1 The set-up

Assume \( f \) on \((\infty, \infty)\) satisfies:

- For \( u \neq 0, f(-u) = f(u) > 0 \), \( f(0) = 0 \), \( f(\infty) = \infty \);
- \( f \in C^1((0, \infty) \cup (-\infty, 0)) \) with a strictly positive first derivative on \((0, \infty)\);
- There exists \( q > 0 \) such that
  \[
  f'(u) = O \left( |u|^{q-1} \right), \quad u \to 0, \quad (f' = df/du)
  \]

Define two linear spaces of functions:

\[
H_1 = \{ u : u \in AC[0, 1], u' \in L^2[0, 1], u(0) = 0 \}, \quad H_2 = \{ u : u \in H_1, u(1) = 0 \}.
\]

and put

\[
M_i = \left\{ u : u \in H_i, \int_0^1 f(u(x)) \, dx = 1 \right\}, \quad i = 1, 2.
\]

Define

\[
\kappa_i = \inf \left\{ \int_0^1 u'^2 : u \in M_i \right\}.
\]

We call a function \( u \) a minimizer for the problem \( \kappa_i \), if \( u \in M_i \) and \( \int u'^2 = \kappa_i \).

The purpose of this note is to elucidate the constants \( \kappa_i \) with special attention to the case in which \( f' \) may have a singularity at \( u = 0 \) if \( q < 1 \) in (1).

For convenience we shall write \( \mathcal{J}(u) \) and \( \mathcal{R}(u) \) for the functionals \( \int_0^1 u'^2 \, dx \) and \( \int_0^1 f(u) \, dx \), respectively.

**Theorem 1** (Minimizers exist). *For each \( i = 1, 2 \) there exists a function \( u \in M_i \) such that \( u > 0 \) on \((0, 1)\) and \( \kappa_i = \mathcal{J}(u) = \int_0^1 u'^2 > 0. \) Moreover, if \( \kappa_1 < \kappa_2 \) and \( u \) is a minimizer for \( \kappa_1 \) positive on \((0, 1)\), then \( u(1) > 0. \)
For \( u \in H_i \) let

\[
S_u = \{ \varphi : \varphi \in AC[0,1], \varphi' \in L^2[0,1], \varphi = O(u) \text{ uniformly on } [0,1] \}
\]

Clearly, the space \( S_u \) for \( u \neq 0 \) on \((0,1)\) includes all continuously differentiable functions \( \varphi \) on \([0,1]\) vanishing off of a compact subinterval of \((0,1)\) because the ratio \( |\varphi|/|u| \) is continuous on \([0,1]\).

**Theorem 2** (Euler-Lagrange). If \( u \in H_i \) minimizes \( J \) on \( M_i \) and \( u > 0 \) on \((0,1)\), then there is a number \( \lambda = \lambda_i \) such that for \( \varphi \) in the space \( S_u \)

\[
\delta J(u, \varphi) - \lambda \delta R(u, \varphi) = 2 \int_0^1 u' \varphi' \, dx - \lambda \int_0^1 f'(u) \varphi \, dx = 0 \quad (1)
\]

where \( \delta \) denotes variational derivative:

\[
\delta J(u, \varphi) = \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon \varphi) - J(u)}{\varepsilon} \quad (3a)
\]

\[
\delta R(u, \varphi) = \lim_{\varepsilon \to 0} \frac{R(u + \varepsilon \varphi) - R(u)}{\varepsilon} \quad (3b)
\]

**Theorem 3** (Regularity and differential equations). Let \( u \) be a minimizer for the problem \( \kappa_i \) satisfying the conclusions of Theorem 1. Then the Lebesgue equivalence class of \( u' \) contains a unique representative, also denoted \( u' \), such that (i) \( u' \) is non-negative, decreasing, continuous on \((0,1)\), with finite right, left limits at 0, 1, respectively, and for problem \( \kappa_1 \) we also have \( u'(1) = 0 \). (ii) For both problems the minimizers are solutions to

\[
u'^2 = K^2 - \lambda f(u) \quad \text{on } [0,1],
\]

where \( K = u'(0) > 0 \) and \( \lambda \), the Lagrange parameter of Theorem 2, satisfies

\[
\kappa_i = \frac{\lambda \mu}{2} = K^2 - \lambda, \quad \mu = \int_0^1 uf'(u) \, dx, \quad i = 1, 2.
\]

Finally \( \kappa_2 = 4\kappa_1 \).

Before going to the proofs, the following preliminary result seems appropriate here:

**Proposition 1.** The numbers \( \kappa_i \) are strictly positive (and obviously finite) and \( \kappa_1 \leq \kappa_2 \). If \( M_i^+ = \{ u \in M_i : R(u) > 1 \} \); then \( \kappa_i = \inf \{ J(u) : u \in M_i^+ \} \).

(The last assertion is not used in the sequel, but seems mentionable.)

---

1. The functional \( R \) has unbounded variation in those most interesting cases in which \( f'(u) \approx |u|^{q-1} \) for \( 0 < q < 1 \) and it is for these cases that the spaces \( S_u \) are introduced.
Clearly $\kappa_1 \leq \kappa_2$ because $M_2 \subset M_1$. From (1) and $f(0) = 0$, $f(u) \leq c|u|^q$ for some $c < \infty$, and for $u \in H_i$

$$|u(x)| \leq \int_0^x |u'| \, dx \leq x^{1/2}||u'||_2 = x^{1/2}J(u)^{1/2}.$$ 

Hence $1 = R(u) \leq c \int |u|^q \, dx \leq c(q/2 + 1)^{-1/2}J(u)^{q/2}$ which implies $\kappa_i > 0$.

