Divergent sums over excursions *

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Abstract
Criteria for the almost sure divergence or convergence of sums of functions of excursions away from a recurrent point in the state space of a Markov process are proved and then applied to the excursions from 0 of reflecting linear diffusions, in particular reflecting Brownian motion, to derive some sample path properties of these processes.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a canonical standard Markov process and 0 a recurrent point of the state space which is regular for itself, but is neither a trap nor a holding point. Introduce the following additional objects:

- $\sigma$ the hitting time of $X$ at 0: $\inf\{t : t > 0, X(t) = 0\}$
- $L$ the local time at 0 normalized so that $E e^{-\sigma} = E \int_0^\infty e^{-s} dL_s$, $E \equiv E^0$
- $\beta$ the right continuous inverse of $L$: $\beta_t = \inf\{s : L_s > t\}$
- $\beta^{-}$ the left continuous inverse of $L$: $\beta^{-} = \lim_{r \uparrow t} \beta_r$, $\beta^{-}_0 = 0$
- $G$ the random set of strictly positive $s$ at which $s = \beta^{-}_t$ for some $t$ with $\beta^{-}_t < \beta_t$. Or, more simply, the left-hand end points of the excursion intervals of $X$ away from 0.
- $\tau$ a truncation map taking $\Omega$ to itself: $\tau X = X \circ \tau = X'$ where $X'(\omega)(t) = X_t(\omega)$ for $t < \sigma(\omega)$, $X'(\omega)(t) = 0$ for $t \geq \sigma(\omega)$. Note that $\sigma \circ \tau = \sigma$.
- $\hat{p}$ the excursion measure: $\hat{p}(A)$ is the expected number of excursions of $X$ which have the property $A$ during any interval of time that the local time increases by a unit amount.

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*This is a slightly expanded, slightly corrected, version of a paper published in *Stochastic Processes and their Applications* 54 (1994) 175-182. This version includes a new example, applying Theorem 4, and more details of the proof of Theorem 4. The Theorem numberings here differ slightly from the original.
The reader is referred to Blumenthal’s (1992) book, in particular Chapter III, Section 3, for a detailed study of the excursions of a Markov process.

Let $f$ be a bounded nonnegative function on $[0, \infty) \times \Omega$ which has the property that $f(\cdot, \tau(\cdot))$ is $\mathcal{B}([0, \infty)) \times \mathcal{F}$ measurable where $\mathcal{F}$ is the $\hat{\rho}$–completion of the $\sigma$–algebra generated by the coordinate functions. Put

$$
\Sigma_1(f; a, b) = \sum_{s \in G, \beta(a) < s \leq \beta(b)} f(L_s, \tau \circ \theta_s), \quad \Sigma_2(f; a, b) = \sum_{s \in G, a < s \leq b} f(s, \tau \circ \theta_s)
$$

**Theorem 1.** (i) For every $b > 0$, $\Sigma_1(f; 0, b) = \infty$ a.s. if and only if $E\{\Sigma_1(f; 0, b)\} = \infty$. (ii) Assume that $s \mapsto f(s, \omega)$ is decreasing for every $\omega$ and that every $a > 0$

$$
\Sigma_2(f; \delta, a) \text{ is finite a.s. for every } \delta, \ 0 < \delta \leq a. \quad \text{(0.1)}
$$

Then $\Sigma_2(f; 0, a) = \infty$ a.s. if and only if $E\{\Sigma_2(f; 0, a)\} = \infty$.

Excursion formulas relate expectations of sums such as those occurring in Theorem 1 to integrals involving $\hat{\rho}$. In particular, if $0 \leq a < b < \infty$,

$$
E\{\Sigma_1(f; a, b)\} = \int_a^b \hat{\rho}(f_t) \, dt, \quad \text{(0.2)}
$$

$$
E\{\Sigma_2(f; a, b)\} = E\left\{ \int_a^b \hat{\rho}(f_s) dL_s \right\} \quad \text{(0.3)}
$$

For a proof of (0.3) in the case that $f$ does not depend on $s$, see Blumenthal (1992, III.3); for the general case, see Maisonneuve (1975); for (0.2), see Section 2 below.

