The non-random nature of random limit points

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**Theorem 1** If for a sequence \( \{Z_t, t = 1, 2, \ldots\} \) of random variables on a complete probability space \((\Omega, \mathcal{F}, P)\) with values in a compact metric space \((E, d)\) we have

\[
(0) \quad P\{Z_t \in V \ i. \ o.\} = 0 \text{ or } 1 \text{ for every open set } V,
\]

and if

\[
G = \{x : P\{d(Z_t, x) < \varepsilon \ i. \ o.\} = 1, \ \forall \ \varepsilon > 0\},
\]

then \( G \neq \emptyset \) and

\[
(1) \quad P\{\omega : A(\omega) = G\} = 1,
\]

where

\[
A(\omega) = \{x : \liminf_{t \to \infty} d(Z_t(\omega), x) = 0\}
\]

\[
= \bigcap_{n=1}^{\infty} \{Z_t(\omega) : t \geq n\}.
\]

1 Proof.

First \( G \) is closed. For if \( G \) is non-empty and if \( y \) is a limit point of \( G \) which we may suppose is not isolated, then for any neighborhood \( V \) of \( y \) we can find a neighborhood \( V_0 \) of some point of \( G \) which is contained in \( V \). But then \( P\{Z_t \in V \ i. \ o.\} \geq P\{Z_t \in V_0 \ i. \ o.\} = 1 \). Since \( V \) is any neighborhood of \( y \) the result follows.

Next we show \( G \) is non-empty. Without loss of generality we may suppose that the diameter of \( E \) is 1. Let \( N(x, r) \) denote the open ball centered at \( x \) with radius \( r \). Since \( E \) is compact, we can cover \( E \) with a finite number
of balls of radius $1/2$, and since $Z_t$ stays in $E$ for all $t$, we can find at least one of these balls of radius $1/2$ that will be entered infinitely often with positive probability and therefore with probability 1 by (1). Call this ball $V_1 = N(z_1, 1/2)$. Cover the compact set $V_1$ with a finite number of balls of radius $1/4$ each having a non-empty intersection with $V_1$. Then $Z_t$ must enter at least one of these, i. o., with probability 1. Write it as $V_2 = N(z_2, 1/4)$. We continue inductively. At the $k$th stage we get a ball of radius $1/2^k$ of the form $V_k = N(z_k, 1/2^k)$ where $z_k \in V_{k-1}$.

Also at each stage we can find a null set of paths, say $\Xi_k$, such that if $\omega \notin \Xi_k$, then $Z_t(\omega) \in V_k$ i. o.. The sequence of centers of these balls, $\{z_k\}$, clearly form a Cauchy sequence. Let $z = \lim_k z_k$. If $r > 0$ then $V_k \subset N(z, r)$ for all $k$ sufficiently large. If $\omega \notin \cup_k \Xi_k$, then $Z_t(\omega)$ will enter all $V_k$ infinitely often and therefore will enter $N(z, r)$ infinitely often. In as much as $\cup_k \Xi_k$ is a null set, $z$ must be in $G$ by definition of $G$.

Let $z_1^n, \ldots, z_{k(n)}^n$ be points in $G$ such that every point of $G$ is at a distance strictly less than $1/2^n$ of one of these points. This can be done for every $n$ because $G$ is compact. Also put

$$O_n = \bigcup_{i=1}^{k_n} N(z_i^n, 1/2^n).$$

Then $G \subset O_n$ for every $n$; moreover $\min(d(x, y) : x \in G, y \in O_n) > 0$. By definition of $G$ for each $y \in O_n$ there is a $r = r(y)$ such that $P\{Z_t \in N(y, r) \text{ i. o.}\} = 0$, or, put another way,

$$P\{Z_t \notin N(y, r) \text{ for all } t \geq \ell \text{ for some } \ell\} = 1$$

The sets $\{N(y, r(y))\}_{y \in O_n}$ make a covering of $O_n$, so there exists $y_1^n, \ldots, y_{m_n}^n$ in $O_n$ such that

$$O_n^c \subset \bigcup_{i=1}^{m_n} N(y_i^n, r_i^n), \quad r_i^n = r(y_i^n).$$

Define

$$\Omega_n = \{w : Z_t(\omega) \in N(z_i^n, 2^{-n}) \text{ i. o. } \forall i = 1, \ldots, k_n\}$$

$$\Lambda_\ell = \{\omega : Z_t(\omega) \notin N(y_i^n, r_i^n) \forall t \geq \ell \forall i = 1, \ldots, m_n\} \quad \Lambda = \bigcup_{\ell=1}^\infty \Lambda_\ell$$

Then $P(\Omega_n \cap \Lambda) = 1$ for every $n$, hence, if we put

$$\Omega_\infty = \bigcap_{n \geq 1} \Omega_n \Lambda,$$
then \( P(\Omega_\infty) = 1 \). Finally let us define

\[ \tilde{\Omega}_r = \{ w : Z_t(w) \in N(z, r) \text{ i. o. for every } z \in G \} \].

Clearly \( \Omega_\infty \subset \tilde{\Omega}_r \) for every \( r > 0 \). Indeed \( \Omega_n \subset \tilde{\Omega}_r \) as soon as \( 1/2^n < r/2 \). It follows that \( P(\tilde{\Omega}_{0+}) = 1 \) where

\[ \tilde{\Omega}_{0+} = \bigcap_{r > 0, \text{ r rational}} \tilde{\Omega}_r. \]

Recalling the definition of \( A(\omega) \) it is now clear that

\[ \omega \in \Omega_\infty \implies A(\omega) \subset G \]
\[ \omega \in \tilde{\Omega}_{0+} \implies G \subset A(\omega). \]

These implications and \( P\{\Omega_\infty \triangle \tilde{\Omega}_{0+}\} = 0 \) yield (1).

For a proof of a similar result for normalized sums of independent random variables, see Theorem 1 on page 1174 of the paper by H. Kesten, (970), *The limit points of normalized random walk. Ann. Math. Statist.* 41.