Let $M$, $M^+$ stand for either of $M_i$, $M_i^+$ and $\kappa$, $\kappa^+$ for inf $\mathcal{J}$ over the corresponding spaces. Let $\varepsilon > 0$ be fixed but arbitrary. Suppose $v \in M^+$ satisfies $\mathcal{J}(v) \leq \kappa^+ + \varepsilon$. Put $u = Cv$ where $C$ is chosen so that $R(u) = 1$; then because $\int f(v) \, dx = R(v) > 1$, properties of $f$ imply $C \leq 1$ and then $\kappa \leq \mathcal{J}(u) = C^2\mathcal{J}(v) \leq \mathcal{J}(v) < \kappa^+ + \varepsilon$. Hence $\kappa \leq \kappa^+$ because $\varepsilon$ is arbitrary. Next suppose $u \in M$ and $\mathcal{J}(u) \leq \kappa + \varepsilon$. We may suppose that $|u| > 0$ on some open $\Delta \subset (0, 1)$. Let $v = (1 + \varepsilon)u$; then $\int_{\Delta} f(v) \, dx > \int_{\Delta} f(u) \, dx \Rightarrow R(v) > R(u) = 1$ so $v \in M^+ \Rightarrow \kappa^+ \leq \mathcal{J}(v) = (1 + \varepsilon)^2\mathcal{J}(u) \leq (1 + \varepsilon)^2(\kappa + \varepsilon) \Rightarrow \kappa^+ \leq \kappa$. $\therefore \kappa = \kappa^+$.

\section{Proof of Theorem 1}

Fix $i = 1$ or 2. Let $\{u_n\} \subset M$, $\lim \mathcal{J}(u_n) = \kappa$.

\textbf{Step (a)} There is a subsequence $\{u_n; n \in \mathcal{N}\}$, $\mathcal{N} = \{\text{an infinite increasing sequence of positive integers, and a continuous function } u \text{ such that } u \text{ satisfies either } u(0) = 0 \text{ or } u(0) = u(1) = 0, \text{ depending on the problem, and (i) } \lim_\mathcal{N} u_n = u \text{ uniformly on } [0, 1], \text{ and (ii) there exists } v \in L^2(0, 1) \text{ such that } \lim_\mathcal{N} \int u'_n w = \int v w \forall w \in L^2(0, 1).$ 

The last two assertions follow from $\sup_n \|u'_n\| = \sup_\mathcal{N} \mathcal{J}(u)^{1/2} \equiv C < \infty$ via Arzela-Ascoli and weak conditional sequential compactness of norm bounded sets in $L^2[0, 1]$, [5] p. 209. The boundary conditions should be plain.

\textbf{Step (b)} The subsequence $\{u_n, \ n \in \mathcal{N}\}$ is uniformly absolutely continuous, and, therefore, $u$ is also absolutely continuous.

For any function $p$ and any interval $\Delta = (a, b)$, $p(\Delta)$ denotes $p(b) - p(a)$. Let $\{\Delta_k, \ k = 1, 2, \ldots, m\}$ be any finite collection of pairwise disjoint intervals in $[0, 1]$. Then

$$\sum_{k=1}^m |u_n(\Delta_k)| = \sum_{k=1}^m \int_{\Delta_k} u'_n \, dx \leq \int_{\bigcup_{k=1}^m \Delta_k} |u'_n| \, dx \leq \left(\sum_{k=1}^m |\Delta_k|\right)^{1/2} \times C$$ 

where $|\Delta_k|$ denotes the length of the interval $\Delta_k$ and $C$ is the constant of (a).

Given any $\varepsilon > 0$, choose $\delta = \varepsilon^2/C^2$. Then for every $n$ and any collection of intervals as above with $\sum_{k=1}^m |\Delta_k| < \delta$, we have $\sum_{k=1}^m |u_n(\Delta_k)| < \varepsilon$. Passing to the limit shows that $u \in AC[0, 1]$.

\textbf{Step (c)} The primitives $u'$ are in the equivalence class $[v]$ (mod Lebesgue measure) and hence are square integrable. In particular this implies that the weak $\lim_\mathcal{N} u'_n = u'$. 

3
Let \( u' \in L^1 \) satisfy: 
\[
\int_x^y u'(t) \, dt = u(y) - u(x)
\]
for all \( x, y \). Then 
\[
\int u \varphi' = -\int u' \varphi
\]
for any \( \varphi \in C_c^1(0,1) \). (To see this, take \( g \in C_c^\infty \) such that \( \|u' - g\|_1 < \varepsilon \). If \( G = \int_0^x g \), then \( |u(x) - G(x)| \leq \int_0^1 |u' - G| < \varepsilon \) and the result follows from ordinary integration by parts applied to \( \int G \varphi \) upon letting \( \varepsilon \to 0 \).)

Weak convergence of \( \{u'_n, n \in \mathcal{N}\} \) to \( v \) in \( L^2 \) and uniform convergence of \( \{u_n, n \in \mathcal{N}\} \) shows that \( \int u \varphi' = \lim_N \int u_n \varphi' = -\lim_N \int u'_n \varphi = -\int v \varphi \) for any \( \varphi \in C^1(0,1) \). Therefore \( \int (v - u') \varphi = 0 \) for such \( \varphi \Rightarrow u' = v \) a.e.

