University of Washington Math 523A Lecture 5

LECTURER: EYAL LUBETZKY

Monday, April 13, 2009

1 Review: Doob's martingales on $\mathcal{G}(n,p)$

Let $G = (V, E)$ be a graph on n vertices and f a function on such graphs. The vertex set of G is $V = \{v_1, \ldots, v_n\}$, the edge set E is a subset of $\{e_1, \ldots, e_m\}$, where $m = \binom{n}{2}$. Suppose that $G \sim \mathcal{G}(n,p)$, i.e. the edges of G are IID Bernoulli(p). Last time we saw two special cases of Doob's martingale process applied to $\mathcal{G}(n,p)$, namely the **edge exposure martingale**

$$X_t = \mathbb{E}[f(G) \mid \mathbf{1}_{\{e_1 \in G\}}, \ldots, \mathbf{1}_{\{e_t \in G\}}],$$

and the **vertex exposure martingale**

$$Y_t = \mathbb{E}[f(G) \mid G|_{\{v_1,\ldots,v_{t+1}\}}].$$

Here $G|_{\{v_1,\ldots,v_{t+1}\}}$ denotes the induced subgraph on $\{v_1, \ldots, v_{t+1}\}$. Note that for the vertex exposure martingale Y_t, the vertex v_{t+1} is revealed at time t, along with the t edges (or nonedges) connecting v_{t+1} to each vertex in $\{v_1, \ldots, v_t\}$.

Last time we saw that for $f = \chi$ (the chromatic number of G), the vertex exposure martingale Y_t satisfies $|Y_t - Y_{t-1}| \leq 1$, which allowed us to apply the Hoeffding-Azuma inequality to prove that $\forall n, p$, if $G \sim \mathcal{G}(n,p)$, then $\mathbb{P}\left(|\chi(G) - \mathbb{E}\chi(G)| > a\sqrt{n}\right) \leq 2e^{-a^2/2}$. We now prove an even sharper concentration for small enough p.

2 A four-value concentration for χ

Theorem 2.1 (Shamir-Spencer '87)**.** *If $p = n^{-a}$ for some $a > \frac{5}{6}$, then $\exists c = c(n,p)$ such that*

$$c \leq \chi(\mathcal{G}(n,p)) \leq c + 3$$

asymptotically almost surely (a.a.s.).

Remarks:

- (Luczak '91) improved Theorem 2.1 to a 2-value concentration.
- (Alon, Krivelevich '97) proved a 2-value concentration for $a > \frac{1}{2}$.
- (Achlioptas, Naor '03) showed where the two-value concentration is located for $\mathcal{G}(n, d/n)$, d a constant.
- For $p = p(n)$ in the range $[n^{-1/2}, 1)$ it is still unknown what the best concentration bound is. It is even open whether there is an $n^{1/2-\epsilon}$ concentration for $p = \frac{1}{2}$ and $\epsilon > 0$.

To prove Theorem 2.1 we will use the following lemma.

Lemma 2.2. *Let $G = (V, E)$ be $\sim \mathcal{G}(n, p)$, $c > 0$ a constant. For $a > \frac{5}{6}$, $p = n^{-a}$, the following holds with high probability (i.e tending to 1 as $n \to \infty$): Any $S \subseteq V$ of size $|S| \leq c\sqrt{n}$ is 3-colorable.*

Proof. Let S be a "bad" subset (i.e. $|S| \leq c\sqrt{n}$ but S is not 3-colorable) which is minimal with respect to size (i.e. there is no bad subset smaller than S).

Observation: The induced subgraph on S, $G|_S$, has minimum degree ≥ 3. Why? Suppose some $u \in S$ has at most two neighbors in S. Remove u. By minimality, we can color $S \setminus \{u\}$ using 3 colors. Now return u. Then u can be colored using one of the 3 colors we used for $S \setminus \{u\}$ since u has at most two neighbors in S. But then S is 3-colorable, which is a contradiction.

