BASIC VON NEUMANN ALGEBRA THEORY

FARBOD SHOKRIEH

Contents

1. Introduction

The theory of von Neumann algebras and von Neumann dimensions allows one to measure some infinite-dimensional subspaces in a Hilbert space, by assigning to them a notion of "dimension" (not necessarily an integer). We quickly review some very basic notions in the theory. We follow the presentation in [Shu93]. See also [Lüc02] and [Pan96] for proofs and a more thorough treatment.

2. von Neumann algebras and factors

Let H be a Hilbert space over $\mathbb{C}$ with the (Hermitian) inner product $\langle \cdot, \cdot \rangle$.

Let $\mathcal{B}(\mathrm{H})$ be the algebra of all bounded linear operators on H. This is Banach space with respect to the operator norm. Moreover, this is a C^*-algebra:

(i) $\mathcal{B}(\mathrm{H})$ is a Banach algebra, meaning $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in \mathcal{B}(\mathrm{H})$.
(ii) $\mathcal{B}(\mathrm{H})$ is a $*$-algebra, i.e. it is closed under the operation of taking adjoints of operators, where $A \mapsto A^*$ is defined by the usual identity $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y \in \mathrm{H}$.
(iii) $(AB)^* = B^* A^*$ for all $A, B \in \mathcal{B}(\mathrm{H})$.
(iv) $\|A^* A\| = \|A\|^2$ for all $A \in \mathcal{B}(\mathrm{H})$.

For any subset $\mathcal{M} \subseteq \mathcal{B}(\mathrm{H})$, its *commutant* is the subalgebra

$$\mathcal{M}' = \{A \in \mathcal{B}(\mathrm{H})\colon AB = BA, \forall B \in \mathcal{M}\}$$

in $\mathcal{B}(\mathrm{H})$. Clearly, the identity operator I is in $\mathcal{M}'$.

Definition 2.1. A *von Neumann algebra* on H is a subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathrm{H})$ such that

(i) $\mathcal{M}$ is a $*$-subalgebra.
(ii) $\mathcal{M}'' = \mathcal{M}$.

Date: October 30, 2017.

It follows from $\mathcal{M}'' = \mathcal{M}$ that $I \in \mathcal{M}$.

Remark 2.2. In Definition 2.1, the slightly unmotivated condition $\mathcal{M}'' = \mathcal{M}$ can be replaced by either of the following two more natural conditions:

(a) $I \in \mathcal{M}$, and $\mathcal{M}$ is closed in the weak operator topology.
(b) $I \in \mathcal{M}$, and $\mathcal{M}$ is closed in the strong operator topology.

This is usually called the *von Neumann Double Commutant Theorem*.

Definition 2.3. Let $\mathcal{M}$ be a von Neumann algebra on H.

(i) $\mathcal{M}$ is called a *factor* if its center is trivial, i.e. $\mathcal{M} \cap \mathcal{M}' = \{\lambda I \colon \lambda \in \mathbb{C}\}$.
(ii) The set of all *orthogonal projections* in $\mathcal{M}$ is denoted by $\mathrm{Pr}(\mathcal{M})$. In other words

$$\mathrm{Pr}(\mathcal{M}) = \{A \in \mathcal{M} \colon A^2 = A = A^*\}.$$

Factors can be thought of as "indecomposable" von Neumann algebras: every von Neumann algebra on a (separable) Hilbert space is isomorphic to a direct integral of factors. For our purpose, their significance lies in the fact that if $\mathcal{M}$ is a factor, then there is *at most one trace* function (up to a constant factor) on $\mathcal{M}$.

3. von Neumann trace

Definition 3.1. Let $\mathcal{M}$ be a von Neumann algebra on H. Let

$$\mathcal{M}^+ = \{A \in \mathcal{M} \colon A \geq 0\}.$$

(Here $A \geq 0$ means $\langle Ax, x\rangle \in \mathbb{R}^{\geq 0}$, $\forall x \in \mathrm{H}$).

