University of Washington Math 523A
Homework assignment 2
Due by: May 11

Answer at least 3 of the following questions:

1. Let $\{X_n\}_{n\geq 0}$ be a martingale with $\mathbb{E}X_0 = 0$ and $\mathbb{E}X_n^2 < \infty$. Show that
$$\mathbb{P}\left(\max_{1\leq k\leq n} X_k > r\right) \leq \frac{\mathbb{E}[X_n^2]}{\mathbb{E}[X_n^2] + r^2} .$$

2. Let $\{S_n\}$ be a simple random walk on $\mathbb{Z}$ with $S_0 = 0$.
   (a) For which $\alpha > 0$ can you find a stopping time τ such that
   $$\mathbb{E}\tau^\alpha < \infty \quad \text{yet} \quad \mathbb{E}S_\tau \neq 0$$
   (b) For which $\alpha > 0$ can you prove that
   $$\mathbb{E}\tau^\alpha < \infty \quad \Longrightarrow \quad \mathbb{E}S_\tau = 0$$
   Hint: try bounding $\mathbb{E}\max k \leq \tau|S_k|$.

3. In a sequence of fair coin tosses, let τ_w be the number of tosses until w appears, so $\tau_{01} = 5$ for the sequence 11001.
   (a) For $0 < z < 1$, determine $g(z) = \mathbb{E}z^{\tau_{00}}$.
   (b) For $0 < z < 1$, determine $\tilde{g}(z) = \mathbb{E}z^{\tau_{010}}$.

4. Solve any problem you skipped from the previous HW.
   Hint: use symmetry.