For some diffusions on $[0, \infty)$, in particular reflecting Brownian motion and excursions away from the origin, it is possible to calculate the excursion measure of some events quite explicitly. In conjunction with Theorem 1 and the excursion formulas, these calculations lead to some interesting zero–one type laws for the small time behavior of the sample paths. Below are several such results including the two discovered by Knight using different methods. No doubt the reader can think of other possibilities.

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1. $f(s, \tau \circ \theta_s)(\omega) = f(s, \theta(s)(\omega))$, and $f(L_s, \tau \circ \theta_s)(\omega) = f(L_s(\omega), \theta(s)(\omega))$. Also $f_t(\cdot) = f(t, \cdot)$, $X_t(\cdot) = X(\cdot, t)$, etc.
In the first two applications $X$ is a standard reflecting Brownian motion on $[0, \infty)$. In Theorem 2 we get a 0–1 law for the maximums over excursion intervals compared with their lengths. In Theorem 3 we get a lower function 0–1 law for the “areas” under excursions. In Theorem 4 we have a 0–1 law for the excursion areas compared with their lengths.

Throughout we suppose that $h$ is a continuous, positive, increasing function on $[0, \infty)$, $h(0+) = 0$. In Theorem 2 we also suppose that $r \mapsto h(r)/\sqrt{r}$ is decreasing and tends to $\infty$ as $r$ tends to 0. As usual, i.o. is the abbreviation for “ininitely often,” and a.s. is for “almost surely.”

**Theorem 2.** For $s > 0$ put $\sigma_s = \sigma \circ \theta_s$ = the first time back to the origin after time $s$, and $M_s = \max\{X_r : s \leq r \leq s + \sigma_s\}$; then

$$P\{M_s \geq h(\sigma_s) \text{ i.o. as } s \downarrow 0, s \in G\} = 0 \text{ or } 1$$

according as

$$\int_{0+} t^{-5/2} h(t)^2 e^{-2h(t)^2/t} \, dt$$

is finite or not.

**Example.** $M_s \geq \sqrt{\frac{4}{\sigma_s} \log 1/\sigma_s + \kappa \sigma_s \log \log 1/\sigma_s} \text{ i.o. as } s \downarrow 0 \text{ in } G, \text{ a.s., if and only if } \kappa \leq 1$.

**Theorem 3.** Let $A_s = \int_s^{s + \sigma_s} X_r \, dr$. Then

$$P\{A_s \geq h(s) \text{ i.o. as } s \downarrow 0\} = 0 \text{ or } 1$$

according as

$$\int_{0+} s^{-1/2} h(s)^{-1/3} \, ds$$

is finite or infinite.

**Example.** $A_s \geq s^{3/2} |\log s|^3 |\log |\log s|| r \text{ i.o., as } s \downarrow 0, \text{ a.s., if and only if } r \leq 3$.

**Theorem 4.** Assume that $h$ is increasing and that $b^3/h(b)^2 \to \infty$ as $b \to 0$. Let $A_s = \int_s^{s + \sigma_s} X_t \, dt$. Then

$$P\{A_s \leq h(\sigma_s) \text{ i.o. as } s \downarrow 0 \text{ in } G\} = 0 \text{ or } 1$$

\[\text{In the journal version of this paper, this theorem appeared at the end of the paper as it was a last minute addition.}\]
according as the integral

$$\int_{0^+} b^{3/2} h(b) - 2 e^{-\gamma b^3/h(b)^2} \, db$$

is finite or not where \( \gamma = 2|a_1|^3/27 \) and \( a_1 \) is the largest (negative) zero of the standard Airy function. \(^3\)

Example. \( P\{A_s \leq \kappa \sigma_s^{3/2} \} \) i.o., \( s \downarrow 0 \) in \( G \) = 0 or 1 according as \( \kappa < \sqrt{2\gamma} \) or \( \kappa > \sqrt{2\gamma} \). \(^4\)

For the next two results, let \( X \) be a persistent nonsingular diffusion on \([0, \infty)\) on natural scale. We will suppose that 0 is both an exit and an entrance boundary point. Then 0 is regular for itself and is neither a trap nor a holding point; Itô and Mckean (1974; Chapter 3). We denote by \( q(t, x, y) \) the transition density of \( X \) with respect to the speed measure. Theorems 5 and 6 are due to Frank Knight: Theorems 0 and 2 in Knight (1973). Given the excursion theory our proofs seem to be a little shorter than his.

