1 reflection principle

we wish to compute the distribution of $\tau_1$ where the simple random walk start from zero.

note that we have:

$$P_0(\tau_1 = k) = P_0(X_0 = -1, S_{k-1} = 0, X_k = 1) - P_0(X_0 = -1, S_{k-1} = 0, X_k = 1, \tau_1 < k)$$

$$= \frac{1}{4}P(S_{k-1} = 0 | S_1 = -1) - \frac{1}{4}P(\tau_1 < k, S_{k-1} = 0 | S_1 = -1)$$

In order to compute the later probability, we consider the reflected mapping:

$$\varphi : (X_1, X_2, ..., X_{k-1}) \mapsto (X_1, X_2, ..., X_{\tau_1}, -X_{\tau_1+1}, ..., -X_{k-1} := (Y_1, ..., Y_{k-1})$$

Note $\varphi$ is a bijection and is identity if $\tau_1 \geq k$.

$\varphi$ map the event $\{X_1 = -1, \tau_1 < k, S_{k-1} = 0\}$ to $\{Y_1 = -1, \tau_1 < k, S_{k-1} = 2\}$
which is $\{Y_1 = -1, S_{k-1} = 2\}$.

Thus we have:

$$P_0(\tau_1 = k) = \frac{1}{4}P(S_{k-1} = 0 | S_1 = -1) - \frac{1}{4}P(\tau_1 < k, S_{k-1} = 0 | S_1 = -1)$$

$$= \frac{1}{4}P(S_{k-1} = 0 | S_1 = -1) - \frac{1}{4}P(S_{k-1} = 2 | S_1 = -1)$$

$$= \frac{1}{4} \left(\frac{1}{2}\right)^{k-2} \frac{k-2}{k-1} - \frac{1}{4} \left(\frac{1}{2}\right)^{k-2} \frac{k-2}{k+1}$$

which can be estimated using Stirling’s Formula. Finally we get $P_0(\tau_1 = k) \sim \frac{C}{k^{3/2}}$.

**Corollary 1.1.** For simply random walk start from zero, we have $E_0\tau_1 = \infty$. 

2 Probability for a Martingale to stay positive

Our next goal is to estimate the probability that a Martingale stay positive for \( k \) steps.

For simple random walk (which is a Martingale), we have:

\[
P_1(S_t > 0, t = 1, 2, ..., k) = P_1(S_k > 0) - P_1(S_k < 0) = P_1(S_k > 0) - P_1(S_k < 0)
\]

\[
= P_1(S_k > 0) - P_1(S_k < 2)
\]

\[
= P_1(S_k = 1 \text{ or } 2)
\]

\[
= 2^{-k} \left( \frac{k}{k/2} \right) \sim \frac{1}{\sqrt{\pi k/2}}
\]

Now we consider general Martingale \( \{M_t\}_{t \geq 0} \) with \( M_0 = 1 \) and \( M_t \geq 0 \) for any \( t \).

We assume that:

\begin{enumerate}
  \item \( |M_{t+1}| \leq D|M_t| \) for some \( D \geq 1 \) when \( M_t \neq 0 \). (this can be weakened.)
  \item \( E((M_{t+1} - M_t)^2|F_t) \geq \sigma^2 > 0 \) for any \( t \) when \( M_t \neq 0 \).
\end{enumerate}

The example of simple random walk may suggest the probability that \( M_t \) stay positive for \( k \) steps is of order \( k^{1/2} \). In fact this is true:

**Theorem 2.1.** Under previous assumption, we have \( \Pr(M_t > 0, t = 1, ..., k) \leq \frac{C(D, \sigma)}{\sqrt{k}} \) where \( C(D, \sigma) \) is a constant related to \( D \) and \( \sigma \).

To prove the theorem we let \( \tau = \tau_{[0,\infty[} \wedge k \text{ and } k = \min \{ t : M_t = 0 \text{ or } M_t \geq h \} \wedge k \).

By Markov inequality and optional stopping we have \( h \Pr(M_\tau \geq h) \leq E M_\tau = 1 \).

**Claim:** If \( Y_t := M_t^2 - \sigma^2 t \) then \( Y_t \wedge \tau \) is a submartingale.

To see why this is true, observe that:

\[
E(M_{t+1}^2 - M_t^2 | F_t)
\]

\[
= E((M_{t+1} - M_t + M_t)^2 - M_t^2 | F_t)
\]

\[
= E(M_{t+1} - M_t)^2 | F_t) + E(2(M_{t+1} - M_t)M_t | F_t)
\]

Note that the second term is 0 since \( M_t \) is a martingale so we have \( E(Y_{t+1} - Y_t | F_t) \geq 0 \) on \( \{ M_t \neq 0 \} \). Thus \( Y_t \wedge \tau \) is a submartingale.
Note $\tau$ is integrable so applying optional stopping we have $1 = Y_0 \leq EY_{\tau} = EM^2_{\tau} - \sigma^2E\tau$.