Since the minimum degree of S is at least 3, we have $|E(G|_S)| \geq \frac{3}{2}|S|$ (by counting the edges at each vertex). We will show that this has small probability, i.e.

$$\mathbb{P}\left(\exists S \subseteq V \text{ of size } |S| \leq c\sqrt{n} \text{ with } |E(G|_S)| \geq \frac{3}{2}|S|\right) = o(1)$$

as $n \to \infty$. Let B denote the above event. Using the union bound and the inequality $\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$, we have

$$\begin{aligned}
\mathbb{P}(B) &\leq \sum_{s=4}^{c\sqrt{n}} \binom{n}{s} \binom{\binom{s}{2}}{\frac{3s}{2}} p^{\frac{3s}{2}} && \text{union bound, neglect the } (1-p) \text{ term} \\
&\leq \sum_{s=4}^{c\sqrt{n}} \left[\frac{en}{s}\right]^s \cdot \left[\frac{ps(s-1)e}{2 \cdot \frac{3s}{2}}\right]^{\frac{3s}{2}} && \text{using } \binom{a}{b} \leq \left(\frac{ea}{b}\right)^b \\
&= \sum_{s=4}^{c\sqrt{n}} \left[c' \sqrt{s} \cdot n \cdot p^{\frac{3}{2}}\right]^s \\
&= \sum_{s=4}^{c\sqrt{n}} \left[c'' \cdot n^{\frac{5}{4}} \cdot n^{-\frac{3}{2}a}\right]^s && \text{since } s \leq c\sqrt{n} \text{ and } p \leq n^{-a} \\
&= o(1) && \text{since } \frac{3}{2}a > \frac{3}{2} \cdot \frac{5}{6} = \frac{5}{4}.
\end{aligned}$$

□

Proof of Theorem 2.1. Let $\epsilon \in (0,1)$, and let $c \in \mathbb{N}$ be minimal such that

$$\mathbb{P}\left[\chi(\mathcal{G}(n,p)) \leq c\right] > \epsilon. \tag{2.1}$$

Note that c exists because $h(c) = \mathbb{P}\left[\chi(\mathcal{G}(n,p)) \leq c\right]$ is an increasing function of c. E.g. $h(1) = (1-p)^{\binom{n}{2}}$, $h(2) = \mathbb{P}(G \text{ is bipartite}), \ldots, h(n) = 1$. Observe also that c is the largest integer such that $\mathbb{P}\left[\chi(\mathcal{G}(n,p)) < c\right] \leq \epsilon$.

Let $G \sim \mathcal{G}(n,p)$, and let $y = y(G)$ be the size of the smallest possible set S such that $\chi(G \setminus S) \leq c$. Then by the definition (2.1) of c, we have

$$\mathbb{P}(y = 0) = \mathbb{P}[\chi(G) \leq c] > \epsilon. \tag{2.2}$$

Now choose λ so that $2e^{-\lambda^2/2} < \epsilon$. We will use the vertex exposure martingale on y. The value of y changes by ≤ 1 as we expose each new vertex, so we can apply Hoeffding-Azuma:

$$\mathbb{P}\left(|y - \mathbb{E}y| > \lambda\sqrt{n}\right) \leq 2e^{-\lambda^2/2} < \epsilon. \tag{2.3}$$

The inequalities (2.2) and (2.3) together imply that $\mathbb{E}y \leq \lambda\sqrt{n}$. (Otherwise we would have $\mathbb{P}(y \leq 2\lambda\sqrt{n}) \geq \mathbb{P}(y = 0) + \mathbb{P}(\mathbb{E}y - \lambda\sqrt{n} < y < \mathbb{E}y + \lambda\sqrt{n}) > \epsilon + (1-\epsilon) = 1$.) Combining this with (2.3) we get

$$\mathbb{P}(y > 2\lambda\sqrt{n}) < \epsilon. \tag{2.4}$$

Thus, by (2.4) and the definition of y, with probability $> 1-\epsilon$ we can find a set S of size $|S| \leq 2\lambda\sqrt{n}$ such that $G \setminus S$ is c-colorable. Then, by Lemma 2.2 we can color S with 3 colors with probability $> 1-\epsilon$ (for large enough n). Hence, $\mathbb{P}[\chi(G) \leq c+3] > 1-2\epsilon$. On the other hand, from the definition of c we have $\mathbb{P}[\chi(G) < c] \leq \epsilon$, so for large enough n,

$$\mathbb{P}(c \leq \chi(G) \leq c+3) \geq 1 - 3\epsilon.$$

□