A *trace* on $\mathcal{M}$ is a function $\tau \colon \mathcal{M}^+ \to [0, +\infty]$ satisfying:

- (Linear) $\tau(aA + bB) = a\tau(A) + b\tau(B)$, $\forall a, b \in \mathbb{R}^{\geq 0}$ and $\forall A, B \in \mathcal{M}^+$.
- (Tracial) $\tau(A^*A) = \tau(AA^*)$, $\forall A \in \mathcal{M}$.
- (Faithful) $\tau(A) = 0 \Rightarrow A = 0$.
- (Normal) If $\{A_\alpha\}$ is an increasing net on $\mathcal{M}^+$ converging (in operator norm topology) to A, then $\{\tau(A_\alpha)\}$ is an increasing net on $\mathbb{R}^{\geq 0}$ converging to $\tau(A)$.
- (Semifinite) $\tau(A) = \sup\{\tau(B) \colon B \in \mathcal{M}^+, \tau(B) < +\infty, A - B \geq 0\}$, $\forall A \in \mathcal{M}^+$.

The most basic example is the following:

Example 3.2. Clearly, $\mathcal{M} = \mathcal{B}(\mathrm{H})$ is a von Neumann algebra. Moreover, it is a factor, hence there is at most one trace function Tr (up to scaling) associated to it. Let $\{u_\alpha\}_{\alpha \in J}$ be an orthonormal basis for H. For $A \in \mathcal{B}(\mathrm{H})^+$ define:

$$\mathrm{Tr}(A) = \sum_{\alpha \in J} \langle A u_\alpha, u_\alpha \rangle, \tag{1}$$

where the sum is understood as the supremum of finite partial sums. Then Tr defines a trace function according to Definition 3.1, and it is independent of the choice of the orthonormal basis. Any other trace on $\mathcal{B}(\mathrm{H})$ is of the form $c\,\mathrm{Tr}$ for some $c > 0$.

4. von Neumann dimension

Now we can describe the beautiful approach to assigning a notion of "dimension" to certain infinite dimensional subspaces of H.

Definition 4.1. Let $\mathcal{M}$ be a von Neumann algebra on H.

(i) A subspace $\mathrm{L} \subseteq \mathrm{H}$ is *admissible* with respect to $\mathcal{M}$ (or $\mathcal{M}$-admissible) if there is an orthogonal projection $A \in \mathrm{Pr}(\mathcal{M})$ such that $\mathrm{L} = \mathrm{Image}(A)$. The set of all $\mathcal{M}$-admissible subspaces of H is the $\mathcal{M}$-*admissible Grassmannian*:

$$\mathbf{Gr}(\mathrm{H}, \mathcal{M}) = \{\mathrm{L} \subseteq \mathrm{H} \colon \mathrm{L} = \mathrm{Image}(A) \text{ for some } A \in \mathrm{Pr}(\mathcal{M})\}.$$

(ii) Given a trace τ on $\mathcal{M}$, define the corresponding *dimension* function

$$\dim_\tau \colon \mathbf{Gr}(\mathrm{H}, \mathcal{M}) \to [0, +\infty]$$

by

$$\mathrm{L} = \mathrm{Image}(A) \mapsto \tau(A).$$

It is elementary to check that this notion of dimension generalizes the dimension theory for finite-dimensional vector spaces.

The following (expected) properties of $\dim_\tau$ are easy to verify:

(i) $\dim_\tau(\mathrm{L}) = 0 \iff \mathrm{L} = \{0\}$.
(ii) $\mathrm{L}_1 \subseteq \mathrm{L}_2 \Rightarrow \dim_\tau(\mathrm{L}_1) \leq \dim_\tau(\mathrm{L}_2)$.
(iii) $\mathrm{L}_1 \perp \mathrm{L}_2 \Rightarrow \dim_\tau(\mathrm{L}_1 \oplus \mathrm{L}_2) = \dim_\tau(\mathrm{L}_1) + \dim_\tau(\mathrm{L}_2)$.
(iv) If $U \in \mathcal{M}$ is *unitary* (i.e. $U^{-1} = U^*$), then $\dim_\tau(U(\mathrm{L})) = \dim_\tau(\mathrm{L})$.

