University of Washington Math 523A
Lectures 12 and 13

LECTURER: YUVAL PERES

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1 More application of exploration process

We have showed that for bond percolation on b-ary tree with $P = 1/b$,

$$P(|\mathcal{C}(root)| \geq k) \leq \frac{6}{\sqrt{k}}$$

, by using exploration process. In particular, the set of active set A_t where $A_0 = \emptyset$ satisfies

$$|A_t| = \begin{cases} 0, & if |A_{t-1}| = 0 \\ |A_{t-1}| - 1 + Y_t, & otherwise \end{cases}$$

, where Y_t and $|A_t|$ are independent, Y_t is the number of open neighbors of w_t.

Now we suppose $d = b + 1$, G is a graph with n vertices and all degrees of them are less than or equal to d.

Let $\widetilde{A_t}$ denote active set in G started with $\widetilde{A_0} = \{v_0, v_0'\}$ where $\{v_0, v_0'\}$ are two neighbor vertices in G.

Consider the exploration process as before. $\widetilde{A_0} = \{v_0, v_0'\}$, and

$$\widetilde{\tau} = min\{t : |\widetilde{A_t}| = 0\} = |\mathcal{C}(v_0) \cup \mathcal{C}(v_0')|$$

. Note that for a suitable coupling, we can always have

$$|\widetilde{A_t}| \leq 2|A_t|$$

, where A_t is the active set of exploration process in b-ary tree. This can be obtained by induction: if $|\widetilde{A_{t-1}}| = 0$ we are done, otherwise, $\widetilde{w_t} \in \widetilde{A_{t-1}}$ has at most $d-1$ neutral neighbors while $w_t \in A_t$ has exactly $d-1$ neutral neighbors.

So we obtain

$$P(\widetilde{\tau} \geq k) \leq P(\tau \geq k) \leq \frac{6}{\sqrt{k}}$$

.

Theorem 1.1. *(Nachmias, Peres 05) Let G be a graph with n vertices and all degrees less than or equal to d, Let $|C_1| \geq |C_2| \geq ...$ be the sizes of connected components for percolation on G with $p = 1/(d-1)$, then*

$$P(|C_1| \geq An^{2/3}) \leq \frac{const}{A^{3/2}}$$

.

Remark: A special case of this theorem is when G is the complete graph with n vertices, which is the classical Erdos-Renyi random graph $G(n, 1/n)$.

Proof. Let

$$N_k := \#\{v \in G : |\mathcal{C}(v)| \geq k\} = \sum_{v \in G} 1_{\{|\mathcal{C}(v)| \geq k\}}$$

. Then we have

$$EN_k \leq n \frac{6}{\sqrt{k}}$$

. So

$$P(|C_1| \geq k) \leq P(N_k \geq k) \leq \frac{EN_k}{k} \leq \frac{6n}{k^{3/2}}$$

. Now we set $k = An^{2/3}$. □

Open Problem: Can we get sharper upper bound on $P(|C_1| \geq An^{2/3})$? In particular, prove or disprove

$$P(|C_1| \geq An^{2/3}) \leq exp(-CA^3)$$

. This is known to be true when G is complete graph.

2 Martingale upcrossing inequality

Definition 2.1. *Given $a < b$ and a supermartingale $\{X_t\}$, define*

$$\sigma_1 = min\{t \geq 0 : X_t \leq a\}, \tau_1 = min\{t \geq 0 : X_t \geq b\}$$

, and σ_j and τ_j inductively by

$$\sigma_{j+1} = min\{t \geq \tau_j : X_t \leq a\}, \tau_{j+1} = min\{t \geq \sigma_{j+1} : X_t \geq b\}$$

. Now define

$$U_n[a,b] := max\{j : \tau_j \leq n\}$$

to be the number of upcrossing of the interval $[a,b]$ for process $\{X_t\}$ before time n.

Theorem 2.1. *(Doob's upcrossing inequality) For the supermartingale $\{X_t\}$,*

$$(b-a)EU_n[a,b] \leq E(a-X_n)_+ - E(a-X_0)_+$$

, where

$$Z_+ = \begin{cases} 0 & if Z \leq 0 \\ Z & if Z > 0 \end{cases}$$

.

Proof. we define $Y_t := \sum_{l=1}^{t} \psi_l (X_l - X_{l-1})$ where

$$\psi_l = \begin{cases} 1 & \sigma_j \leq l-1 \leq \tau_j \\ 0 & otherwise \end{cases}$$

. Note that ψ_l is previsible, i.e. ψ_l is F_{l-1} measurable. Note also that under our definition, Y_t is a supermartingale. Then we have

$$Y_n \geq X_{\tau_1} - X_{\sigma_1} + ... + X_{\tau_k} - X_{\sigma_k} + X_n - X_{\sigma_{k+1}} 1_{\{X_n \leq a\}}$$

where $k = U_n[a,b]$. Note that $X_n - X_{\sigma_{k+1}} 1_{\{X_n \leq a\}} \geq (X_n - a) \wedge 0$, so

$$Y_n \geq (a-X_0)_+ + (b-a)U_n[a,b] + (X_n - a) \wedge 0 = (a-X_0)_+ + (b-a)U_n[a,b] + (a-X_n)_+$$

. Since $Y_0 = 0$, so we have

$$0 = EY_0 \geq EY_n \geq (b-a)EU_n[a,b] + E(a-X_0)_+ - E(a-X_n)_+$$

which imply

$$EU_n[a,b] \leq \frac{E(a-X_0)_+ - E(a-X_n)_+}{b-a}$$

. □

Corollary 2.2. *(martingale convergence theorem) X_n be a supermartingale(or submartingale) with $sup_n E|X_n| \leq M < \infty$, then X_n converges almost surely to an integrable limit.*