**Step (d) \( \mathcal{J}(u) = \kappa_i \).**

By (a)-(c), \( u \in H_c \) and \( \mathcal{R}(u) = \int f(u) = \lim u' \geq \kappa_i \) by definition of \( \kappa_i \). Fix \( \varepsilon > 0 \) and put \( z = u'/\|u'\|_2 \). Weak convergence of \( \{u'_n\}_\mathcal{N} \) implies \( \exists n_1 \in \mathcal{N} \) so that \( n \geq n_1 \Rightarrow \|u(u' - u'_n)\|_2 \leq \varepsilon \). Find \( n_2 > n_1 \in \mathcal{N} \) so that \( \mathcal{J}(u'_{n_2})^{1/2} = \|u'_{n_2}\|_2 \leq \sqrt{\kappa_i} + \varepsilon, n \in \mathcal{N} \geq n_2 \). Then \( n_2 \geq n_2 \Rightarrow \sqrt{\kappa_i} \leq \|u'\|_2 = \int u' z = \int (u' - u'_n) z + \int u'_n z \leq \varepsilon + \|u'_{n_2}\|_2 \leq 2 \varepsilon + \sqrt{\kappa_i} \)
and \( \mathcal{J}(u) = \kappa_i \) as \( \varepsilon \) is arbitrary.

**Step (e) \( u \) can be chosen non-negative.**

If \( u \) of (a)-(d) is already non-negative, then we are done. Otherwise replace \( u \) by \( |u| \). Clearly \( |u| \) is continuous, satisfies the appropriate boundary conditions, and \( \mathcal{R}(|u|) = \mathcal{R}(u) = 1 \) because \( f \) is even.

To finish, it must be shown that \( |u| \) is absolutely continuous with a square integrable primitive. To this end, let \( g \) be a bounded continuous function on the line with \( g(0) = 0 \) and which satisfies a uniform Lipschitz condition. Put \( G(t) = \int_0^t g(s) \, ds \) (the sign convention, \( \int_0^x = -\int_x^0 \), is in effect). Then the composite function \( G \circ u \) is absolutely continuous and \( G(u(x)) = \int_0^x g(u(t))u'(t) \, dt, 0 \leq x \leq 1 \). Apply this to \( g = g_n(s) = (2/\pi) \arctan(ns) \). The sequence \( \{g_n\} \) is uniformly bounded and converges pointwise to the signum function. By LDC (Lebesgue’s Dominated Convergence theorem), \( G_n(t) = \int_0^t g_n(s) \, ds \to |t| \) boundedly on bounded intervals. Also \( |g_n(u)u'| \leq \|g\|_\infty |u'| \in L^1(0,1) \). So

\[
|u(x)| = \lim_n G_n(u(x)) = \lim_n \int_0^x g_n(u(t))u'(t) \, dt = \int_0^x \lim_n g_n(u(t))u'(t) \, dt = \int_0^x \text{sgn}(u(t))u'(t) \, dt, \quad \forall x,
\]
and \( |u| \in AC, |u'| = \text{sgn}(u)u', \text{a.e., and } \mathcal{J}(|u|) = \int (\text{sgn}(u)u')^2 = \kappa_i \).

---

\(^2\)Choose smooth \( h \) so that \( \|u' - h\|_2 < \varepsilon \). Then \( \|u - H\|_\infty < \varepsilon, H(x) = \int_0^x h \). Hence

\[
G \circ u = \int_0^x g + G \circ H = O(\|H - u\|_\infty) + \int_0^x (g \circ H)(h - u') + \int_0^x (g \circ H)u'
\]

\[
= O(\varepsilon) + \int_0^x g[H - g \circ u]u' + \int_0^x (g \circ u)u' = O(\varepsilon) + \int_0^x g(u)u'
\]
Step (f) There exists a strictly positive on $(0, 1)$ minimizer for each problem, and, if $\kappa_1 < \kappa_2$, then the minimizer for the problem $\kappa_1$ does not vanish at $x = 1$.

For an arbitrary interval $\Delta = [a, b]$ and a number $\ell > 0$, denote by $H_i(\Delta)$, $M_i(\Delta, \ell)$, the same spaces as defined for $[0, 1]$ and with integrals over $\Delta$. In $M_i(\Delta, \ell)$ the side condition is $R_\Delta(u) = \ell$, and $\kappa_i(\Delta, \ell) = \inf_{u \in M_i(\Delta, \ell)} J_\Delta(u)$.

**Proposition 2.** Non-negative minimizers for $\kappa_i(\Delta, \ell)$ exist. If $u_*$ is a minimizer for $\kappa_i(\Delta, \ell)$, then $\hat{u}$ is a minimizer for $\kappa_i = \kappa_i([0, 1], 1)$ where

$$
\hat{u}(x) = m_{\mu_*}(nx + a), \ x \in [0, 1], \ n = |\Delta| = b - a,
$$

and $m$ is the unique solution to: $\int_{\Delta} f(\mu_*(t)) dt = |\Delta|$. This implies

$$
\kappa_i(\Delta, \ell) = \kappa_i([0, 1], 1)/m^2n
$$

The proof is omitted. (The existence of minimizers is proved as in (a)-(d). Existence of a unique $m$ is a consequence of the hypotheses imposed on $f$.)

Back to (f). Let $u$ be as in Step (e).

*Case 1:* $u$ is a minimizer for the problem $\kappa_2$.

