Math 4370 – Homework 1
Due: February 11 (beginning of class)

Read:

Read Chapter 1 from the textbook as well as the course notes.

Turn in:

1. Let R be a ring with two operations $+$ and $\times$. Let 0 and 1 denote the identity elements for $+$ and $\times$. If $1 = 0$ show[1] that $R = \{0\}$.

2. Let n be any integer. Consider the set $\mathbb{Z}/n\mathbb{Z}$ consisting of integers *modulo* n with the addition and multiplication modulo n.

(a) Show $\mathbb{Z}/n\mathbb{Z}$ forms a ring with n elements.

(b) Let p be a prime (integer) number. Show $\mathbb{Z}/p\mathbb{Z}$ is indeed a field. We often denote this field by $\mathbb{F}_p$.

(c) Prove that $a^p = a$ for all $a \in \mathbb{F}_p$.

(d) Find a nonzero polynomial in $\mathbb{F}_p[x]$ which vanishes at every point of $\mathbb{F}_p$.

3. Let R be an integral domain. Prove:

(a) $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$ for all $p(x), q(x) \in R[x]$.

(b) $R[x]^\times = R^\times$, i.e., R and $R[x]$ have the same *units*.

(c) $R[x]$ is an integral domain.

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[1] *Show* = *Prove*.

If R contains zero divisors, could any of (a), (b), (c) still be true?

4. Recall a commutative ring R with 1 (identity) is called *Noetherian* if every ideal of R is finitely generated. Prove the following are equivalent:

(1) R is a Noetherian ring.

(2) R has no infinite increasing chains of ideals, i.e., whenever

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

is an increasing chain of ideals of R, then there is a positive integer m such that for all $k \geq m$ we have $I_k = I_m$. (this is called *ascending chain condition* on ideals).

5. An integral domain R is said to be a *Euclidean domain* (or possess a *division algorithm*) if there is a function $N\colon R \to \mathbb{Z}_{\geq 0}$ with $N(0) = 0$, such that for any two element $a \in R$ and $0 \neq b \in R$ there exist elements q and r in R with

$$a = qb + r \quad \text{with} \quad r = 0 \text{ or } N(r) < N(b)\ .$$

(a) Show that the ring of integers $\mathbb{Z}$ and the polynomial ring in one variable $k[x]$ over a field k are examples of Euclidean Domains.

(b) Prove that every ideal in a Euclidean Domain is *principal*. More precisely, show that if I is a nonzero ideal in the Euclidean Domain R, then $I = \langle d \rangle$, where d is any nonzero element of I of minimum $N(\cdot)$-value.

6. Let R be a commutative ring with identity, and $I \subseteq R$ be an ideal. Prove

(a) $I = R$ if and only if I contains a unit of R.

(b) R is a field if and only if $\{0\}$ and R are the only ideals of R.

7. Let R be a commutative ring and let $I \neq R$ be an ideal. Recall I is called a *prime ideal* if whenever $ab \in I$ for some $a, b \in R$ then $a \in I$ or $b \in I$. Also recall I is called a *maximal ideal* if the only ideals containing I are I and R. Prove:

(a) I is a prime ideal in R if and only if the quotient ring R/I is an integral domain.

(b) I is a maximal ideal in R if and only if the quotient ring R/I is a field.

(c) Every maximal ideal is a prime ideal.

8. Let I be an ideal of a commutative ring R and define the *radical* of I

$$\mathrm{rad}(I) = \{r \in R\colon r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\} .$$

(a) Prove that $\mathrm{rad}(I)$ is an ideal of R containing I.

(b) If $\mathrm{rad}(I) = I$ we call the ideal I *radical*. Show that prime ideals are radical.

(c) Prove that for any ideal I we have $\mathrm{rad}(\mathrm{rad}(I)) = \mathrm{rad}(I)$, i.e., $\mathrm{rad}(I)$ is radical.

Note: $\mathrm{rad}(I)$ is sometimes denoted by $\sqrt{I}$. It is a theorem that $\mathrm{rad}(I) = \bigcap P$ where the intersection is over all prime ideals P containing I.

Notes:

(i) *Collaboration*: On the homework sets, collaboration is both allowed and encouraged. However, you must write up yourself and understand your own homework solutions. You should give credit to any outside sources or collaborations.

(ii) *Academic honesty*: All students are expected to comply with the Code of Academic Integrity. The institute honor code is available at: www.theuniversityfaculty.cornell.edu/AcadInteg/code.html

(iii) Please turn in your homework in class. If there are multiple pages, please staple them. Emailed homework will not be accepted unless you have a special reason, and you have discussed your situation with me a priori. In that case, please send a single .pdf file, put your name on all pages, and make sure to number the pages.