University of Washington Math 523A

Final exam

Due by: June 9

Answer exactly 4 of the following questions:

1. Let S_t be a simple random walk on $\mathbb{Z}$ and set $\tau = \min\{t : |S_t| \geq a\}$ for some integer $a > 0$. Calculate $\mathbb{E}\tau^2$.

   Hint: Consider a martingale involving the fourth power of S_t.

2. Consider a biased random walk on the integers: for some $0 < p < 1$,

$$S_{t+1} = \begin{cases} S_t + 1 & p \\ S_t - 1 & 1-p \end{cases} .$$

   Let $\tau_a = \min\{t : S_t = a\}$, and let a, b be two positive integers.

   (a) Calculate $\mathbb{P}(\tau_{-a} < \tau_b)$.

   (b) Compute $\mathbb{E}[\min\{\tau_{-a}, \tau_b\}]$.

3. In a sequence of fair coin tosses, let τ_{011} and τ_{000} denote the number of tosses until witnessing the corresponding pattern.

   (a) Find the value of $\mathbb{E}\max\{\tau_{011}, \tau_{000}\}$.

   (b) Find $\mathbb{E}\tau_{01}^2$.

4. A gambler plays the following game. In each round, he can pay any $0 < p < 1$ dollars, and win \$1 with probability p (independently). Show that the probability that the gambler's net gain exceeds h at any of the first n rounds is at most $\exp(-h^2/2n)$.

5. Consider percolation on a binary tree with parameter $p < \frac{1}{2}$. Let $\mathcal{C}$ denote the cluster of the root. Show that

$$\mathbb{P}(|\mathcal{C}| > k) \leq \exp(-k\alpha(p))$$

   for some $\alpha(p) > 0$ (find $\alpha(p)$ explicitly).

6. Let S_t be a simple random walk on $\mathbb{Z}$, and for $c, \alpha > 0$ define

$$\tau_{c,\alpha} = \min\{t : |S_t| \geq ct^\alpha + 1\} .$$

   (a) For which c, α is $\tau_{c,\alpha} < \infty$ almost surely?

   (b) For which c, α is $\mathbb{E}\tau_{c,\alpha} < \infty$ almost surely?