If $u > 0$ on $(0, 1)$, then there is nothing to prove. Suppose there is a point $c \in (0, 1)$ at which $u(c) = 0$. Because $u$ is continuous the set $\{u > 0\}$ is a union of disjoint relatively open intervals in $[0, 1]$. But we also have $u(0) = 0$. Hence there must be an open interval $(a, b) \subset (0, 1)$ such that $u > 0$ on $(a, b)$ but $u(a) = u(b) = 0$. Put $\ell = \int_a^b f(u) dx$.

The restriction $u_*$ of $u$ to $\Delta = [a, b]$ must be a minimizer of the problem $\kappa_2(\Delta, \ell)$. Suppose not. Let $w$ be an actual minimizer for this problem, so that $\int_a^b w'^2 dx < \int_a^b u'^2 dx$. Define $u_1 = u$ for $x \notin [a, b]$ and $u_1 = w$ for $x \in [a, b]$. Then $u_1 \in AC[0, 1]$. $u_1$ satisfies the boundary conditions of the problem $\kappa_2$ and, most importantly, $\int_0^1 f(u_1) dx = 1$, so $u_1$ is in $M_2$. But then $J(u_1) = (\int_0^a f + \int_a^b f + \int_b^1) u_1'^2 dx < J(u)$ which contradicts the hypothesis that $u$ is a minimizer for the problem $\kappa_2$.

The function $\hat{u}$ defined in Proposition 2 is a minimizer for $\kappa_2$ and this function is strictly positive on $(0, 1)$.

*Case 2:* $u$ is a minimizer for the problem $\kappa_1$.

If $\kappa_1 = \kappa_2$ then we can take $\hat{u}$ from the last paragraph.

Suppose $\kappa_1 < \kappa_2$ which is the only alternative. Then $J(u) = \kappa_1$ implies $u$ cannot vanish at $x = 1$ for otherwise, $J(u) \geq \kappa_2$ which is impossible. If this function is positive on $(0, 1)$, then we are done.

If $u$ vanishes at some point in $(0, 1)$, let $a = \sup \{x : u(x) = 0\}$. Then $u(1) > 0$ and continuity implies that $a < 1$ and that $u(x) > 0$ for $a < x \leq 1$. Arguing as in the preceding case, the restriction $u_*$ of $u$ to $[a, 1]$ is a minimizer for the problem $\kappa_1([a, 1], \ell)$ where $\ell = \int_a^1 f(u)$. But then, by Proposition 2, $\hat{u}(x) = m_{\mu_*}((1-a)x + a)$ is a minimizer for $\kappa_1$ strictly positive on $(0, 1)$. ▲

---

3 $u_1$ is uniformly continuous on $[0, 1]$ and absolutely continuous on each of $[0, a], [a, b]$, and $[b, 1]$. $\Rightarrow u \in AC[0, 1]$.
3 Proof of Theorem 2

Lemma 1. Let \( u \in H_i \). For any \( \varphi \in S_u \) (\( S_u \) defined in \( \S 1 \)), we have

\begin{align}
\delta J(u, \varphi) &= 2 \int_0^1 u' \varphi' \, dx \\
\delta R(u, \varphi) &= \int_0^1 f'(u) \varphi \, dx
\end{align}

Remark. The not quite obvious problem that needs to be faced is that \(|f'(u)|\) may well be unbounded as \( u \to 0 \) if \( q < 1 \) in (1).

Proof. First \( J(u + \varepsilon \varphi) - J(u) = 2 \varepsilon \int_0^1 u' \varphi' \, dx + \varepsilon^2 \int_0^1 \varphi'' \, dx \). Dividing by \( \varepsilon \) and making it go to 0, gives the formula for \( \delta J \). (Singularities of \( u', u' + \varepsilon \varphi' \), can only occur on a null sets (independent of \( \varepsilon \neq 0 \)) and do not affect the integrals because these functions are integrable.)

\( \delta R \) in the case \( q \geq 1 \).

The function \( f' \) is odd so \( f'(u) = f'(|u|) \text{sgn}[u], \ u \neq 0 \). The continuous functions \(|u| \) and \(|\varphi| \) are bounded on \([0, 1] \) and \(|u + \varepsilon \varphi| - |u| \leq |\varepsilon \varphi| \). Hence

\[
\left| \frac{f(u + \varepsilon \varphi) - f(u)}{\varepsilon} \right| = \left| \frac{(1/\varepsilon) \int_{|u|}^{u + \varepsilon \varphi} f'(s) \, ds}{\varepsilon} \right| = O\left( \int_{|u|}^{|u + \varepsilon \varphi|} s^{q-1} \, ds \right) = O(\varphi) \in L^1[0, 1], \ q \geq 1.
\]

This allows evaluating

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [R(u + \varepsilon \varphi) - R(u)] = \lim_{\varepsilon \to 0} \int_0^1 \frac{f(u + \varepsilon \varphi) - f(u)}{\varepsilon} \, dx
\]

on bringing \( \lim \) inside the integral and applying the easily verified:

\begin{equation}
\frac{d}{d\varepsilon} \frac{f(u + \varepsilon \varphi)}{\varepsilon} \bigg|_{\varepsilon = 0} = \lim_{\varepsilon \to 0} \frac{f(u + \varepsilon \varphi) - f(u)}{\varepsilon} = f'(u) \varphi
\end{equation}

valid for all \( q > 0 \) in (1) and all \( \varphi \in S_u \). (For \( \varphi \notin S_u \), (7) can only be asserted for \( u \neq 0 \).)

