University of Washington Math 523A Lecture 3

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1 Optional Stopping

Problems:

1. Suppose $\{S_t\}_{t\geq 0}$ is simple random walk on Z, i.e. $P(S_{t+1} - S_t = 1|S_t) = P(S_{t+1} - S_t = -1|S_t) = 1/2$, Find $E_k\tau_{\{0,n\}}$, where $\tau_A = min\{t : S_t \in A\}$. Here E_k means the random walk start at location k.

2. For $\{Y_t\}$ simple random walk on $[0, n]$, $E_k\tau_n =?$. Here the random walk reflects with probability 1 when at location 0 and n.

3. Find $E_k(\tau_n|\tau_n < \tau_0)$ for $\{Y_t\}$ or $\{S_t\}$. Note that this expectation is the same for $\{Y_t\}$ or $\{S_t\}$.

Definition 1.1. *Given a filtration $\{F_t\}$, a sequence of random variable $\{X_t\}$ is called **Submartingale** if*

(i) $E|X_t| < \infty$.

(ii) $E(X_{t+1}|F_t) \geq X_t$.

Theorem 1.1. *Suppose $\{X_t\}_{t\geq 0}$ takes values in (a, b) where we allow $a = -\infty$ or $b = \infty$. $\psi : (a, b) \to R$ is a convex function. If $\{X_t\}_{t\geq 0}$ is an $\{F_t\}_{t\geq 0}$ martingale then $\{\psi(X_t)\}_{t\geq 0}$ is a submartingale. Moreover, if we assume in addition that ψ is increasing, then the conclusion still holds when $\{X_t\}_{t\geq 0}$ is a submartingale.*

Proof. This follows from conditional expectation case of Jensen's inequality. For any random variable X, since ψ is convex, we may take $y = cx + b \leq \psi(x)$ for any $x \in (a, b)$ but $c(EX) + d = \psi(EX)$. So $E\psi(X) \geq E(cX + b) = cEX + b = \psi(EX)$. Here we can take $c = \psi_t'(EX)$ and $d = \psi(EX) - c(EX)$ if ψ is differentiable.

In conditional version, $E(\psi(X_{t+1})|F_t) \geq E(cX_{t+1} + d|F_t) = cE(X_{t+1}|F_t) + d = cX_t + d = \psi(X_t)$. If ψ is increasing, then c is positive, so the inequality still hold in case of submartingale.Note that in this case c and d are random so we have to verify that they are measurable w.r.t F_t. This can be observed by noting $c = \psi_t'(X_t)$ and $d = \psi(X_t) - cX_t$. $\square$

Definition 1.2. *$\tau : \Omega \to \{0, 1, ...\} \cup \{\infty\}$ is a **Stopping Time** for $\{F_t\}$ if for any $t \geq 0$, $\{\tau \leq t\} \in F_t$. Note that this is equivalent to for any $t \geq 0$, $\{\tau = t\} \in F_t$.*

Now we state the **Optional Stopping Theorem** in bounded case.

Theorem 1.2. *Suppose $\{X_t\}$ is an $\{F_t\}$ submartingale and $\sigma \leq \tau$ are both $\{F_t\}$ stopping times. $P(\tau \leq M) = 1$ for some M constant, then $EX_\sigma \leq EX_\tau$.*

Proof. We will show $EX_{\sigma\Lambda k} \leq EX_{\tau\Lambda k}(\star)$ by induction on k. Since τ is bounded by M, this will imply the theorem.

When $k = 0$, it is clear.

The induction step: Assuming $(\star)$ for k, we have

$$\begin{aligned} & E(X_{\tau\Lambda(k+1)} - X_{\sigma\Lambda(k+1)}) - E(X_{\tau\Lambda k} - X_{\tau\Lambda k}) \\ = \;& E(X_{\tau\Lambda(k+1)} - X_{\tau\Lambda k}) - E(X_{\sigma\Lambda(k+1)} - X_{\sigma\Lambda k}) \\ = \;& E(X_{k+1} - X_k)(1_{\{\tau \geq k+1\}} - 1_{\{\sigma \geq k+1\}}) \\ = \;& E(X_{k+1} - X_k)(1_{\{\sigma \leq k\}} - 1_{\{\tau \leq k\}} \\ = \;& E(X_{k+1} - X_k)1_A. \end{aligned}$$

where $A = \{\sigma \leq k\} \backslash \{\tau \leq k\} \in F_k$. So we have:

$$E[(X_{k+1} - X_k)1_A|F_k] = E(X_{k+1} - X_k|F_k)1_A \geq 0.$$

$\square$

In optional stopping theorem, we can replace $\tau \leq M$ by one of the following:

(i) $\sup_{j\leq\tau} |X_j| \leq C$.

(ii) More generally, $\sup_{j\leq\tau} |X_j| \leq \varphi \in L^1$(Dominate convergence).

(iii) $|X_{j+1} - X_j| \leq C$ for any $j \geq 0$ and $E\tau < \infty$. In this case, for any $k \leq \tau$, $|X_k| \leq |X_0| + \sum_{j=1}^k |X_j - X_{j-1}| \leq X_0 + \tau C$.

Now we use the optional stopping theorem to solve the problems in the beginning of the section.

First note for simple random walk $\{S_t\}$, $\{S_t\}$ itself is a martingale. Note also $\sup_{j \le \tau_{\{0,n\}}} |S_j| \le n$. Thus we can apply optional stopping theorem, get $ES_0 = ES_{\tau_{\{0,n\}}}$, i.e.

$$k = ES_0 = ES_{\tau_{\{0,n\}}} = nP(\tau_n < \tau_0)$$

this means $P(\tau_n < \tau_0) = k/n$.

To get $E\tau_{\{0,n\}}$ we need a somehow more complicated martingale. Consider $\{M_t = S_t^2 - t\}$ where $\{S_t\}$ is the usual simple random walk. Note that:

$$\begin{aligned} & E(S_{t+1}^2|F_t) = E((S_t + X_{t+1})^2|F_t) \\ = \quad & E(S_t^2 + 2S_tX_{t+1} + X_{t+1}^2|F_t) = S_t^2 + 2S_tE(X_{t+1}) + E(X_{t+1}^2) = S_t^2 + 1. \end{aligned}$$

This suggests that $\{M_t\}$ is a martingale. To apply optional stopping we need to prove $E\tau < \infty$ where $\tau = \tau_{\{0,n\}}$. To prove this we first consider $\tau \wedge N$ where N is an integer. This is a bounded stopping time. Applying optional stopping we get $k^2 = EM_0 = ES_{\tau\wedge N}^2 - E\tau \wedge N$. Thus $E\tau \wedge N = ES_{\tau\wedge N}^2 - k^2 \le n^2 - k^2$. Note that $0 \le \tau \wedge N \uparrow \tau$ as $N \to \infty$. Applying Monotone convergence theorem we get $E\tau \le n^2 - k^2$.

Now we have $\sup_{j \le \tau_{\{0,n\}}} |M_j| \le n^2 + \tau$ where is integrable, which satisfies (ii) of conditions that can apply optional stopping. This leads to

$$k^2 = EM_0 = EM_\tau = ES_\tau^2 - E\tau = n^2P(\tau_n < \tau_0) - E\tau = nk - E\tau.$$

which means $E\tau_{\{0,n\}} = k(n-k)$.