**Theorem 5.** \( P\{X(t) > h(\beta - (L_t)) \} \) i.o. as \( t \downarrow 0 \) = 0 or 1 according as

$$\int_{0^+} h(t)^{-1} q(t, 0, 0) \, dt$$

is finite or not.

**Theorem 6.** Let \( X_t^* = \max\{X_r : 0 \leq r \leq t\} \), then

$$P\{X_t^* > h(t) \} \text{ i.o. as } t \downarrow 0 = 0 \text{ or } 1$$

according as \( \int_{0^+} h(t)^{-1} \, dt \) is finite or not.

## 1 Proof of Theorem 1

It is easy to see that type 1 sums are integrals of a convenient sort:

$$\Sigma_1(f; a, b) = \int_{(a, b) \times U} f(t, u) Y_\omega(dt, du),$$

\(^3\gamma \approx .9468.\)

\(^4\sqrt{2\gamma} \approx 1.376.\)

\(^5\)Knight calls a function \( h \) lower in local time when this probability is 1.
where $Y$ is the Poisson point process of excursions regarded as a random measure on $[0, \infty) \times U$, $U$ being the excursion path space; see Blumenthal (1992, Section III.2–3). The intensity measure of $Y$ is the product of Lebesgue measure and $\tilde{p}$. The conclusion of part (i) and formula (1.2) may thus be seen as a consequence of Campbell’s theorem; see Kingman (1993, p.28).

Part (ii) of theorem 1 does not have such a simple proof. Fix $\varepsilon > 0$ and define two increasing sequences of positive random variables as follows: $T_0 = 0$ and for $j \geq 1$,

$$S_j = \inf \{ s : s \geq T_{j-1}, L(s + \varepsilon) = L(s) \} = T_{j-1} + S_1 \circ \theta_{T(j-1)},$$

$$T_j = \inf \{ s : s \geq S_j, X(s) = 0 \} = S_j + \sigma \circ \theta_{S(j)}.$$

In words the $S_j$ are the successive times that $X$ begins an excursion of length at least $\varepsilon$ and the $T_j$ are the successive (stopping) times that $X$ is at 0 at the ends of these excursions. It is easily seen that for $j$ and $k$ with $j < k \geq 1$ we have

$$S_j = T_k + S_{j-k} \circ \theta_{T(k)}.$$  \(1.1\)

Write $G(\varepsilon)$ for the point set $\{ S_1, S_2, \ldots \}, V_k = I(S_k \leq a) f(S_k, \tau \circ \theta_{S(k)}), and$$\Sigma_{2, \varepsilon}^2 = \sum_{s \in G, s \leq a} f(s, \tau \circ \theta_s) = \sum_{k=1}^{\infty} V_k.$$

The sets $G(\varepsilon)$ increase to $G$ as $\varepsilon$ decreases to 0 so the random variables $\Sigma_{2, \varepsilon}$ increase to $\Sigma_2 = \Sigma_2(f; 0, a)$. Furthermore, because $S_{k+1} - S_k \geq \varepsilon$, $E\{ \Sigma_{2, \varepsilon} \} \leq \|f\|\infty a/\varepsilon$ is finite.