\( \delta R \) for \( 0 < q < 1, \ \varphi \in S_u \).

The difficulty in this case is the presence of the negative exponent \( q - 1 \) which allows the possibility that the right-hand side of (6b) is infinite. Restricting \( \varphi \) to the class \( S_u \) obviates this annoyance.

Let \( \varphi \in S_u, \ u \in H_i, \neq 0 \). There exists \( m > 0 \) such that \(|\varphi(x)| \leq m |u(x)|\) for all \( x \in [0, 1] \). Then \(|u + \varepsilon \varphi| \geq |u| - |\varepsilon \varphi| \geq \frac{1}{2} |u|\) for \(|\varepsilon| \leq \varepsilon_0 = 1/(2m) \). Fix \( x \) and suppose \( u = u(x) \neq 0 \). As before, \( I = I(x) \) denotes the interval with
and similar formulas for $G$ have a derivative at $(C, \psi)$, the functions must also vanish on $\{u \geq 0\}$ uniformly in $0 < |\varepsilon| \leq \varepsilon_0$. This is correct at least for the calculation of the variation of derivatives on this Lemma. Fix $S$ that I needed the larger class $S_u$ because, as noted earlier, the derivation of the differential equations and boundary conditions for $u$, I found that I needed the larger class $S_u$ to make it work. See the next subsection.

**Remark.** The reader might suppose that one could make a proof using only the functions $C^1_{\varepsilon}(0, 1)$ provided one assumed that $u$ does not vanish on $(0, 1)$. This is correct at least for the calculation of the variational derivatives on this smaller class because, as noted earlier, $\varphi = \mathcal{O}(u)$ for such $\varphi$. However, in the derivation of the differential equations and boundary conditions for $u$, I found that I needed the larger class $S_u$ to make it work. See the next subsection.

**Corollary 1.** For $\varphi, \psi \in S_u$, the functions $F(\varepsilon, \tilde{\varepsilon}) = \mathcal{J}(u + \varepsilon \varphi + \tilde{\varepsilon} \psi)$ and $G(\varepsilon, \tilde{\varepsilon}) = \mathcal{R}(u + \varepsilon \varphi + \tilde{\varepsilon} \psi)$ are of class $C^1$ on some neighborhood of $(0, 0)$.

This result is reasonably clear for $F$ because the formula for $\delta \mathcal{J}$ is valid for any $\varphi \in \mathcal{AC}[0, 1], \varphi' \in L^2$. Care must be exercised in the case of $G$ because the spaces $S_u$ change with $u$ and there could be trouble when $q < 1$ in (1).

**Lemma.** Fix $\varphi \in S_u$ and choose $m > 0$ so that $|\varphi| < m |u|$ on $[0, 1]$ and put $u_1 = u + \varepsilon_1 \varphi$. Then for any $\varepsilon_1, S_{u_1} \subset S_u$. If also $|\varepsilon_1| < 1/m$, then $S_{u_1} = S_u$ and $u$ and $u_1$ have the same zeros.

The first assertion is obvious from the definition of $S_u$. The second follows from $|u_1| \geq (1 - m |\varepsilon_1|) |u| = \mathcal{O}(u) = \mathcal{O}(u_1)$ for $|\varepsilon_1| < 1/m$.

Clearly, $u_1 \in H_1$. Lemma 1 and $|\varepsilon_1| < 1/m$ shows that $\varepsilon \mapsto \mathcal{R}(u_1 + \varepsilon \tilde{\varphi})$ has a derivative at $\varepsilon = 0$ for any $\tilde{\varphi} \in S_u = S_{u_1}$ given by $\delta \mathcal{R}(u_1, \tilde{\varphi}) = \int f'(u + \varepsilon_1 \varphi) \tilde{\varphi} \, dx$ which is obviously continuous in $\varepsilon_1$. By applying a similar argument to $\varepsilon \mapsto G(\varepsilon_1, \tilde{\varepsilon})$ one arrives at the desired conclusion.

The partial derivative, $G_2$, of $G$ w.r.t. $\tilde{\varepsilon}$ equals $\int f'(u + \varepsilon \varphi + \tilde{\varepsilon} \psi) \psi \, dx$ and similar formulas for $G_1, F_1, F_2$. These expressions are clearly continuous in $(\varepsilon, \tilde{\varepsilon})$. 

7
We now finish the proof of Theorem 2. We adapt here the proof given of a similar result in [3], page 23, but our assumptions differ. It’s a nice proof and worth repeating.

First choose any $\psi \in C^1_c(0,1) (\Rightarrow \psi \in S_u)$ such that $\delta R(u,\psi) > 0$. This is possible because $u > 0$ on $(0,1)$. Next, let $\varphi$ be any function in $S_u$ and let $F$ and $G$ be as in Corollary 1 with this $\psi$ and $\varphi$. Because $G_2(0,0) = \delta R(u,\psi) \neq 0$, and $G(0,0) = 1$, the implicit function theorem implies there exists a differentiable function $\tilde{\varepsilon} = \rho(\varepsilon)$, and a number $a > 0$ such that $\rho(0) = 0$ and

$$G(\varepsilon, \rho(\varepsilon)) = \int_0^1 f(u + \varepsilon \varphi + \rho(\varepsilon) \psi) \, dx \equiv 1, \quad -a \leq \varepsilon \leq a,$$

$$\rho'(0) = \frac{G_1(0,0)}{G_2(0,0)} = \frac{\delta R(u,\varphi)}{\delta R(u,\psi)}.$$

