University of Washington Math 523A
Lectures 10 and 11

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1 reflection principle

we wish to compute the distribution of τ_1 where the simple random walk start from zero.

note that we have:

$$\begin{aligned} P_0(\tau_1 = k) &= P_0(X_0 = -1, S_{k-1} = 0, X_k = 1) - P_0(X_0 = -1, S_{k-1} = 0, X_k = 1, \tau_1 < k) \\ &= \frac{1}{4}P(S_{k-1} = 0 | S_1 = -1) - \frac{1}{4}P(\tau_1 < k, S_{k-1} = 0 | S_1 = -1) \end{aligned}$$

In order to compute the later probability, we consider the reflected mapping:

$$\varphi : (X_1, X_2, ..., X_{k-1}) \longmapsto (X_1, X_2, ..., X_{\tau_1}, -X_{\tau_1+1}, ..., -X_{k-1} := (Y_1, ..., Y_{k-1})$$

Note φ is a bijection and is identity if $\tau_1 \geq k$.

φ map the event $\{X_1 = -1, \tau_1 < k, S_{k-1} = 0\}$ to $\{Y_1 = -1, \tau_1 < k, \widetilde{S}_{k-1} = 2\}$ which is $\{Y_1 = -1, \widetilde{S}_{k-1} = 2\}$.

Thus we have:

$$\begin{aligned} P_0(\tau_1 = k) &= \frac{1}{4}P(S_{k-1} = 0 | S_1 = -1) - \frac{1}{4}P(\tau_1 < k, S_{k-1} = 0 | S_1 = -1) \\ &= \frac{1}{4}P(S_{k-1} = 0 | S_1 = -1) - \frac{1}{4}P(S_{k-1} = 2 | S_1 = -1) \\ &= \frac{1}{4}(\frac{1}{2})^{k-2}\binom{k-2}{\frac{k-1}{2}} - \frac{1}{4}(\frac{1}{2})^{k-2}\binom{k-2}{\frac{k+1}{2}} \end{aligned}$$

which can be estimated using Stirling's Formula. Finally we get $P_0(\tau_1 = k) \sim \frac{C}{k^{\frac{3}{2}}}$.

Corollary 1.1. *For simply random walk start from zero, we have $E_0\tau_1 = \infty$.*

2 Probability for a Martingale to stay positive

Our next goal is to estimate the probability that a Martingale stay positive for k steps.

For simple random walk(Which is a Martingale), we have:

$$\begin{aligned}
& P_1(S_t > 0, t = 1, 2, ..., k) \\
= \; & P_1(S_k > 0) - P_1(S_k > 0, \tau_0 < k) \\
= \; & P_1(S_k > 0) - P_1(S_k < 0) \\
= \; & P_1(S_k > 0) - P_1(S_k > 2) \\
= \; & P_1(S_k = 1 \, or \, 2) \\
= \; & 2^{-k}\binom{k}{k/2} \sim \frac{1}{\sqrt{\pi k/2}}
\end{aligned}$$

Now we consider general Martingale $\{M_t\}_{t\geq 0}$ with $M_0 = 1$ and $M_t \geq 0$ for any t.

We assume that:

(i) $|M_{t+1}| \leq D|M_t|$ for some $D \geq 1$ when $M_t \neq 0$. (this can be weakened.)

(ii) $E((M_{t+1} - M_t)^2|F_t) \geq \sigma^2 > 0$ for any t when $M_t \neq 0$.

The example of simple random walk may suggest the probability that M_t stay positive for k steps is of order $k^{1/2}$. In fact this is true:

Theorem 2.1. *Under previous assumption, we have $P(M_t > 0, t = 1, ..., k) \leq \frac{C(D,\sigma)}{\sqrt{k}}$ where $C(D, \sigma)$ is a constant related to D and σ.*

To prove the theorem we let $\tau = \tau_{\{0,[h,\infty]\}} \wedge k = min\{t : M_t = 0 \, or \, M_t \geq h\} \wedge k$.

By Markov inequality and optional stopping we have $hP(M_\tau \geq h) \leq EM_\tau = 1$.

Claim: If $Y_t := M_t^2 - \sigma^2 t$ then $Y_{t\wedge\tau}$ is a submartingale.

To see why this is true, observe that:

$$\begin{aligned}
& E(M_{t+1}^2 - M_t^2|F_t) \\
= \; & E((M_{t+1} - M_t + M_t)^2 - M_t^2|F_t) \\
= \; & E(M_{t+1} - M_t)^2|F_t) + E(2(M_{t+1} - M_t)M_t|F_t)
\end{aligned}$$

Note that the second term is 0 since M_t is a martingale so we have $E(Y_{t+1} - Y_t|F_t) \geq 0$ on $\{M_t \neq 0\}$. Thus $Y_{t\wedge\tau}$ is a submartingale.

Note τ is integrable so applying optional stopping we have $1 = Y_0 \leq EY_\tau = EM_\tau^2 - \sigma^2 E\tau$.