Proof. Let $U_\infty[a,b] := \lim_{n\to\infty} U_n[a,b]$. Doob's upcrossing theorem implies that for any fixed $a < b$, $EU_\infty[a,b] < \infty$. so we get

$$P(U_\infty[a,b] < \infty \; for\ any\ rational\ numbers\ a < b,) = 1$$

, since the choice of rational numbers $a < b$ is countable. This implies

$$P(liminf\, X_n = limsup\, X_n) = 1$$

, which means $lim\, X_n$ exists almost surely. Now by Fatou's Lemma,

$$E\, liminf|X_n| \leq liminf E|X_n| \leq M$$

, so $liminf|X_n|$ is integrable, i.e. $lim|X_n|$ is integrable. In particular, $lim\, X_n$ is finite almost surely. □

In the convergence theorem, we have $E|lim\, X_n| \leq M < \infty$, but the convergence need not hold in L^1, so we might have $EX_n \nrightarrow E(lim\, X_n)$.

Example: consider the simple random walk on Z. Let τ to be the hitting time of 1, then $X_{n\wedge\tau}$ as a martingale converge almost surely to 1 while $0 = EX_{n\wedge\tau} \nrightarrow E1 = 1$. So it suffices to check $sup_n E|X_{n\wedge\tau}| < \infty$.

Example(Double or Nothing): Let

$$Y_j = \begin{cases} 2 & with\ probability\ 1/2 \\ 0 & with\ probability\ 1/2 \end{cases}$$

being i.i.d sequence. Let $X_n := \prod_{i=1}^n Y_i$. It is easy to verify that X_n is a non-negative martingale with the property $X_n \to 0$ almost surely. But we have $EX_n = 1$ for all n.

Theorem 2.3. *(L^2-bounded martingale convergence theorem) Suppose S_n is a martingale with $ES_n^2 \leq B < \infty$ for any n, then S_n converges almost surely and in L^2 to the same limit.*

Proof. First note L^2 boundedness implies L^1 boundedness, so the previous theorem implies S_n converges almost surely to a finite limit S. The L^2 convergence follows from

Cauchy criterion and orthogonality: write $X_j = S_j - S_{j-1}$, then we have

$$B \geq ||S_n||_2^2 = \sum_{j=1}^{n} ||X_j||_2^2$$

. So $||S_n - S_m||_2^2 = \sum_{j=n+1}^{m} ||X_j||_2^2$, which means S_n is a Cauchy series in L^2 space, so $S_n \to \widetilde{S}$ in L^2.

Now it remains to show S and $\widetilde{S}$ are the same. This follows from the fact that almost sure convergence and L^2 convergence both imply convergence in probability. We have

$$P(|S - \widetilde{S}| > 2\epsilon) \leq P(|S_n - S| > \epsilon) + P(|S_n - \widetilde{S}| > \epsilon) \to 0$$

as $\epsilon \to 0$. □

Using exactly the same proof, we can generalize the result:

Corollary 2.4. *If X_n are independent variable with $EX_n = 0$ and $\sum Var(X_n) < \infty$, then $\sum X_j$ converges almost surely and in L^2.*

Doob's upcrossing inequality is pretty sharp in some examples.

Example Consider the simple random walk on Z. By Doob's upcrossing inequality, we have

$$EU_n[0,1] \leq E(-S_n)_+ = E(S_n)_+ = \frac{1}{2}E|S_n|$$

. By central limit theorem,

$$\frac{E(S_n)_+}{\sqrt{n}} \to \frac{1}{\sqrt{2\pi}} \int_0^\infty x\, exp(-\frac{x^2}{2}) dx = \frac{1}{\sqrt{2\pi}}$$

. On the other hand, we can compute explicitly by

$$\begin{aligned} EU_n[0,1] &= \sum_{k=0}^{n-1} \frac{1}{2} P(S_k = 0) \\ &= \frac{1}{2} \sum_{l<n/2} \binom{2l}{l} 2^{-2l} \sim \frac{1}{2} \sum_{l<n/2} \frac{1}{\sqrt{\pi l}} \\ &\sim \frac{1}{2\sqrt{\pi}} \int_1^{n/2} \frac{1}{\sqrt{x}} dx = \frac{1}{\sqrt{2\pi}} \sqrt{n}. \end{aligned}$$

3 Doob's maximal inequality in L^p

Theorem 3.1. *: $\{X_k\}$ being submartingale with $X_k \geq 0$. Let $\widetilde{X_n} := max_{1\leq k\leq n} X_k$. Then we have $||\widetilde{X_n}||_p \leq q||X_n||_p$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.*

Remark: This can be extended to martingales that are not necessarily ≥ 0 by considering submartingale $|X_n|$.

Proof. we have

$$
\begin{aligned}
E\widetilde{X_n}^p &= E\int_0^{\widetilde{X_n}} pr^{p-1}dr \\
&= E\int_0^{\infty} 1_{\{r\leq \widetilde{X_n}\}} pr^{p-1}dr \\
&= \int_0^{\infty} pr^{p-1}P(\widetilde{X_n} \geq r)dr \\
&\leq \int_0^{\infty} pr^{p-2}E(X_n 1_{\{r\leq \widetilde{X_n}\}})dr \\
&= pE(X_n \int_0^{\widetilde{X_n}} r^{p-2}dr) \\
&= \frac{p}{p-1}E(X_n \widetilde{X_n}^{p-1}) \\
&\leq q||X_n||_p||\widetilde{X_n}^{p-1}||_q
\end{aligned}
$$

The last step follows from Holder's inequality. Thus we have

$$(E\widetilde{X_n}^p)^{1/p} \leq q||X_n||_p$$

which means $||\widetilde{X_n}||_p \leq q||X_n||_p$. □