Hence $u + \varepsilon \varphi + \rho(\varepsilon) \psi \in M_i(q)$, so

$$F(\varepsilon, \rho(\varepsilon)) \geq F(0,0) = \mathcal{J}(u) = \kappa_i, \quad -a < \varepsilon < a$$

and this implies

$$0 = \frac{dF(\varepsilon, \rho(\varepsilon))}{d\varepsilon} \bigg|_{\varepsilon = 0} = \delta \mathcal{J}(u, \varphi) + \delta \mathcal{J}(u, \psi) \rho'(0)$$

$$= \delta \mathcal{J}(u, \varphi) - \left\{ \frac{\delta \mathcal{J}(u, \psi)}{\delta R(u, \psi)} \right\} \delta R(u, \varphi).$$

Put $\lambda = \delta \mathcal{J}(u, \psi)/\delta R(u, \psi)$, to get (2).

### 4 Proof of Theorem 3

The problem with the non-positive exponent $q - 1$ for $q \leq 1$ necessitates a somewhat roundabout method to avoid considering separate cases. (The subscript $i$ might be omitted when no confusion seems likely.)

Define, for $0 \leq x \leq 1$,

$$L(x) = \int_0^x \left[ 2u'^2 - \lambda uf'(u) \right] \, dt$$

Then $L' = 2u'^2 - \lambda uf'(u)$, a.e., and

$$L(1) = 2\kappa_i - \lambda \mu, \quad \mu = \mu(u) = \int_0^1 u(x)f'((u(x)) \, dx$$

We now make maximum use of the space $S_u$ as promised: Let $\varphi = \gamma u \in S_u$, $\gamma \in C^1[0,1]$. Then (2) transforms thus:

$$\delta \mathcal{J} - \lambda \delta R = \int_0^1 \left[ 2uu'\gamma + uu' \gamma \right] \, dx - \lambda \int_0^1 uf'(u) \gamma \, dx$$

$$= \int_0^1 L' \gamma \, dx + \int_0^1 2uu' \gamma' \, dx = \gamma(1)L(1) + \int_0^1 [2uu' - L] \gamma' \, dx = 0.$$
The DuBois-Reymond Lemma and the arbitrariness of the value $\gamma(1)$ justify the conclusion that $L(1) = 2\kappa_i - \lambda\mu = 0 \Rightarrow$ first equality in (5)) and that a constant $C_1$ exists such that $2uu' - L = C_1$, a.e. Hence

$$
\gamma(1)L(1) + \int_0^1 [2uu' - L]\gamma' \, dx = C_1 \int_0^1 \gamma' \, ds = [\gamma(1) - \gamma(0)]C_1 = 0
$$

which implies $C_1 = 0$ and then $2uu' = L$ a.e. In as much as $L$ is continuous and $u > 0$ on $(0, 1)$ it follows that there is a uniquely determined continuous version of $u'$ on $(0, 1)$ such that

$$
(8) \quad 2uu' = L \quad \text{on} \quad (0, 1)
$$

Because $L(1) = 0$, we get from (8) the boundary condition:

$$
(9) \quad u(1)u'(1-) = 0
$$

Equation (5) and the positivity of both $\kappa_i$ (Proposition 1) and the integral $\int uf'(u) \, dx$ imply that $\lambda > 0$.

If $0 < a < b < 1$, then $u$ is bounded away from 0 on $[a, b]$ and (8) implies $u'$ itself is absolutely continuous on $[a, b]$ with a uniquely determined continuous derivative $u''$. Differentiating (8), simplifying, and passing to a limit $a \downarrow 0, b \uparrow 1$, we find that

$$
(10) \quad 2u'' = -\lambda f'(u) \quad \text{on} \quad (0, 1)
$$

(In many cases, $u''$ will be unbounded at the zeros of $u$. This is the case, for example, if $f(u) = \sqrt{|u|}$.)

Because $u > 0$ on $(0, 1)$, $\lambda > 0$, and $f'(u) > 0$, (10) entails $u'$ strictly decreases on $(0, 1)$. This fact and $u(0) = 0$ then implies $K := u'(0+) > 0$ for otherwise the positivity of $u$ could not hold.

A multiplication of (10) by $u'$ leads to $(d\,dx)(u'^2) = -\lambda (d\,dx)f(u)$ and this is valid even at points where $u' = 0$ if any. Therefore $u'^2 = \text{Const.} - \lambda f(u)$ on $(0, 1)$. Continuity of $x \mapsto f(u(x))$ implies $u'(0+) = u'(1-) = 0$ both exist and thus it is permissible to say that (4) holds on $[0, 1]$ and $\text{Const.} = u'(0+)^2 = K^2$.

(Recall also that $f(0) = 0$.) Note that the continuous, absolute and uniform, of $u$ also imply that the right and left-hand derivatives of $u$ exist at $x = 0$ and $x = 1$ and that $u'$ is right and left continuous at $x = 0$ and $x = 1$, respectively.

Integrating (4) gives the second equality in (5).