We now establish that

$$E\{ (\Sigma_{2, \varepsilon})^2 \} \leq C \max[1, (E\{ \Sigma_{2, \varepsilon} \})^2],$$  \(1.2\)

where $C$ is a constant independent of $\varepsilon$. Let $j$ and $k$ be integers, $j > k \geq 1$. It is clear from (1.1) that the event $\{ S_j \leq a \}$ implies the event $\{ S_{j-k} \circ \theta_{T(k)} \leq a \}$. Also, since the function $f$ is decreasing in its first variable, we have

$$f(S_j, \tau \circ \theta_{S(j)}) = f(T_k + S_{j-k} \circ \theta_{T(k)}, \tau \circ \theta_{S(j-k)} \circ \theta_{T(k)}) \leq f(S_{j-k} \circ \theta_{T(k)}, \tau \circ \theta_{S(j-k)} \circ \theta_{T(k)})$$

and consequently $V_j \leq V_{j-k} \circ \theta_k$. Applying this inequality and the strong Markov property we get

$$E(V_k V_j) \leq E[V_k V_{j-k} \circ \theta_{T(k)}] = E[V_k E[V_{j-k} \circ \theta_{T(k)}|\mathcal{F}]]$$

$$= E[V_k E^X(T_k) \{ V_{j-k} \}] = (EV_k)(EV_{j-k}).$$
(The necessary $\mathcal{F}_{T(k)}$-measurability of $V_k$ is easily established and its proof is safely omitted.) Hence,

$$E(\Sigma_2, \varepsilon)^2 = E \sum_k V_k^2 + 2 \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} E(V_k V_{j-k})$$

$$\leq \|f\| E\Sigma_2, \varepsilon + 2 \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} (E V_k)(E V_{j-k})$$

$$= \|f\| E\Sigma_2, \varepsilon + 2 [E\Sigma_2, \varepsilon]^2,$$

and (1.2) follows with $C = 2 + \|f\|$.

If $\Sigma_2 = \infty$ a.s., then $E \Sigma_2 = \infty$, so let us suppose $E \Sigma_2 = \infty$. In this case, $\lim_{\varepsilon \to 0} E(\Sigma_2, \varepsilon) = \infty$ by monotone convergence. From this and (1.2) it is straightforward to show (see Kochen and Stone (1964) for a typical argument) that the event

$$\left\{ \limsup_{\varepsilon \to 0} \frac{\Sigma_2, \varepsilon}{E(\Sigma_2, \varepsilon)} > r \right\}$$

has nonzero probability for some strictly positive number $r$ from which fact it follows that the event $\{\Sigma_2 = \infty\}$, which contains the above event, also has positive probability. But clearly, condition (1.1) of Theorem 1 demands that $\{\Sigma_2 = \infty\}$ is in $\cap_{r>0} \mathcal{F}_r = \mathcal{F}_0$ and therefore this event must have probability 1 because $\mathcal{F}_0$ is a.s. trivial. This completes the proof.

2 Proof of Theorem 2

We apply part (i) of Theorem 1. Let $f(s, \omega) = f(\omega) = I\{M \geq h(\sigma)\}(\omega)$ where $M = \max\{X_s : 0 < s \leq \sigma\}$. Then since $\beta_a \downarrow 0$ as $a \downarrow 0$, we have

$$\{M_s \geq h(\sigma_a) \text{ i.o. as } s \downarrow 0 \text{ in } G\} = \{\Sigma_1(f; 0, a) = \infty \text{ for some } a > 0\}.$$

From (1.2) and the fact that $f$ does not depend on $s$ and $\Sigma_1(f; 0, a)$ increases with $a$, we have that $P\{\Sigma_1(f; 0, a) = \infty \text{ for some } a > 0\} = 1$ if and only if $\hat{p}(M > h(\sigma)) = \infty$. From Williams (1979, p.99), exercise at the bottom of the page, we find the formula (in slightly different notation):

$$\hat{p}(M > x \mid \sigma = t) = 2 \sum_{k=1}^{\infty} [t^{-1}(2kx)^2 - 1] e^{-2k^2x^2/t}.$$
Also, \(^6\) see Blumenthal (1992, p. 112),
\[
\hat{p}\{\sigma > t\} = 1/\sqrt{\pi t}
\]
(2.1)