5. Tensor products

Let $(\mathrm{H}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathrm{H}_1, \langle\cdot,\cdot\rangle_1)$ be two Hilbert spaces. Let $\mathrm{H}_1 \otimes_{\mathrm{alg}} \mathrm{H}_2$ denote the algebraic (vector space) tensor product. The vector space $\mathrm{H}_1 \otimes_{\mathrm{alg}} \mathrm{H}_2$ is endowed with an inner product defined by $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle\langle x_2, y_2\rangle$. The Hilbert space tensor product $\mathrm{H}_1 \otimes \mathrm{H}_2$ is, by definition, the (metric space) completion of $\mathrm{H}_1 \otimes_{\mathrm{alg}} \mathrm{H}_2$ with respect to the inner product.

If $A_1 \in \mathcal{B}(\mathrm{H}_1)$ and $A_2 \in \mathcal{B}(\mathrm{H}_2)$, we can form the algebraic tensor product $A_1 \otimes_{\mathrm{alg}} A_2 \colon \mathrm{H}_1 \otimes_{\mathrm{alg}} \mathrm{H}_2 \to \mathrm{H}_1 \otimes_{\mathrm{alg}} \mathrm{H}_2$. It can be extended (by continuity) to an operator $A_1 \otimes A_2 \in \mathcal{B}(\mathrm{H}_1 \otimes \mathrm{H}_2)$.

If $\mathcal{M}_1 \subseteq \mathcal{B}(\mathrm{H}_1)$ and $\mathcal{M}_2 \subseteq \mathcal{B}(\mathrm{H}_2)$ are von Neumann algebras we may form the von Neumann tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$. This is the strong (or weak) completion of the set of all finite linear combinations of tensor products $A_1 \otimes A_2$ where $A_1 \in \mathcal{M}_1$ and $A_2 \in \mathcal{M}_2$. Equivalently, $\mathcal{M}_1 \otimes \mathcal{M}_2$ is the von Neumann algebra generated by the algebraic tensor product $\mathcal{M}_1 \otimes_{\mathrm{alg}} \mathcal{M}_2$ inside $\mathcal{B}(\mathrm{H}_1) \otimes \mathcal{B}(\mathrm{H}_2)$.

6. von Neumann algebras associated with a discrete group

We will now focus on the von Neumann algebras and dimensions that arise in the presence of group actions.

6.1. **Group von Neumann algebras.** Let $\mathbf{G}$ be a discrete group (topological group equipped with the discrete topology). Consider the Hilbert space:

$$\ell^2(\mathbf{G}) = \{f \mid f\colon \mathbf{G} \to \mathbb{C},\ \|f\|^2 = \sum_{g \in \mathbf{G}} |f(g)|^2 < +\infty\},$$

where, as usual, the sum is understood as the supremum of finite partial sums. An orthonormal basis is given by

$$\{\delta_g \colon g \in \mathbf{G}\},$$

where $\delta_g(h) = 1$ if $g = h$ and 0 otherwise.

Translations operators generate (two) natural von Neumann algebra on $\ell^2(\mathbf{G})$, which we now describe. For each $g \in \mathbf{G}$, we have the following left and right translation operators $L_g, R_g \in \mathcal{B}(\ell^2(\mathbf{G}))$:

$$L_g(f)(h) = f(g^{-1}h) \quad , \quad h \in \mathbf{G},$$

$$R_g(f)(h) = f(hg) \quad , \quad h \in \mathbf{G}.$$

In terms of the orthonormal basis $\{\delta_g \colon g \in \mathbf{G}\}$ we have

$$L_g(\delta_h) = \delta_{gh} \quad , \quad R_g(\delta_h) = \delta_{hg^{-1}}.$$

Clearly, both L_g and R_g are unitary:

$$(L_g)^* = (L_g)^{-1} = L_{g^{-1}} \quad , \quad (R_g)^* = (R_g)^{-1} = R_{g^{-1}}.$$

It is easy to see

$$L_{g_1 g_2} = L_{g_1} L_{g_2} \quad , \quad R_{g_1 g_2} = R_{g_1} R_{g_2}.$$

In fact, the assignments $g \mapsto L_g$ and $g \mapsto R_g$ are the usual left and right regular representations of $\mathbf{G}$.

Now we are ready to define the relevant group von Neumann algebras:

Definition 6.1.

(i) $\mathcal{M}_l = \mathcal{M}_l(\mathbf{G})$ is the von Neumann algebra generated by $\{L_g \colon g \in \mathbf{G}\} \subseteq \mathcal{B}(\ell^2(\mathbf{G}))$.

