G$.
b. $(cf:\phi)=c(f:\phi)$ for all $c\in\mathbb{R}$.
c. $(f+g:\phi)\leq(f:\phi)+(g:\phi)$.
d. $(f:\phi)\leq(f:g)(g:\phi)$.

Proof: (a) Notice that $f\leq\sum_{i=1}^{n}c_iL_{x_i}\phi$ if and only if $L_x(f)\leq L_x\big(\sum c_iL_{x_i}\phi\big)=\sum c_iL_x(L_{x_i}\phi)=\sum c_iL_{xx_i}\phi$. Thus $(f:\phi)\leq\sum c_i$ if and only if $(L_x(f):\phi)\leq\sum c_i$, thus $(f:\phi)=(L_x(f):\phi)$. Similarly, $f\leq\sum c_iL_{x_i}\phi$ if and only if $f\leq\sum c_iL_{x_ix^{-1}}L_x\phi$, thus $(f:\phi)=(f:L_x(\phi))$.

(b) Notice $cf\leq\sum c_iL_{x_i}\phi$ if and only if $f\leq\sum\frac{c_i}{c}L_{x_i}\phi$. Thus, $(cf:\phi)\leq\sum c_i$ if and only if $(f:\phi)\leq\frac{1}{c}\sum c_i$. Thus $(cf:\phi)=c(f:\phi)$.

(c) Notice if $f\leq\sum_{i=1}^{n}c_iL_{x_i}\phi$ and $g\leq\sum_{j=1}^{m}d_jL_{y_j}\phi$, then (as $f,g,\phi\geq0$) $f+g\leq\sum_{i=1}^{n}c_iL_{x_i}\phi+\sum_{j=1}^{m}d_jL_{y_j}\phi$. Thus if $(f:\phi)\leq\sum c_i$ and $(g:\phi)\leq\sum d_j$ then $(f+g:\phi)\leq\sum c_i+\sum d_j$. Notice we can pick $(c_i),(d_j)$ s.t. $(f:\phi)+\epsilon\geq\sum c_i$ and $(g:\phi)+\epsilon\geq\sum d_j$, thus $(f+g:\phi)\leq(f:\phi)+(g:\phi)+2\epsilon$. As we can make ϵ as small as we like, $(f+g:\phi)\leq(f:\phi)+(g:\phi)$.

(d) Notice if $f\leq\sum_{i=1}^{n}c_iL_{x_i}\phi$ and $g\leq\sum_{j=1}^{m}d_jL_{y_j}\phi$, then $f\leq\sum_{i=1}^{n}\sum_{j=1}^{m}c_id_jL_{x_iy_j}\phi$. Thus, if $(f:g)\leq\sum c_i$ and $(g:\phi)\leq\sum d_j$, then $(f:\phi)\leq\big(\sum c_i\big)\cdot\big(\sum d_j\big)$. Then as we can pick $(c_i),(d_j)$ s.t. $(f:g)+\epsilon\geq\sum c_i$ and $(g:\phi)+\epsilon\geq\sum d_j$, then we have $(f:\phi)\leq((f:g)+\epsilon)\cdot((g:\phi)+\epsilon)=(f:g)(g:\phi)+\epsilon((f:g)+(g:\phi)+\epsilon)$. As $(f:g),(g:\phi)$ are finite and we can make ϵ as small as we like, $(f:\phi)\leq(f:g)(g:\phi)$.

3.3 An Almost-Linear Functional

Now, using the above lemma, we can normalize $(f : \phi)$ as we did for the Haar covering number of sets to get a pseudo-functional I_ϕ. That is, we pick f_0 for which we want $I_\phi(f_0) = 1$ and define:

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)}$$

Notice I_ϕ is left invariant by (a) of the above lemma. Furthermore I_ϕ is almost linear (it is linear under scalar multiplication by (b) and sub-additive by (c) of the above lemma). We also have:

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0)$$

This follows as by (d), $(f_0 : \phi) \leq (f_0 : f)(f : \phi)$, thus $(f_0 : f) \geq \frac{(f_0:\phi)}{(f:\phi)}$, thus $(f_0 : \phi)^{-1} \leq \frac{(f:\phi)}{(f_0:\phi)} = I_\phi(f)$. By (d) we also have $(f : \phi) \leq (f : f_0)(f_0 : \phi)$¡ thus $I_\phi(f) \leq (f : f_0)$.