Whatever be the specific form of the quantities, we have for \(\varepsilon > 0\),
\[
\hat{p}\{M > h(\sigma)\} = \left( \int_0^\varepsilon + \int_\varepsilon^\infty \right) \hat{p}\{M > h(t) \mid \sigma = t\} \hat{p}\{\sigma \in dt\} \equiv J_1 + J_2.
\]

Since \(x \mapsto \hat{p}\{M > h(t) \mid \sigma = t\}\) is decreasing and bounded by 1 and \(h\) is increasing and strictly positive, it follows that \(J_2 \leq \hat{p}\{\sigma > t\} = 1/\sqrt{\pi t} < \infty\), and \(\hat{p}\{M > h(\sigma)\}\) is finite or not according as \(J_1\) is finite or not. From the above formulas we have
\[
J_1 = \int_0^\varepsilon \sum_k v_k(t) dt/2\pi
\]
where \(v_k(t) = t^{-3/2}[4k^2t^{-1}h(t)^2 - 1] e^{-2k^2h(t)^2/t}\). The function \(h(t)^2/t\) increases to \(\infty\) as \(t\) decreases to 0, so we can find a constant \(C_1\) and \(\varepsilon\) such that for all \(t\), \(0 < t \leq \varepsilon\) and every \(k \geq 2\),
\[
0 < [4k^2t^{-1}h(t)^2 - 1] e^{-2k^2h(t)^2/t} \leq [C_1/k^2] [h(t)^2/t] e^{-2h(t)^2/t}.
\]
Consequently, for another constant \(C_2\),
\[
\sum_{k \geq 2} v_k(t) \leq C_2 t^{-5/2} h(t)^2 e^{-2h(t)^2/t} \equiv C_2 u(t)
\]
for \(0 < t \leq \varepsilon\). Also \(v_1(t) \sim 4u(t)\) as \(t \downarrow 0\). From these calculations it follows that for \(\varepsilon > 0\) sufficiently small \(J_1\) is equivalent to \(\int_0^\varepsilon u(t) dt\). This completes the proof of Theorem 2.

3 Proof of Theorems 3 and 4

Here we apply Theorem 1(ii) and formula (1.3). Put
\[
f(s, \omega) = I\left\{ \int_0^\sigma X_r \: dr > h(s) \right\}(\omega).
\]

\(^6\)Multiplying by \(\sqrt{2}\) gives Williams’ formula for \(\hat{p}\{M > x \mid \sigma = t\}\). The reason for the discrepancy is that the normalization of local time chosen here entails that \(L\) is \(\sqrt{2}\) times the Itô–McKean–Williams Brownian local time. From this it follows that the constant \(C_0\) which appears in Section 4 has the value \(\sqrt{2}\). This is also the value of \(C_0\) in Section 5 as can be seen from the formula for \(\ell\) in terms of a time change of Brownian local time in Itô–McKean (1992, §5.4).
Then, provided (1.1) holds,

\[{A_s > h(s) \text{ i.o. } s \downarrow 0, s \in G} \equiv \{\Sigma_2(f; a, 0) = \infty \forall a > 0 \text{ sufficiently small}\}\]

Let \(W\) denote the BES(3) process, \(Z\) the process \(t \mapsto Z_t = (1 - t)W(t/(1 - t)), 0 < t \leq 1, Z_1 = 0\), and let \(\mu_b\) denote the law of the scaled process \(r \mapsto \sqrt{b}Z_{r/b}, 0 \leq r \leq b\). Then \(\bar{\mu} = \int_0^\infty \mu_b \, d\bar{\mu}\{\sigma \in db\} = \int_0^\infty \mu_b \, d\bar{\mu}\{\sigma \in db\}/2\pi^{1/2}\) (i.e., \(\mu_b(\cdot) = \bar{\mu}\{\sigma = b\}\) in the notation of the previous section). See Blumenthal (1992, p. 42 and p. 112). Now if \(q(t, x, y)\) denotes the transition density of \(X\) with respect to Lebesgue measure, then there exists a constant \(C_0\) such that \(E\{dL(s)\} = C_0q(s, 0, 0)\) (see the next section). In our case \(q(t, 0, 0) = (2/\pi t)^{1/2}\). Put \(\xi = \int_0^t Z_t \, dt, \mu = \mu_1 = \text{law of } Z\), and \(C_1 = (2/\pi)^{1/2}/(1/2^{1/2})C_0 = C_0/\pi^{1/2}.\) Then