(ii) $\mathcal{M}_r = \mathcal{M}_r(\mathbf{G})$ is the von Neumann algebra generated by $\{R_g \colon g \in \mathbf{G}\} \subseteq \mathcal{B}(\ell^2(\mathbf{G}))$.

It turns out

$$\mathcal{M}_l = (\mathcal{M}_r)' \quad , \quad \mathcal{M}_r = (\mathcal{M}_l)'.$$

so one might alternatively define $\mathcal{M}_l$ (respectively $\mathcal{M}_r$) as the algebra of $\mathbf{G}$-equivariant bounded operators on $\ell^2(\mathbf{G})$.

Also, $\mathcal{M}_l$ and $\mathcal{M}_r$ are factors if and only if all non-trivial conjugacy classes in $\mathbf{G}$ are infinite. For example, this is the case when G is a free group on at least two generators.

Finally, a trace on $\mathcal{M}_r$ (and also on $\mathcal{M}_l$) is defined by

$$\tau(A) = \langle A\delta_h, \delta_h \rangle, \tag{2}$$

for $h \in \mathbf{G}$. This is independent of the choice of h, so it is convenient to use the $h = \mathrm{id}$, the group identity.

6.2. **Hilbert G-modules, and G-dimensions.** By a Hilbert $\mathbf{G}$-module we mean a Hilbert space together with a (left) unitary action of the discrete group $\mathbf{G}$.

Definition 6.2. A free Hilbert $\mathbf{G}$-module is a Hilbert $\mathbf{G}$-module which is unitarily isomorphic to

$$\ell^2(\mathbf{G}) \otimes \mathrm{H},$$

where H is a Hilbert space with the trivial $\mathbf{G}$-action and the action of $\mathbf{G}$ on $\ell^2(\mathbf{G})$ is by left translations. In other words, the representation of $\mathbf{G}$ on $\ell^2(\mathbf{G}) \otimes \mathrm{H}$ is given by $g \mapsto L_g \otimes I$.

Let $\{u_\alpha\}_{\alpha \in J}$ be an orthonormal basis for H. Then we have an orthogonal decomposition

$$\ell^2(\mathbf{G}) \otimes \mathrm{H} = \bigoplus_{\alpha \in J} \ell^2(\mathbf{G})^{(\alpha)},$$

where $\ell^2(\mathbf{G})^{(\alpha)} = \ell^2(\mathbf{G}) \otimes u_\alpha$ is just a copy of $\ell^2(\mathbf{G})$. This means

$$\ell^2(\mathbf{G}) \otimes \mathrm{H} = \{\sum_{\alpha \in J} f_\alpha \otimes u_\alpha \colon f_\alpha \in \ell^2(\mathbf{G}), \sum_{\alpha \in J} \|f_\alpha\|^2 < +\infty\}.$$

Moreover, an orthonormal basis for $\ell^2(\mathbf{G}) \otimes \mathrm{H}$ is $\{\delta_g \otimes u_\alpha \colon g \in \mathbf{G},\, \alpha \in J\}$.

We are interested in the following von Neumann algebra on $\ell^2(\mathbf{G}) \otimes \mathrm{H}$:

$$\mathcal{M}_r(\mathbf{G}) \otimes \mathcal{B}(\mathrm{H}).$$

By combining (1) and (2) in, we obtain the following trace function.

Definition 6.3. Let $\{u_\alpha\}_{\alpha \in J}$ be an orthonormal basis for H. For every $A \in (\mathcal{M}_r(\mathbf{G}) \otimes \mathcal{B}(\mathrm{H}))^+$, define

$$\mathrm{Tr}_{\mathbf{G}}(A) = (\tau \otimes \mathrm{Tr})(A) = \sum_{\alpha \in J} \langle A(\delta_h \otimes u_\alpha), \delta_h \otimes u_\alpha \rangle,$$

for $h \in \mathbf{G}$. This is independent of the choice of h, so it is convenient to use the $h = \mathrm{id}$, the group identity. It can be checked that this is a trace function according to Definision 3.1.

Definition 6.4. A projective Hilbert $\mathbf{G}$-module is a Hilbert $\mathbf{G}$-module V which is unitarily isomorphic to a closed submodule of a free Hilbert $\mathbf{G}$-module, i.e. a closed $\mathbf{G}$-invariant subspace in *some* $\ell^2(\mathbf{G}) \otimes \mathrm{H}$.