What we have now is a class of functionals I_ϕ which are left-invariant, linear under scalar multiplication, sub-additive, and uniformly bounded. From this class we want to construct a left-invariant linear functional from which we will get a left Haar measure by invoking the Riesz Representation Theorem. The last major lemma we will state will show that for a specific choice of $f, g \in C_c^+$, we can choose $\phi \in C_c^+$ s.t. I_ϕ is almost linear. Specifically:

Proposition 3.1. *If $f_1, f_2 \in C_c^+$ and $\epsilon > 0$, then there is a neighborhood V of e such that $I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \epsilon$ for all $\phi \in C_c^+$ s.t. supp$(\phi) \subset V$.*

Combined with the sub-additivity of I_ϕ $(I_\phi(f_1 + f_2) \leq I_\phi(f_1) + I_\phi(f_2))$, this tells us FINISH THIS

3.4 Every LCH Group G Has a Left Haar Measure

Now we have the tools to prove the existence of a left Haar Measure. I will provide an outline of the proof. For the full proof, see Folland's Real Analysis.

Proposition 3.2. *Let G be an LCH group. Then there exists a left Haar Measure μ on $(G, \mathcal{B}_G)$.*

Fix an $f_0 \in C_c^+$. This function will have integral 1 with respect to the measure we construct - it is our unit. Define the produce space:

$$\prod_{f \in C_c^+} [(f_0 : f)^{-1}, (f : f_0)^{-1}]$$

By Tychonoff's Theorem, this space is compact. A brief digression: Tychonoff's Theorem is a theorem of Topology which states that arbitrary products of compact spaces are compact under the product topology. Tychonoff's

Theorem requires (and is in fact equivalent to) the axiom of choice. A proof of the existence of Haar measure that does not depend on the axiom of choice exists, though we do not discuss it here.

Note that the elements of the product space defined above are functionals on C_c^+. This follows as uncountable products are defined as 'functions' from the index space C_c^+ to the element space $\mathbb{R}$, which in this case is precisely a subset of the set of functionals on C_c^+.

Then for any open set $V \subset G$ containing e, define $K(V)$ to be the closure of $\{I_\phi : \text{supp}(\phi) \subset V\}$ in the product space. Then notice finite intersections of the $K(V)$s are non-empty. It therefore follows from the compactness of $\prod_{f \in C_c^+}[(f_0 : f)^{-1}, (f : f_0)]$ that $\bigcap_{V \text{ open}, e \in V} K(V)$ is non-empty.

Then let I be a member of this intersection. Them I is left invariant and linear. This follows by the previous lemma and the fact that for any $f_1, f_2, ..., f_n$ and $\epsilon > 0$, you can find a $\phi \in C_c^+$ s.t. I_ϕ is within ϵ of I on $f_1, ..., f_n$. Consult Folland's book for a more in depth proof.

Then we have a left invariant linear functional I on C_c^+. This can be extended to be a positive left invariant linear functional by observing, for an $f \in C_c$, $I(f^+) - I(f^-)$. Thus we have a positive left invariant linear functional on G. Thus the Riesz representation theorem gives us a Radon measure μ corresponding to I. Proposition 2.2 gives us that μ is a left Haar measure.

Thus we have a left Haar measure μ on G - the existence of the right Haar measure follows from proposition 2.2. It is important to note that μ is not unique. At the beginning of this section, we fixed $f_0 \in C_c^+$. f_0 was a function we picked arbitrarily to have integral 1 - we could have just as easily picked any other function. Thus there are many other possible left Haar measures on G - we can easily see this by scaling μ by a constant. The important result presented in the next section states that such scalings are the only other left Haar measures on G.

4 Uniqueness of Haar Measure

Not only can we show the existence of left and right Haar measures, but we can show uniqueness (up to scalar multiplication as well). This gives us a very strong result - that is that on any LCH group G, there exists a canonical measure that obeys the topological structure of G (it is radon) as well as the group structure of G (it is left or right translation invariant). The statement of this result is as follows:

Proposition 4.1. *Let G be an LCH group. Then if μ, ν are left/right Haar measures, $\mu = c\nu$ for some $c > 0$.*

We do not provide a proof of this, but a proof can be found in Folland's Real Analysis (the proof does not carry with it much intuition - it is largely computation).