\[E\Sigma_2(f; \delta, a) = E\left\{\int_0^a \bar{\mu}\left[\int_0^a X_r \, dr > h(s)\right] \, dL(s)\right\}\]

\[= C_1 \int_0^a \left(\int_0^\infty \mu\left\{\int_0^a \sqrt{b}Z_{r/b} \, dr > h(s)\right\} b^{-3/2} \, db\right) s^{-1/2} \, ds\]

\[= C_1 \int_0^a \left(\int_0^\infty \mu\{\xi > b^{-3/2}h(s)\} b^{-3/2} \, db\right) s^{-1/2} \, ds\]

\[= \frac{2}{3} C_1 \int_0^a \left(\int_0^\infty \mu\{\xi > z^{-2/3}\} z^{-2/3} \, dz\right) h(s) s^{-1/3} \, ds\]

\[= C_2 \int_0^a h(s) s^{-1/3} \, ds\]

where \(C_2 = 2C_1E_\mu(\xi)1/3 < \infty\). The last displayed integral is finite since the integrand is bounded on \([\delta, a]\) for \(0 < \delta \leq a < \infty\); thus, (1.1) is satisfied since the exhibited sum there has a finite expectation. Clearly, \(s \mapsto f(s, \omega)\) is nonincreasing for each \(\omega\) because \(h\) is nonincreasing. Therefore, Theorem 1(ii) applies. Taking \(\delta = 0\) in the preceding computation, gives \(E\{\Sigma_2(f; 0, a)\} = C_2 \int_0^a h(s) s^{-1/2} \, ds\) and the conclusion of Theorem 3 is apparent.

The process \(Z\) defined above process is identical in law to the process \(\{\|B^0_t\|, t \geq 0, P^0\}\) where \(B^0\) is the standard tied–down three dimensional Brownian motion and \(\| \cdot \|\) is the customary Euclidean norm. Takács (1992) has computed the asymptotics of the distribution of the integral \(\int_0^1 \|B^0_t\| \, dt\) and this integral coincides with \(\xi\) defined above. More precisely, it follows from his work that

\[\mu\{\xi \leq x\} \sim \sqrt{6\gamma} x^{-2} e^{-\gamma/x^2}, \text{ as } x \to 0+,\]
where \( \gamma \) is the number defined in Theorem 4 and \( \mu = \mu_1 \) is the measure introduced in the proof of Theorem 3. Let \( g(s, \omega) = g(\omega) = I\{A \leq h(\sigma)\} \), \( A = \int_0^\sigma X_r \, dr \). Then

\[
\{A_s \leq h(\sigma_s) \ \text{i.o.} \} = \{\Sigma_1(g, 0, a) = \infty \ \text{for some} \ a > 0\}.
\]

Applying Theorem 1 and (1.3) we find that \( P\{A_s \leq h(\sigma_s) \ \text{i.o.} \} = 0 \) or 1 according as

\[
\int_0^a \hat{p}(g_t) \, dt = a \hat{p}(g)
\]

is finite or not. Computing as in the proof of Theorem 3, we get

\[
\hat{p}(g) = \hat{p}\{A \leq h(\sigma)\} = \int_0^\infty \hat{p}\{A \leq h(b) \mid \sigma = b\} \hat{p}\{\sigma \in db\}
\]

\[
= \left( \int_0^\varepsilon + \int_\varepsilon^\infty \right) \mu\{\xi \leq b^{-3/2}h(b)\} b^{-3/2} \, db = I_1 + I_2.
\]

