University of Washington Math 523A

Homework assignment 1

Due by: April 20

Answer at least 3 of the following questions:

1. A sequence of random variables (A_i) is called *previsible* with respect to a filtration $(\mathcal{F}_i)$ if A_i is $\mathcal{F}_{i-1}$-measurable for each i.

   (i) Show that for any sequence of random variables (X_t), adapted to a filtration $(\mathcal{F}_t)$ and satisfying $\mathbb{E}|X_t| < \infty$ for all t, there is a unique decomposition of the form

   $$X_t = M_t + A_t \ ,$$

   where (M_t) is a martingale and (A_t) a previsible sequence (both w.r.t. $(\mathcal{F}_t)$) with $A_0 = 0$.

   (ii) Show that (X_t) is a sub-martingale iff the above sequence (A_t) is non-decreasing with probability 1.

2. Let $P_1, \ldots, P_n$ be uniformly and independently chosen points in the $m \times m$ square $[0,m]^2$ with $m = \sqrt{n}$, and let $X = |\cup_i B_{P_i}(1)|$, where $B_P(r)$ denotes a ball of radius r about the point P. Show that there exists some fixed $c > 0$ such that

   $$\mathbb{P}(|X - \mathbb{E}X| > \varepsilon\sqrt{n}) \leq \exp(-c\varepsilon^2) \text{ for any } \varepsilon > 0\,.$$

3. Let $S_t = S_0 + \sum_{j=1}^{t} X_j$, where $X_j \sim \begin{cases} 1 & p \\ -1 & q = 1-p \end{cases}$ are i.i.d. random variables with $p > \frac{1}{2}$. Let $\mathbb{P}_k$ and $\mathbb{E}_k$ denote the probability and expectation resp. for the process started at $S_0 = k$.

   (i) Find λ such that $\mathrm{e}^{\lambda S_t}$ is a martingale.

   (ii) Calculate $\mathbb{P}_k(\tau_n < \tau_0)$.

   (iii) Calculate $\mathbb{E}_0(\tau_1)$.

   (iv) Find $\mathbb{E}_0(\tau_1)$ for $p = \frac{1}{2}$.

4. Let $S_0, S_1,$ be simple random walk on $\mathbb{Z}$ with $S_0 = 0$. Fix $a \in (0,1)$.

   (i) Show that there is a unique $\lambda > 0$ such that $\cosh(\lambda) = \mathrm{e}^{\lambda a}$.

   (ii) For $a \in (0,1)$ and $b > 0$, define

   $$\tau \triangleq \inf\{t \geq 1 : S_t \geq at + b\}\,,$$

   with the convention that $\tau = \infty$ if the set under the infimum is empty. Show that

   $$\mathrm{e}^{-\lambda(b+1)} \leq \mathbb{P}(\tau < \infty) \leq \mathrm{e}^{-\lambda b}\,.$$

5*. Let G be a graph whose chromatic number is $\chi(G) = 600$, and let S be a random uniformly chosen subset of of its vertices. Show that the induced subgraph of G on S, denoted by $G|_S$, is *not* 200-colorable except with probability at most 2^{-10}:

   $$\mathbb{P}\left(\chi(G|_S) \leq 200\right) < 2^{-10}\ .$$