Note that the embedding of V into $\ell^2(\mathbf{G}) \otimes \mathrm{H}$ is *not* part of the structure; only its existence is required.

For the moment, we will fix such an embedding. Let P_{V} denote the orthogonal projection from $\ell^2(\mathbf{G}) \otimes \mathrm{H}$ onto V. Note that $P_{\mathrm{V}} \in \mathcal{M}_r(\mathbf{G}) \otimes \mathcal{B}(\mathrm{H})$ because it commutes with all $L_g \otimes I$.

Definition 6.5. Let V be a projective Hilbert $\mathbf{G}$-module. Fix an embedding into a free Hilbert $\mathbf{G}$-module $\ell^2(\mathbf{G}) \otimes \mathrm{H}$, and let $P_{\mathrm{V}} \in \mathcal{M}_r(\mathbf{G}) \otimes \mathcal{B}(\mathrm{H})$ denote the orthogonal projection onto V. The $\mathbf{G}$-dimension of V is defined as

$$\dim_{\mathbf{G}}(\mathrm{V}) = \mathrm{Tr}_{\mathbf{G}}(P_{\mathrm{V}}).$$

The amazing (but elementary) fact is that $\dim_{\mathbf{G}}(\mathrm{V})$ does *not* depend on the choice of the embedding of V into a free Hilbert $\mathbf{G}$-module. Therefore it is a well-defined invariant of the projective Hilbert $\mathbf{G}$-module V. Furthermore, it satisfies the following properties:

(i) $\dim_{\mathbf{G}}(\mathrm{V}) = 0 \iff \mathrm{V} = \{0\}$.

(ii) $\dim_{\mathbf{G}}(\ell^2(\mathbf{G})) = 1$.
(iii) $\dim_{\mathbf{G}}(\ell^2(\mathbf{G}) \otimes H) = \dim_{\mathbb{C}}(H)$.
(iv) $\dim_{\mathbf{G}}(\mathrm{V}_1 \oplus \mathrm{V}_2) = \dim_{\mathbf{G}}(\mathrm{V}_1) + \dim_{\mathbf{G}}(\mathrm{V}_2)$.
(v) $\mathrm{V}_1 \subseteq \mathrm{V}_2 \Rightarrow \dim_{\mathbf{G}}(\mathrm{V}_1) \leq \dim_{\mathbf{G}}(\mathrm{V}_2)$. Equality holds here if and only if $\mathrm{V}_1 = \mathrm{V}_2$.
(vi) If $0 \to \mathrm{U} \to \mathrm{V} \to \mathrm{W} \to 0$ is a short *weakly exact sequence* of projective Hilbert $\mathbf{G}$-modules, then $\dim_{\mathbf{G}}(\mathrm{V}) = \dim_{\mathbf{G}}(\mathrm{U}) + \dim_{\mathbf{G}}(\mathrm{W})$.
(vii) V and W are *weakly isomorphic*, then $\dim_G(\mathrm{V}) = \dim_G(\mathrm{W})$.

We call a sequence of $\mathrm{U} \xrightarrow{i} \mathrm{V} \xrightarrow{p} \mathrm{W}$ of projective Hilbert $\mathbf{G}$-modules *weakly exact* at V if $\mathrm{Kernel}(p) = \mathrm{cl}(\mathrm{Image}(i))$. A map of projective Hilbert $\mathbf{G}$-modules $\mathrm{V} \to \mathrm{W}$ is a *weak isomorphism* if it is injective and has dense image.

References

[Lüc02] Wolfgang Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 44, Springer-Verlag, Berlin, 2002. MR1926649 ↑1

[Pan96] Pierre Pansu, *Introduction to L^2 Betti numbers*, Riemannian geometry (Waterloo, ON, 1993), 1996, pp. 53–86. MR1377309 ↑1

[Shu93] Mikhail A. Shubin, *von Neumann algebras and L^2-invariants*, 1993. Lecture Notes, Northeastern University. Draft of the book available at http://ium.mccme.ru/f02/vNa.html. ↑1

Cornell University, Ithaca, New York 14853-4201, USA
E-mail address: farbod@math.cornell.edu