University of Washington Math 523A Lecture 9

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1 Biased and unbiased random walks on $\mathbb{Z}$

We first discuss some ways to approach Problem 3 from the first homework assignment. Let $S_t = S_0 + \sum_{j=1}^t X_j$, where $X_j \sim \begin{cases} 1, & p \\ -1, & q = 1-p \end{cases}$ are i.i.d. random variables with $p \geq \frac{1}{2}$. Let $\mathbb{P}_k$ and $\mathbb{E}_k$ denote the probability and expectation, respectively, for the process started at $S_0 = k$. We want to show

$$\mathbb{E}_0(\tau_1) = \begin{cases} \frac{1}{2p-1}, & p > \frac{1}{2} \\ \infty, & p = \frac{1}{2}. \end{cases}$$

Biased walk:

First consider the case $p > \frac{1}{2}$. Then we can use the Optional Stopping Theorem to compute $\mathbb{E}_k(\tau_n)$ whenever $k \leq n$. It is easy to verify that $M_t := S_t - (2p-1)t$ is a martingale, and if $M_0 = S_0 = k$, then $\sup_{t \leq \tau_n} |M_t| \leq (k + \tau_n) + (2p-1)\tau_n = k + 2p\tau_n$. To apply optional stopping to M_{τ_n}, we need to verify that $\mathbb{E}_k(\tau_n) < \infty$. To prove this, consider $\tau_n \wedge N$, where $N \in \mathbb{N}$. This is a bounded stopping time, so we can apply optional stopping to obtain

$$k = \mathbb{E}_k M_0 = \mathbb{E}_k M_{\tau_n \wedge N} = \mathbb{E}_k S_{\tau_n \wedge N} - (2p-1)\mathbb{E}_k(\tau_n \wedge N).$$

Thus, $\mathbb{E}_k(\tau_n \wedge N) = \frac{1}{2p-1}[\mathbb{E}_k S_{\tau_n \wedge N} - k] \leq \frac{n-k}{2p-1}$, where the inequality $\mathbb{E}_k S_{\tau_n \wedge N} \leq n$ holds for any $N < \infty$ because if $k \leq n$, then $S_{\tau_n \wedge N} \leq n$ almost surely. Since $\tau_n \wedge N$ is nonnegative and increases a.s. to τ_n as $N \to \infty$, we have $\mathbb{E}_k \tau_n \leq \frac{n-k}{2p-1}$ by monotone convergence. Since $\mathbb{E}_k(\tau_n) < \infty$, we can now apply the Optional Stopping Theorem at τ_n to obtain

$$k = \mathbb{E}_k M_0 = \mathbb{E}_k M_{\tau_n} = \mathbb{E}_k[S_{\tau_n} - (2p-1)\tau_n] = n - (2p-1)\mathbb{E}_k(\tau_n),$$

or

$$\mathbb{E}_k(\tau_n) = \frac{n-k}{2p-1}.$$

Note that in the above calculation we used the fact that $\tau_n < \infty$ a.s. (which follows from $\mathbb{E}_k(\tau_n) < \infty$) to conclude that $S_{\tau_n} = n$ a.s.

Unbiased walk:

Now we want to show that $\mathbb{E}_0(\tau_1) = \infty$ when $p = \frac{1}{2}$. There are various ways to see this. Here are a few methods:

Method 1: Coupling. We can couple a biased and an unbiased random walk by using a single sequence $\{U_i\}_{i\geq 0}$ of i.i.d. $\text{Uniform}\big([0,1]\big)$ random variables to determine the steps for both walks simultaneously: If $U_i \leq p$ (resp. $U_i \leq \frac{1}{2}$), then the ith step of the biased walk (resp. unbiased walk) is to the right (i.e. $X_i = +1$); otherwise the ith step is to the left (i.e. $X_i = -1$).

With this coupling we always have $X_i^{(p)} \geq X_i^{(\frac{1}{2})}$, hence $S_t^{(p)} \geq S_t^{(\frac{1}{2})}$. Therefore, the biased walk is always to the right of the unbiased walk, so it must reach 1 first, i.e. $\tau_1^{(p)} \leq \tau_1^{(\frac{1}{2})}$. Thus we have $\mathbb{E}_0(\tau_1^{(\frac{1}{2})}) \geq \mathbb{E}_0(\tau_1^{(p)}) = 1/(2p-1)$ for any $p > \frac{1}{2}$, so $\mathbb{E}_0(\tau_1^{(\frac{1}{2})}) = \infty$.