5 The Modular Function

Let μ be a left Haar Measure on G. Then for any $x \in G$, we define μ_x by $\mu_x(E) = \mu(Ex)$. Notice μ_x is a left Haar Measure, as by associativity

$$\mu_x(yE) = \mu(yEx) = \mu(Ex) = \mu_x(E)$$

Therefore by Haar measures' uniqueness property, there is a positive number $\Delta(x)$ s.t. $\mu_x = \Delta(x)\mu$. We call this function the modular function:

Definition 5.1 (The Modular Function)**.** *The modular function $\Delta : G \to \mathbb{R}^+$ takes values s.t. $\mu(Ex) = \Delta(x)\mu(E)$ for all $E \in \mathcal{B}_G$. Note that the module function is independent of the choice of μ as the choice of μ can only differ by a constant factor.*

By proposition 11.10, Δ is a continuous homomorphism from G to $\mathbb{R}^*$ with multiplication. Moreover:

$$\int (R_y f) d\mu = \Delta(y^{-1}) \int f \, d\mu$$

Note that when Δ is identically 1, $\mu(Ex) = \mu_x(E) = \Delta(x)\mu(E) = \mu(E)$, thus μ is right invariant. Thus, left Haar measures are precisely right Haar measures. The converse also holds. In this case, we say:

Definition 5.2 (Unimodularity)**.** *An LCH group G is said to be unimodular when its modular function Δ is identically* 1.

It is easy to see that every abelian group is unimodular. However, there are many non-commutative groups that are unimodular as well. Specifically, let $[G, G]$ is the commutator subgroup of G (it consists of elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$). Then, if the Quotient group $G/[G, G]$ is finite, G is unimodular.

This follows from the fact that $(\mathbb{R}^+, \cdot)$ is abelian, so any homomorphism $\Delta : G \to \mathbb{R}^+$ must annihilate the commutator subgroup. Then, as $\mathbb{R}^+$ has no finite subgroups except for $\{1\}$, if what remains is finite it must all be mapped to 1. Intuitively what this means is that groups with finite non-commutativity are unimodular.

Also, any compact group G must be unimodular. This follows from the fact that a left Haar measure is finite on compact sets and the fact that $Gx = G$, giving us $\mu(G) = \mu(Gx) = \Delta(x)\mu(G)$. Dividing by $0 < \mu(G) < \infty$ gives us that $\Delta(x) = 1$.

The last and most important result we give with respect to the modular function ties together the left and right Haar measures of an LCH group G. We saw in proposition 2.2 that if μ is a left Haar Measure then $\widetilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure. Now we show how to compute $\widetilde{\mu}$ from Δ and μ:

Proposition 5.1. *$d\widetilde{\mu} = \Delta^{-1} d\mu$, or equivalently $\widetilde{\mu}(S) = \int_S \Delta^{-1} d\mu$ for all $S \in \mathcal{B}_G$.*

This leads to the immediate corollary that left and right Haar Measures are mutually absolutely continuous.

6 Examples

Let G be a topological group that is homeomorphic to an open subset of $\mathbb{R}^n$. Then if the group operation on G is a linear transformation, that is the function $f_x(y) = xy$ can be written as $f_x(y) = A_x(y) + b_x$, then a left Haar measure of G is $|\det(A_x)|^{-1}dx$ where dx is Lebesgue measure on $\mathbb{R}^n$.

Note that an alternative formulation of the determinant of a linear transformation is the scaling factor by which the transformation modifies the Lebesgue measure of a set it acts on. Thus, what the above statement is saying is that if G can be viewed as an open subset of $\mathbb{R}^n$ where $\lambda(xS) = g(x)\lambda(S)$ for all measurable S (where λ is Lebesgue measure), then $(g(x))^{-1}d\lambda(x)$ is a left Haar measure on G (This is mostly a formulation to help the reader's intuition - it may not be perfectly correct).

We can apply this idea to many groups. Specifically, we can apply it to the group of non-zero complex numbers under multiplication, for which we get a left and right Haar measure of $\frac{1}{|z|}dz$ where dz is Lebesgue measure in $\mathbb{R}^2$.

For $GL(n, \mathbb{R})$, we get the left and right Haar measure $\frac{1}{|\det(A)|^n}dX$ where dX is Lebesgue measure on $\mathbb{R}^{n^2}$.

For the group of matrices of the form ($x > 0, y \in \mathbb{R}$):

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$$

we get a left Haar measure of $x^{-2}dxdy$ and a right Haar measure of $x^{-1}dxdy$. Verifying this is not too difficult.

7 A Combinatorial Proof of the Existence of Haar Measure

In this section we discuss a proof of the existence of Haar Measure on compact groups G [2]. This proof follows the same broad strategy as the proof discussed earlier in that it constructs a Haar measure on G (left/right are the same as G is compact) by constructing a left invariant linear functional.