Moreover, by assumption (i) we have:

$$EM^2_{\tau} = EM^2_{\tau}1_{\{M_{\tau} \geq h\}} + EM^2_{\tau}1_{\{M_{\tau} = 0\}}$$

$$= EM^2_{\tau}1_{\{M_{\tau} \geq h\}}$$

$$\leq EDhM_{\tau}1_{\{M_{\tau} \geq h\}}$$

$$\leq Dh$$

so $1 \leq Dh - \sigma^2E\tau$ which imply $E\tau \leq \frac{Dh}{\sigma^2}$. Thus:

$$P(\tau > k) \leq \frac{E\tau}{k} \leq \frac{Dh}{\sigma^2k}$$

**Remark:** This is the only place that we apply assumption (i), so assumption (i) can be weakened as (iii) : $EM^2_{\tau} \leq Dh$. This turns out to be helpful when we consider the example in the following section.

To sum up, we have

$$P(M_t > 0, t = 1, ..., k) \leq P(M_t \geq h) + P(\tau > k) \leq \frac{1}{h} + \frac{Dh}{k\sigma^2}$$

To get the optimal upper bound we minimize the right term of the last inequality over $h > 0$ by taking $h = \sqrt{\frac{k\sigma^2}{D}}$. Then we have $P(M_t > 0, t = 1, ..., k) \leq \frac{2\sqrt{D}}{\sigma\sqrt{k}}$.

### 3 Application: Percolation on $b$-ary tree

Consider the bond percolation on $b$-ary tree with parameter $p = 1/b$. Let $C(root)$ be all vertices that can be reached from the root by open path. we will estimate $P(|C(root)| \geq k)$.

Consider the exploration process:

At each time $t$ we have 3 types of vertexes:

(i) $A_t$: active vertexes.

(ii) $N_t$: neutral vertexes.

(iii) $E_t$: explored vertexes.
At the beginning, i.e. \( t = 0 \), we start with a single active vertex, the root, \( A_0 = \{ \text{root} \} \), all other vertices are neutral, and \( E_0 \) is empty.

At time \( t \), If \( A_{t-1} \) is empty then \((A_t, N_t, E_t) = (A_{t-1}, N_{t-1}, E_{t-1})\). If \( A_{t-1} \) is not empty, fix an ordering of \( A_{t-1} \) and pick the front vertex in \( A_{t-1} \), call it \( w_t \). Then:

\[
A_t = A_{t-1} \cup \{ v \in N_{t-1} : \text{edge } w_t v \text{ is open} \} - \{ w_t \}
\]

\[
E_t = E_{t-1} \cup \{ w_t \}
\]

\[
N_t = N_{t-1} - \{ v \in N_{t-1} : \text{edge } w_t v \text{ is open} \}
\]

In this process,

\[
E(|A_t||F_{t-1}) = \begin{cases} 
|A_{t-1}|, & \text{if } A_{t-1} = \emptyset \\
|A_{t-1}| + pb - 1, & \text{if } \text{not}
\end{cases}
\]

In particular, \(|A_t|\) is a martingale when \( p = 1/b \). (If \( p \neq 1/b \) we may modify \(|A_t|\) a little to get a martingale so that the same method can still apply.)

Let \( \tau := \min\{t : |A_t| = 0\} \), (\( \tau = \infty \) if this set is empty.) we have \( E_\tau = C(\text{root}) \).

So:

\[P(|C(\text{root})| \geq k) = P(\tau \geq k) \leq \frac{2\sqrt{D}}{\sigma \sqrt{k}}\]

So it remains to estimate \( \sigma \) and \( D \) in exploration process.

Note that

\[E((|A_{t+1}| - |A_t|)^2|F_t) = bp(1 - p) \geq 1/2\]

since this is just the variance of a binomial random variable. Thus we may take \( \sigma = \sqrt{1/2} \).

Note that if we apply assumption (i) in the main theorem, the trivial bound of \( D \) we can get is \( b \), which is always very big in real applications. So we try to apply assumption (iii) to get estimate of \( D \).

Note that at time \( \tau \) \( A_t \) is obtained from the previous step plus a random variable \( Z \) distributed as a binomial random variable \( B(b, p) \). Thus we have

\[A_\tau 1_{\{A_\tau \geq k\}} \leq (h + Z)1_{\{A_\tau \geq k\}}\]
Since $Z$ and $\{A_r \geq h\}$ are independent, we have
\[
EA^2_r1_{\{A_r \geq h\}} \leq h^2 P(A_r \geq h) + 2hEZ1_{\{A_r \geq h\}} + EZ^21_{\{A_r \geq h\}} \\
\leq h + 2 + 2/h \\
\leq 2h.
\]

For $h \geq 3$.

So we may take $D = 2$ and obtain
\[
P(|C(root)| \geq k) \leq \frac{2\sqrt{D}}{\sigma \sqrt{k}} \leq \frac{4}{\sqrt{k}}
\]

The exploration process can also be applied on percolation on $d$-regular graph or graph with maximal degree less than $d$. 