1 Review: Doob’s martingales on $G(n, p)$

Let $G = (V, E)$ be a graph on $n$ vertices and $f$ a function on such graphs. The vertex set of $G$ is $V = \{v_1, \ldots, v_n\}$, the edge set $E$ is a subset of $\{e_1, \ldots, e_m\}$, where $m = \binom{n}{2}$. Suppose that $G \sim G(n, p)$, i.e., the edges of $G$ are IID Bernoulli($p$). Last time we saw two special cases of Doob’s martingale process applied to $G(n, p)$, namely the edge exposure martingale

$$X_t = \mathbb{E}[f(G) \mid 1_{\{e_1 \in G\}}, \ldots, 1_{\{e_t \in G\}}],$$

and the vertex exposure martingale

$$Y_t = \mathbb{E}[f(G) \mid G|_{\{v_1, \ldots, v_t+1\}}].$$

Here $G|_{\{v_1, \ldots, v_t+1\}}$ denotes the induced subgraph on $\{v_1, \ldots, v_t+1\}$. Note that for the vertex exposure martingale $Y_t$, the vertex $v_{t+1}$ is revealed at time $t$, along with the $t$ edges (or nonedges) connecting $v_{t+1}$ to each vertex in $\{v_1, \ldots, v_t\}$.

Last time we saw that for $f = \chi$ (the chromatic number of $G$), the vertex exposure martingale $Y_t$ satisfies $|Y_t - Y_{t-1}| \leq 1$, which allowed us to apply the Hoeffding-Azuma inequality to prove that $\forall n, p$, if $G \sim G(n, p)$, then $\mathbb{P}(|\chi(G) - \mathbb{E}\chi(G)| > a\sqrt{n}) \leq 2e^{-a^2/2}$. We now prove an even sharper concentration for small enough $p$.

2 A four-value concentration for $\chi$

Theorem 2.1 (Shamir-Spencer ’87). If $p = n^{-a}$ for some $a > \frac{5}{6}$, then $\exists c = c(n, p)$ such that

$$c \leq \chi(G(n, p)) \leq c + 3$$

asymptotically almost surely (a.a.s.).
Remarks:
- (Luczak ‘91) improved Theorem 2.1 to a 2-value concentration.
- (Alon, Krivelevich ‘97) proved a 2-value concentration for \( a > \frac{1}{2} \).
- (Achlioptas, Naor ‘03) showed where the two-value concentration is located for \( G(n, d/n) \), \( d \) a constant.
- For \( p = p(n) \) in the range \( [n^{-1/2}, 1) \) it is still unknown what the best concentration bound is. It is even open whether there is an \( n^{1/2-\epsilon} \) concentration for \( p = \frac{1}{2} \) and \( \epsilon > 0 \).

To prove Theorem 2.1 we will use the following lemma.

Lemma 2.2. Let \( G = (V, E) \) be \( \sim G(n,p) \), \( c > 0 \) a constant. For \( a > \frac{5}{6} \), \( p = n^{-a} \), the following holds with high probability (i.e. tending to 1 as \( n \to \infty \)): Any \( S \subseteq V \) of size \( |S| \leq c\sqrt{n} \) is 3-colorable.

Proof. Let \( S \) be a “bad” subset (i.e. \( |S| \leq c\sqrt{n} \) but \( S \) is not 3-colorable) which is minimal with respect to size (i.e. there is no bad subset smaller than \( S \)).

Observation: The induced subgraph on \( S \), \( G|_S \), has minimum degree \( \geq 3 \). Why? Suppose some \( u \in S \) has at most two neighbors in \( S \). Remove \( u \). By minimality, we can color \( S \setminus \{u\} \) using 3 colors. Now return \( u \). Then \( u \) can be colored using one of the 3 colors we used for \( S \setminus \{u\} \) since \( u \) has at most two neighbors in \( S \). But then \( S \) is 3-colorable, which is a contradiction.