Moreover, by assumption (i) we have:

$$\begin{aligned} EM_\tau^2 &= EM_\tau^2 1_{\{M_\tau \geq h\}} + EM_\tau^2 1_{\{M_\tau = 0\}} \\ &= EM_\tau^2 1_{\{M_\tau \geq h\}} \\ &\leq EDhM_\tau 1_{\{M_\tau \geq h\}} \\ &\leq Dh \end{aligned}$$

so $1 \leq Dh - \sigma^2 E\tau$ which imply $E\tau \leq \frac{Dh}{\sigma^2}$. Thus:

$$P(\tau > k) \leq \frac{E\tau}{k} \leq \frac{Dh}{\sigma^2 k}$$

Remark: This is the only place that we apply assumption (i), so assumption (i) can be weakened as $(iii) : EM_\tau^2 \leq Dh$. This turns out to be helpful when we consider the example in the following section.

To sum up, we have

$$P(M_t > 0, t = 1, ..., k) \leq P(M_\tau \geq h) + P(\tau > k) \leq \frac{1}{h} + \frac{Dh}{k\sigma^2}$$

To get the optimal upper bound we minimize the right term of the last inequality over $h > 0$ by taking $h = \sqrt{\frac{k\sigma^2}{D}}$. Then we have $P(M_t > 0, t = 1, ..., k) \leq \frac{2\sqrt{D}}{\sigma\sqrt{k}}$.

3 Application: Percolation on b-ary tree

Consider the bond percolation on b-ary tree with parameter $p = 1/b$. Let $\mathcal{C}(root)$ be all vertices that can be reached from the root by open path. we will estimate $P(|\mathcal{C}(root)| \geq k)$.

Consider the **exploration process**:

At each time t we have 3 types of vertexes:

(i) A_t: active vertexes.

(ii) N_t: neutral vertexes.

(iii) E_t: explored vertexes.

At the beginning, i.e. $t = 0$, we start with a single active vertex, the root, $A_0 = \{root\}$, all other vertices are neutral, and E_0 is empty.

At time t, If A_{t-1} is empty then $(A_t, N_t, E_t) = (A_{t-1}, N_{t-1}, E_{t-1})$. If A_{t-1} is not empty, fix an ordering of A_{t-1} and pick the front vertex in A_{t-1}, call it w_t. Then:

$$A_t = A_{t-1} \cup \{v \in N_{t-1} : edge\ w_t v\ is\ open\} - \{w_t\}$$

$$E_t = E_{t-1} \cup \{w_t\}$$

$$N_t = N_{t-1} - \{v \in N_{t-1} : edge\ w_t v\ is\ open\}$$

In this process,

$$E(|A_t||F_{t-1}) = \begin{cases} |A_{t-1}|, & if\ A_{t-1} = \emptyset \\ |A_{t-1}| + pb - 1, & if\ not \end{cases}$$

In particular, $|A_t|$ is a martingale when $p = 1/b$. (If $p \neq 1/b$ we may modify $|A_t|$ a little to get a martingale so that the same method can still apply.)

Let $\tau := min\{t : |A_t| = 0\}$, ($\tau = \infty$ if this set is empty.) we have $E_\tau = \mathcal{C}(root)$. So:

$$P(|\mathcal{C}(root)| \geq k) = P(\tau \geq k) \leq \frac{2\sqrt{D}}{\sigma\sqrt{k}}$$

.

So it remains to estimate σ and D in exploration process.

Note that

$$E((|A_{t+1}| - |A_t|)^2|F_t) = bp(1-p) \geq 1/2$$

since this is just the variance of a binomial random variable. Thus we may take $\sigma = \sqrt{1/2}$.

Note that if we apply assumption (i) in the main theorem, the trivial bound of D we can get is b, which is always very big in real applications. So we try to apply assumption (iii) to get estimate of D.

Note that at time τ A_t is obtained from the previous step plus a random variable Z distributed as a binomial random variable $B(b, p)$. Thus we have

$$A_\tau 1_{\{A_\tau \geq h\}} \leq (h + Z) 1_{\{A_\tau \geq h\}}$$

. Since Z and $\{A_\tau \geq h\}$ are independent, we have

$$\begin{aligned} EA_\tau^2 1_{\{A_\tau \geq h\}} &\leq h^2 P(A_\tau \geq h) + 2hEZ1_{\{A_\tau \geq h\}} + EZ^2 1_{\{A_\tau \geq h\}} \\ &\leq h + 2 + 2/h \\ &\leq 2h. \end{aligned}$$

For $h \geq 3$.

So we may take $D = 2$ and obtain

$$P(|\mathcal{C}(root)| \geq k) \leq \frac{2\sqrt{D}}{\sigma\sqrt{k}} \leq \frac{4}{\sqrt{k}}$$

.

The exploration process can also be applied on percolation on d-regular graph or graph with maximal degree less than d.