Definition 7.1 (Hypergraph)**.** *A Hypergraph $\mathcal{H}$ is a tuple (V, E) where V is a set of vertices and E is a collection of subsets of V called edges.*

If $\mathcal{H} = (V, E)$ is a hypergraph and $f : V \to \mathbb{R}$, let $\delta(f, \mathcal{H}) = \sup\{|f(x) - f(y)| : x, y \in U \text{ for some } U \in E\}$. $\delta(f, \mathcal{H})$ can be intuitively considered to be the maximal difference (w.r.t. f) between pairs of nodes connected by some edge.

Also, if A is a finite set and $f : A \to \mathbb{R}$, let $\overline{f}(A) = \frac{1}{|A|}\sum_{x \in A} f(x)$ ($\overline{f}$ is the average value of f on A w.r.t. the counting measure. Finally:

Definition 7.2. *Let $\mathcal{H} = (V, E)$ be a hypergraph. Then A is a blocking set of H if $U \cap A \neq \emptyset$ for all edges $U \in E$. Another word for A would be a vertex cover. Furthermore, A is called a minimum cardinality blocking set of $\mathcal{H}$ if there does not exist a blocking set B of $\mathcal{H}$ s.t. the cardinality of B is less than that of A.*

Now we can state the following:

Lemma: Let $\mathcal{H} = (V, E)$ be a hypergraph and let A, B be two minimum cardinality blocking sets of $\mathcal{H}$. If A and B are finite, then:

$$|\overline{f}(A) - \overline{f}(B)| \leq \delta(f, \mathcal{H})$$

Note as A, B are finite and both minimum cardinality blocking sets, we have $|A| = |B|$. Then if we view A, B as disjoint sets (i.e. if $x \in A, B$ view the $x \in A$ and the $x \in B$ as different), we can construct a bipartite with nodes $A \sqcup B$ and with edges between each $x \in A, y \in B$ if and only if there exists an edge U of $\mathcal{H}$ s.t. $x, y \in U$.

Then, through some cleverness and use of the Marriage lemma, we see that the bipartite graph above has a perfect matching (see [2]). What this means is that there is a matching $a_i \in A, b_i \in B$ s.t. for each $1 \leq i \leq n = |A|$, there exists an edge U of $\mathcal{H}$ s.t. $a_i, b_i \in U$. But this means:

$$|\overline{f}(A) - \overline{f}(B)| = \Big|\frac{1}{n}\sum_{i=1}^{n} f(a_i) - \frac{1}{n}\sum_{i=1}^{n} f(b_i)\Big| \leq \frac{1}{n}\sum_{i=1}^{n} |f(a_i) - f(b_i)| \leq \frac{1}{n} n\delta(f, \mathcal{H}) = \delta(f, \mathcal{H})$$

Using this lemma, we can derive the existence of a Haar measure on G. First, however, some definitions. Given some open subset U of G, define $\mathcal{H}_U = (G, \{xUy : x, y \in G\})$. That is, $\mathcal{H}_U$ is the graph with vertices G and edges the translates of U. We call a blocking set of $\mathcal{H}_U$ a U net (a U net intersects all translates of U). The compactness of G guarantees the existence of a finite U net for any U.

Fix $f \in C(G)$. Let U, V be open subsets of G. Let A be a minimum cardinality U net and let B be a minimum cardinality V net. Then it follows from the above lemma that:

$$|\overline{f}(A) - \overline{f}(B)| \leq \delta(f, \mathcal{H}_U) + \delta(f), \mathcal{H}_V)$$

Now let U_n be a sequence of open sets such that $\delta(f, \mathcal{H}_{U_n}) \to 0$ (note in this contest, $\delta(f, \mathcal{H}_{U_n}) \to 0$ is equivalent to the uniform continuity of f, which holds as f is a continuous function on a compact set G). Then by the above claim, the sequence $(\overline{f}(U_n))$ is Cauchy. Thus it converges to some value which we call $L(f)$. Note that this limit is independent of the choice of U_n and A_n.

Then it is not too difficult to show that $L(f)$ is a translation invariant linear functional. Thus by the Riesz Representation Theorem the proof of the existence of Haar measure on the compact group G is complete. Something to note about this proof was, although it involved concepts like hypergraphs, the argument was almost entirely an analytic argument. The only place where combinatorics came up was in the use of the Marriage lemma in the proof of the stated lemma.

References

[1] Gerald B Folland. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

[2] László Lovász, L Pyber, DJA Welsh, and GM Ziegler. Combinatorics in pure mathematics. *Handbook of combinatorics, MIT press, North Holland*, pages 2039–2082, 1995.