Since the minimum degree of \( S \) is at least 3, we have \( |E(G|_S)| \geq \frac{3}{2} |S| \) (by counting the edges at each vertex). We will show that this has small probability, i.e.

\[
P(\exists S \subseteq V \text{ of size } |S| \leq c\sqrt{n} \text{ with } |E(G|_S)| \geq \frac{3}{2} |S|) = o(1)
\]
as \( n \to \infty \). Let \( B \) denote the above event. Using the union bound and the inequality \( \binom{a}{b} \leq \left( \frac{ea}{b} \right)^b \), we have

\[
P(B) \leq \sum_{s=4}^{c\sqrt{n}} \binom{n}{s} \left( \frac{s}{2a} \right)^{3a/2} p^{3a/2}
\]
union bound, neglect the \((1-p)\) term

\[
\leq \sum_{s=4}^{c\sqrt{n}} \left[ e^{cn/s} \right]^s \left[ \frac{ps(s-1)e^{2s/2}}{2 \cdot 3s/2} \right]^{3a/2}
\]
using \( \binom{a}{b} \leq \left( \frac{ea}{b} \right)^b \)

\[
= \sum_{s=4}^{c\sqrt{n}} \left[ c' \sqrt{s} \cdot n^{3s/2} \right]^s
\]
\[
= \sum_{s=4}^{c\sqrt{n}} \left[ c'' \cdot n^{3s/2} \cdot n^{-3a/2} \right]^s
\]
since \( s \leq c\sqrt{n} \) and \( p \leq n^{-a} \)

\[
= o(1)
\]
since \( \frac{3a}{2} > \frac{3}{2} \cdot \frac{5}{6} = \frac{5}{4} \).
Proof of Theorem 2.1. Let \( \epsilon \in (0, 1) \), and let \( c \in \mathbb{N} \) be minimal such that

\[
P[\chi(G(n, p)) \leq c] > \epsilon. \tag{2.1}
\]

Note that \( c \) exists because \( h(c) = P[\chi(G(n, p)) \leq c] \) is an increasing function of \( c \). E.g. \( h(1) = (1 - p)^{\binom{n}{2}} \), \( h(2) = P(G \text{ is bipartite}) \), \ldots , \( h(n) = 1 \). Observe also that \( c \) is the largest integer such that \( P[\chi(G(n, p)) < c] \leq \epsilon \).

Let \( G \sim G(n, p) \), and let \( y = y(G) \) be the size of the smallest possible set \( S \) such that \( \chi(G \setminus S) \leq c \). Then by the definition (2.1) of \( c \), we have

\[
P(y = 0) = P[\chi(G) \leq c] > \epsilon. \tag{2.2}
\]

Now choose \( \lambda \) so that \( 2e^{-\lambda^2/2} < \epsilon \). We will use the vertex exposure martingale on \( y \). The value of \( y \) changes by \( \leq 1 \) as we expose each new vertex, so we can apply Hoeffding-Azuma:

\[
P\left(|y - \mathbb{E}y| > \lambda\sqrt{n}\right) \leq 2e^{-\lambda^2/2} < \epsilon. \tag{2.3}
\]

The inequalities (2.2) and (2.3) together imply that \( \mathbb{E}y \leq \lambda\sqrt{n} \). (Otherwise we would have \( P(y \leq 2\lambda\sqrt{n}) \geq P(y = 0) + P(\mathbb{E}y - \lambda\sqrt{n} < y < \mathbb{E}y + \lambda\sqrt{n}) > \epsilon + (1 - \epsilon) = 1 \). Combining this with (2.3) we get

\[
P(y > 2\lambda\sqrt{n}) < \epsilon. \tag{2.4}
\]

Thus, by (2.4) and the definition of \( y \), with probability \( 1 - \epsilon \) we can find a set \( S \) of size \( |S| \leq 2\lambda\sqrt{n} \) such that \( G \setminus S \) is \( c \)-colorable. Then, by Lemma 2.2 we can color \( S \) with 3 colors with probability \( 1 - \epsilon \) (for large enough \( n \)). Hence, \( P[\chi(G) \leq c + 3] > 1 - 2\epsilon \). On the other hand, from the definition of \( c \) we have \( P[\chi(G) < c] \leq \epsilon \), so for large enough \( n \),

\[
P(c \leq \chi(G) \leq c + 3) \geq 1 - 3\epsilon.
\]

\[\square\]