<u>Note:</u> This same coupling can be used for any pair of p values, showing that $\mathbb{E}_0(\tau_1^{(p)})$ must be a monotone function of p.

Method 2: Comparison with $\tau_{\{-k,1\}}$. Based on our previous examples of applying the Optional Stopping Theorem, we know that $\mathbb{E}_0\tau_{\{-k,1\}} = k$ for a simple random walk. Clearly we have $0 \leq \tau_{\{-k,1\}} \nearrow \tau_1$ a.s. as $k \to \infty$, so $\mathbb{E}_0\tau_1 = \lim_{k\to\infty} \mathbb{E}_0\tau_{\{-k,1\}} = \infty$ by the Monotone Convergence Theorem.

Method 3: Contradiction using Optional Stopping. Since the simple random walk $\{S_t\}$ is a martingale with bounded increments, if $\mathbb{E}_0(\tau_1) < \infty$, we could apply optional stopping to obtain $\mathbb{E}_0(S_{\tau_1}) = \mathbb{E}_0(S_0) = 0$. But $S_{\tau_1} = 1$ a.s., which is a contradiction. Therefore we must have $\mathbb{E}_0(\tau_1) = \infty$.

Method 4: Reflection principle. Suppose $\{X_i\}$ are IID random variables taking values in $\{+1, -1\}$. Given a stopping time τ, the mapping

$$(X_1, X_2, \ldots, X_{\tau-1}, X_\tau, X_{\tau+1}, \ldots) \mapsto (X_1, X_2, \ldots, X_{\tau-1}, -X_\tau, -X_{\tau+1}, \ldots)$$

is an involution (i.e. it's its own inverse) on the set of sequences. If we use the right stopping time, we can use this fact to show that for a simple random walk,

$$\mathbb{P}_0(\tau_1 = k) \sim \frac{C}{k^{3/2}}, \quad k \text{ odd},$$

so τ_1 has infinite expectation. We'll see how to do this on Friday, May 1.

2 The O'Donnell–Servedio bound for randomized algorithms

Recall our setup from last time (April 17) for using randomized algorithms to compute monotone Boolean functions:

Consider the probability space $\Omega = \Omega_0^n = \{1, -1\}^n$ equipped with the probability measure $\mathbb{P} = \left(\frac{1}{2}, \frac{1}{2}\right)^n$. Suppose $f : \Omega \to \Omega_0$ is a monotone Boolean function. If $x = (x_1, \ldots, x_n)$ is a random element of Ω, the **influence** of variable j on f is

$$\begin{aligned} I_j(f) = \mathbb{E}[x_j f(x)] &= \mathbb{P}[f(x) = 1 \mid x_j = 1] - \mathbb{P}[f(x) = 1 \mid x_j = -1] \\ &\quad + \mathbb{P}[f(x) = -1 \mid x_j = -1] - \mathbb{P}[f(x) = -1 \mid x_j = 1]. \end{aligned}$$

We are interested in randomized algorithms to compute f exactly. An algorithm is specified by a random sequence of indices $k(1), k(2), \ldots, k(\tau)$ telling us which input variables to look at, where τ is the (random) running time of the algorithm, i.e. the number of steps it takes to finish computing. More explicitly, the index $k(j)$ is computed as some function of the previously revealed variables and an independent source of randomness:

$$k(j) = F_j\left(x_{k(1)}, x_{k(2)}, \ldots, x_{k(j-1)}, U_j\right),$$

where U_j is independent of $\{x, U_1, \ldots, U_{j-1}\}$. The time τ when the algorithm terminates is defined by

$$\tau = \min\left\{j : f(x) \text{ is determined by } x_{k(1)}, x_{k(2)}, \ldots, x_{k(j)}\right\}.$$

That is, if we examine more variables after time τ, the computed value of f will not change. The running time τ is a stopping time with respect to the filtration $\mathcal{F} = (\mathcal{F}_j)$, where

$$\mathcal{F}_j = \sigma\left\{U_1, \ldots, U_j\, ; x_{k(1)}, x_{k(2)}, \ldots, x_{k(j)}\right\}$$

is the information available at the jth step.