The integral \( I_2 \) is bounded by \( \int_\varepsilon^\infty b^{-3/2} \, db \) which is finite. Takàc’s asymptotic formula and the condition \( b^3/h(b)^2 \to \infty \) as \( b \to 0 \) implies that for \( \epsilon > 0 \) sufficiently small, \( I_1 \) is bounded above and below by a constant times

\[
\int_0^\varepsilon \left(b^{-3/2}h(b)\right)^{-2} \exp \left[ -\gamma \left(b^{-3/2}h(b)\right)^{-2}\right] b^{-3/2} \, db
\]

\[
= \int_0^\varepsilon b^{3/2}h(b)^{-2} \exp[ -\gamma b^3/h(b)^2] \, db.
\]

The conclusion of Theorem 4 now easily follows.

### 4 Proof of Theorems 5 and 6

It is well known that for a one-dimensional diffusion such as \( X \) there exists a jointly continuous local time functional \( \ell(t, x) = \ell(t, x, \omega) \) which satisfies

\[
E^x[\ell(t_2, y) - \ell(t_1, x)] = \int_{t_1}^{t_2} q(s, x, y) \, ds,
\]

for \( 0 \leq t_1 \leq t_2 \) and \( x, y \) in the state interval; see Itô and Mckean (1974, §5.4). Moreover the local time at a point is unique up to a constant multiple. It follows that there exists a constant \( C_0 > 0 \) such that for any nonnegative Borel \( g \) on \([0, \infty)\),

\[
E\left\{ \int_0^t g(s) \, dL(s) \right\} = C_0 \int_0^t g(s) q(s, 0, 0) \, ds
\]

\[
= C_0 \int_0^t g(s) q(s, 0, 0) \, ds
\]

(4.1)
We need one additional formula: for \( x > 0 \),
\[
\hat{p}\{M > x\} = \hat{p}\{\sigma_x < \sigma\} = 1/x\sqrt{2}, \tag{4.2}
\]
where \( \sigma_x \) is the hitting time of the point \( x \). The proof of this formula follows verbatim the proof of the same formula in the Brownian motion case; see Blumenthal (1992, pp.110–111). Only the fact that Brownian motion is a natural scale diffusion is used in that proof.

As to the proof of Theorem 5, note first that for each \( t > 0 \) \( \beta^-(L_t) \) is the largest \( s \) in \( G \) which does not exceed \( t \). Hence, \( h(\beta^-(L_t)) \) is constant for \( t \) in the interval \([s, s + \sigma \circ \theta_s]\). Let \( f(s, \omega) = I\{M > h(s)\}(\omega) \), then
\[
\{X_t > h(\beta^-(L_t)) \text{ i.o. } t \downarrow 0\} = \{M_s > h(s) \text{ i.o. } s \downarrow 0, s \in G\}
= \{\Sigma_2(f; 0, a) = \infty, \forall a > 0 \text{ small}\},
\]
provided (1.1) holds. Applying (1.3), (4.1), and (4.2) we get
\[
E\{\Sigma_2(f; \delta, a) = E\left\{ \int_{\delta}^{a} \hat{p}(f_s) \, dL(s) \right\} = C_0 \int_{\delta}^{a} h(s)^{-1}q(s, 0, 0) \, ds.
\]
One may now proceed as in the last part of the proof of Theorem 3.

To prove Theorem 6 observe that, a.s.,
\[
\{X^*(\beta^-) > h(t) \text{ i.o. } t \downarrow 0\} = \{X^*(s) > h(L_s) \text{ i.o. } s \downarrow 0, s \in G\}
= \{\Sigma_1(f; 0, a) = \infty \forall a \text{ small }\},
\]
where \( f \) is the same function used above. (The second equality is a consequence of regularity of 0, and is not as obvious as it might look.) Application of (4.2), (1.2), and Theorem 1 finishes the proof.

References

D. Williams, Diffusions, Markov Processes, and Martingales I (Chichester, 1979).

\[ \text{See previous footnote} \]