Case 1 $u$ is a minimizer for the problem $\kappa_2$. Since $u(1) = u(0) = 0$ and $u'' < 0$ on $(0, 1)$ and $u'(0+) > 0$, there is a unique $c \in (0, 1)$ such that $u'(c) = 0$, $u' > 0$ on $(0, c)$ and $u' < 0$ on $(c, 1)$. From (4) we then obtain:

$$
(11) \quad u'(x) = \begin{cases} \sqrt{K^2 - \lambda f(u)}, & 0 \leq x \leq c, \\ -\sqrt{K^2 - \lambda f(u)}, & c \leq x \leq 1 \end{cases}
$$

Note that

$$
K^2 - \lambda f(u(c)) = 0
$$
The unique solution to (11) is given implicitly by
\[ \int_{0}^{u(x)} \frac{dz}{\sqrt{K^2 - \lambda f(z)}} = x \quad \text{for} \quad 0 \leq x \leq c, \quad \text{and} \]
\[ \int_{u(x)}^{u(c)} \frac{dz}{\sqrt{K^2 - \lambda f(z)}} = x - c \quad \text{for} \quad c \leq x \leq 1. \]

Since \( u(1) = u(0) = 0 \), the preceding formulas imply that \( c = 1 - c \). Hence \( c = 1/2 \) and the graph of \( u \) must be symmetric about the line \( x = 1/2 \). (This could also be established from a uniqueness result in the theory of O.D.E. because \( x \mapsto u(1 - x) \) is also a solution to (11).)

Let \( v(x) = u(x/2) \). Then
\[ \mathcal{J}(v) = \int_{0}^{1} v'^2 \, dx = (1/4) \int_{0}^{1/2} u'(x/2)^2 \, dx = (1/4)\kappa_2 \]

However,
\[ \int_{0}^{1} f(v(x)) \, dx = 2 \int_{0}^{1/2} f(u(t)) \, dt = 1 \]

by the symmetry of \( u \). It follows that \( v \in M_1 \) and \( \mathcal{J}(v) \geq \kappa_1 \). Hence \( \kappa_2 \geq 4\kappa_1 \).

**Case 2.** \( u \) is a minimizer for problem \( \kappa_1 \). Now that we know \( \kappa_2 > \kappa_1 \), we may assume \( u(1) > 0 \), by Theorem 1, and then \( u'(1-) = 0 \) by (9). This and (4) implies
\[ u' = \sqrt{K^2 - \lambda f(u)}, \quad 0 \leq x \leq 1, \quad K = u'(0+) > 0 \]

In this case, \( u \) is given uniquely by
\[ \int_{0}^{u(x)} \frac{dz}{\sqrt{K^2 - \lambda f(z)}} = x, \quad 0 \leq x \leq 1 \]

The vanishing of \( u'(1-) \) also implies \( \lambda f(u(1)) = K^2 \).

Define a function \( w \) by
\[ w(x) = \begin{cases} u(2x), & \text{for } 0 \leq x \leq 1/2 \\ u(2 - 2x), & \text{for } 1/2 \leq x \leq 1 \end{cases} \]

Then, as the reader may easily check, \( \int_{0}^{1} f(w(x)) \, dx = 1, \ w(0) = w(1) = 0, \) and \( w \in M_2 \). Hence \( \mathcal{J}(w) \geq \kappa_2 \). But
\[ \mathcal{J}(w) = 2 \int_{0}^{1/2} w'^2 \, dx = 8 \int_{0}^{1/2} u'(2x)^2 \, dx = 4\mathcal{J}(u) = 4\kappa_1 \]

so \( 4\kappa_1 \geq \kappa_2 \). From Case 1 we have the reverse inequality, and thus the last assertion of Theorem 3 follows.
5 Evaluations

 Ideally, one would like to express the value of $\kappa_1$ as an explicit formula involving one or more integrals of $f$. I have not been able to do this except in the very important special case

$$f(u) = |u|^q$$

for some $q > 0$. Here are the calculations.$^4$

From $\mu = \int u f'(u) \, dx = q \int u^q \, dx = q \, (u \in M_1)$, we have $\lambda = 2\kappa_1/q$ by (5). From (12), one may express the minimizer $u$ for the problem $\kappa_1 = \kappa(q)$ by

$$h(ru(x)) = rKx, \quad 0 \leq x \leq 1, \quad h(y) = \int_0^y \frac{dz}{\sqrt{1 - z^q}}, \quad 0 \leq y \leq 1$$

where $r = (\lambda/K^2)^{1/q}$. Moreover $u'' = K^2(1 - (ru)^q) = 0$ at $x = 1$ implies $ru(1) = 1$ and $h(ru(1)) = h(1) = rK$.

Integrating (4) yields $\kappa_1 = K^2 - \lambda = K^2 - 2\kappa_1/q$, or

$$K = (1 + 2/q)^{1/2}\kappa_1^{1/2}$$

and then

$$h(1) = rK = \lambda^{1/q}K^{1-2/q} = (2\kappa_1/q)^{1/q}(1 + 2/q)^{1/2}\kappa_1^{1/2}]^{1-2/q}$$

$$= \kappa_1^{1/2}(1 + 2/q)^{1/2}(1 + q/2)^{1-1/q}$$

However,

$$h(1) = \int_0^1 \frac{dz}{\sqrt{1 - z^q}} = (1/q) \int_0^1 t^{1/q-1}(1-t)^{1/2-1} \, dt = (1/q)B(1/q, 1/2)$$

where $B$ is the complete beta function. These machinations right readily vomit up the formula:

$$\kappa_1 = (1/2q)(1 + q/2)^{2/q-1} [B(1/q, 1/2)]^2$$

5.1 Two cases

The implicit equation for $u$ found in the last subsection simplifies considerably if $q = 1$ or $q = 2$.