Theorem 2.1 (O'Donnell, Servedio)**.** *If τ is the running time of a randomized algorithm which computes a monotone $f : \{1, -1\}^n \to \{1, -1\}$ exactly, then*

$$\left(\sum_{j=1}^{n} I_j(f)\right)^2 \leq \mathbb{E}\tau.$$

Proof. Let $U = (U_1, \ldots, U_n)$, and let $\mathcal{L}(U)$ denote the law (i.e. distribution) of the random vector U, which we assume takes values in $[0, 1]^n$ without loss of generality. Since U is independent of x, we can work on the probability space $\Omega' = \Omega \times [0, 1]^n$ under the product measure $\nu = \mathbb{P} \otimes \mathcal{L}(U)$.

First we claim that $\mathbb{E}_\nu\left[x_j f(x) \mathbf{1}_{\{j \text{ not examined}\}}\right] = 0$. Intuitively, this says that the expected influence of variables that don't get examined is 0. Here's a proof using the reflection principle:

Note that $\{j \text{ not examined}\} = \{\tau < j\}$. Since τ is a function of x and U, and f and x_j are functions of x, we have $x_j f(x) \mathbf{1}_{\{j \text{ not examined}\}} = \varphi(x, U)$ for some (deterministic) function φ. Let $\iota : \Omega \to \Omega$ be the (random) function which flips all unexamined vertices, i.e. $x_j \mapsto -x_j$ if $j > \tau$ and otherwise x_j stays the same. Then ι is an involution (hence bijection), and $\iota(x)$ has the same law as x since $+1$ and -1 are equally likely for each bit (that is, the marginal distributions of x and $\iota(x)$ are both $\mathbb{P}$). Therefore (using the independence of x and U and the fact that ι is a measure-preserving bijection) we have

$$\mathbb{E}_\nu \varphi(x, U) = \mathbb{E}_\nu \varphi(\iota(x), U).$$

On the other hand, by the definition of τ, the unexamined bits of x do not affect the value of f, and hence the sign of φ changes if we flip all the unexamined bits in x. Therefore we have $\varphi(\iota(x), U) = -\varphi(x, U)$ and hence

$$\mathbb{E}_\nu \varphi(x, U) = -\mathbb{E}_\nu \varphi(\iota(x), U),$$

so we must have $\mathbb{E}_\nu \varphi(x, U) = 0$.

Now, since U is independent of x, and the functions f and x_j depend only on x, not U, the expectation of $x_j f(x)$ with respect to ν is the same as the expectation with respect to $\mathbb{P}$. That is,

$$I_j(f) = \mathbb{E}_{\mathbb{P}}[x_j f(x)] = \mathbb{E}_\nu[x_j f(x)].$$

Thus, using the fact that $\mathbb{E}_\nu \left[x_j f(x) \mathbf{1}_{\{j \text{ not examined}\}}\right] = 0$, we have

$$\sum_{j=1}^n I_j(f) = \mathbb{E}_\nu \sum_{j=1}^n x_j f(x) = \mathbb{E}_\nu \sum_{j=1}^n x_j f(x) \mathbf{1}_{\{j \text{ examined}\}}.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
\left(\sum_{j=1}^n I_j(f)\right)^2 &\leq \mathbb{E}_\nu f(x)^2 \cdot \mathbb{E}_\nu \left(\sum_{j=1}^n x_j \mathbf{1}_{\{j \text{ examined}\}}\right)^2 \\
&= \mathbb{E}_\nu(1) \cdot \mathbb{E}_\nu \left(x_{k(1)} + x_{k(2)} + \ldots + x_{k(\tau)}\right)^2 \\
&= 1 \cdot \mathbb{E}_\nu S_\tau^2 \\
&= \mathbb{E}_\nu \tau.
\end{aligned}$$

Here, $S_t := \sum_{j=1}^t x_{k(j)}$ has the same distribution (for $0 \leq t \leq \tau$) as a simple random walk on $\mathbb{Z}$ since the variables $\{x_{k(j)}\}_{j=1}^\tau$ are IID Unif$\{+1, -1\}$ under the measure ν. Thus, $S_t^2 - t$ is a martingale with respect to the filtration $\mathcal{F}$, so the last step follows by applying the Optional Stopping Theorem at time τ. $\square$