If $q = 1$, then $\int_0^{ru(x)}(1 - z)^{-1/2} \, dz = rKx \Rightarrow u = ax^2 + bx$. From $\int u = 1$ & $u'(1) = 0$, one finds $u = 3(x - x^2/2)$ and $\kappa_1 = \int u'^2 = 3$, $\lambda = 6$.

If $q = 2$ then $u(x) = \sqrt{2}\sin(\pi x/2)$ and $\kappa_1 = \pi^2/4$. Direct methods also apply: Extend $u$ to a periodic function, period 4, antisymmetric about the lines $x = 2n$ and symmetric about lines $x = 2n + 1$, $n = 0, \pm 1, \ldots$. Then

---

$^4$When (13) holds, a much simpler approach is to use [1] which completely sidesteps the direct method of the calculus of variations used here.
\[
u(x) \simeq \sum_k \beta_k \sin(k^*x), \quad k^* = (2k-1)\pi/2, \quad \beta_k = 2 \int_0^1 u(x) \sin(k^*x) \, dx.
\]
Find \(\kappa(2)\) by first minimizing \(J(u|_Z) = \sum_{k \leq n} k^{2} \beta_k^{2}\) on the finite dimensional subspace, \(Z\), spanned by the first \(n\) elements of the sequence \(\{\sin(k^*x) ; \ k = 1, 2, \ldots\}\) with the constraint \(\int u^2 = (1/2) \sum_{k \leq n} \beta_k^2 = 1\) and then taking a limit. See [4], §40, theorem on page 196.

\[6 \text{ Limits at } q = 0, \ q = \infty\]

We have from (14) the interesting:
\[
\lim_{q \to 0^+} \kappa_1 = \frac{\pi e}{2} = 4.269867...
\]
and the less interesting \(\lim_{q \to \infty} \kappa_1 = 1\).

Using \(\|u\|_q = R^{1/q}(u) = 1, \ f = |u|^q\), as the constraint changes nothing in the analysis of \(\kappa_1\) for \(0 < q < \infty\) but does allow for a logical extension of the problem to \(q = 0\) and \(q = \infty\).

**The case \(q = 0\).**

For \(q = 0\), the condition \(\|u\|_q = 1\) changes to
\[
\|u\|_0 = \lim_{q \to 0^+} \|u\|_q = \exp \left( \int_0^1 \log |u| \, dx \right) = 1 \Rightarrow \int_0^1 \log |u| \, dx = 0
\]
and the problem becomes: Compute
\[
\kappa = \inf \{J(u) : u \in AC[0, 1], \log |u| \in L^1[0, 1], \int_0^1 \log |u| \, dx = 0, \ u(0) = 0\}
\]

If a non-negative minimizer \(u\) exists in the admissible space, then a formal (non-rigorous) calculation leads to the equation
\[
2uu'' = -\lambda, \quad \text{for some } \lambda > 0
\]
with boundary conditions \(u(0) = 0, \ u'(1) = 0\). Here we briefly solve this equation.

A first integral, assuming \(u > 0\), equals
\[
u' = K - \lambda \log u.
\]
Integration of this equation shows that \(K^2 = \kappa > 1\). In as much as \(u(0+) = 0\) we see that \(u'(0+) = +\infty\). Put \(r = \lambda/K^2\), and
\[
h(z) = \int_0^z \frac{s^{1/r-1}}{\sqrt{1 - \log s}} \, ds, \quad 0 \leq z \leq e.
\]
Then \(h(u(x)^r) = Krx\), which can be checked by differentiation. At \(x = 1\), \(u'(1) = 0\) and consequently \(u(1) = e^{K^2/\lambda} = e^{1/r}\). Making the change of variables \(Krx = h(t)\), \(dx = (1/Kr)h'(t)\,dt\), with new limits \(t = 0, \ t = e\), we get the following equation:
\[
\int_0^1 \log u \, dx = K \int_0^e \log(t) h'(t) \, dt = K \int_0^e \frac{t^{(1/r)-1}\log(t)}{\sqrt{1 - \log t}} \, dt = 0
\]
Putting $t = e^{-ry + 1}$, the last integral goes to

$$
\int_0^\infty (ry - 1)(ry)^{-1/2} e^{-y} \, dy = 0
$$

which has the solution: $r = 2$. Therefore, $h(z) = \int_0^z s^{-1/2} (1 - \log z)^{-1/2} \, ds$, and then

$$
rK = 2K = 2\sqrt{\kappa} = h(e) = \int_0^e \frac{s^{-1/2} \, ds}{\sqrt{1 - \log s}} = \sqrt{2\pi e},
$$
a straightforward evaluation. So $\kappa = K^2 = \pi e / 2$ as predicted.

**The case $q = \infty$.**

As $q \to \infty$, $\|u\|_q \to \|u\|_\infty$ and the variational problem is to compute

$$
\kappa = \inf \{ J(u) : u \in AC, u(0) = 0, \max_{0 \leq x \leq 1} |u(x)| = 1 \}
$$

The boundary condition $u'(1) = 0$ is no longer imposed because the functional $\|u\|_\infty$ does not have a useful variation. The admissible class includes the function $u(x) = x$ which gives the value $\int u'^2 = 1$. But for any admissible $u$ we have $1 = R(u) = \|u\|_\infty \leq \|u'\|_2$. Therefore $\kappa = 1$ as anticipated by the limit.

**References**


