Continuous Extensions of Skew Product Diffusions

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§1. Introduction

Given a $[0, \infty)$-valued conservative nonsingular diffusion $R$, an additive function- al $K$ of $R$ which is finite and continuous at least on $[0, \sigma_0)$ where $\sigma_0$ is the first hitting time of $R$ at 0, and an independent conservative nonsingular diffusion $\Theta$ on $S^{d-1}$, the unit sphere in $R^d$, we define the skew product (based on $R$, $K$, and $\Theta$) to be the $R^d \setminus \{0\}$-valued stochastic process $Z^0(t) = R(t)\Theta(K)$ for $t < \sigma_0$, $= \infty$ for $t \geq \sigma_0$. As one can easily show, the skew product is a strong Markov process with lifetime $\sigma_0$ and continuous paths up to $\sigma_0$. (One should note that if $K(t) < \infty$ for all $t$, the above product is defined for all $t$ and gives rise to a simple Markov process. Unfortunately it is not a strong Markov process whenever $0$ is accessible and not a trap: the Markov property fails at $\sigma_0$. The simple 2-dimensional process $r \mapsto A(t)\exp(iB(t))$ where $A$ and $B$ are standard independent 1-dimensional Brownian motions convincingly illustrates this failure.)

Let us suppose $d \geq 2$. (The $d = 1$ case is discussed briefly in the last section.) By extension of $Z^0$ we shall mean any $R^d$-valued conservative right continuous strong Markov process which is continuous at $\sigma_0$ and, if killed at $\sigma_0$, agrees in law with $Z^0$. An extension may have jumps out from the origin, but away from the origin it must have continuous paths. Let us say an extension $Z$ is continuous if it also leaves the origin continuously (and hence is a diffusion) and has no sojourn at the origin:

$$E^0 \int_0^\infty 1_{(0)}(Z_t)dt = 0.$$  The problem we pose is to find all continuous extensions of a given skew product. As we will see the radial part $|Z|$ of an extension which leaves the origin continuously must be a diffusion extension of $R^0$, $K$ killed at 0. (See §4.) If 0 is both exit and entrance there are infinitely many such extensions but they differ only in their sojourn time rate at 0. Hence if we assume (as we do from now on) that $R$ has no sojourn at 0, then our problem may be stated thus: find all extensions whose radial part coincides in law with $R$. With or without the no sojourn requirement, this seems to be the most natural statement of the problem in more general situations. Our main results state that if the boundary at 0 is both exit and entrance, and the boundary at $\infty$ is non-exit for $R$, and if certain mild conditions on $\Theta$ are met, see §2, then there is a unique continuous extension if and only if $K(\sigma_0) = \infty$ a.s. on $[\sigma_0 < \infty]$; there must be rapid spinning as $t \uparrow \sigma_0$. If $K(\sigma_0) < \infty$ a.s. on $[\sigma_0 < \infty]$ then there is an infinite family of continuous extensions which is
in a simple one-to-one correspondence with the family of probability measures on $\mathcal{B}(S^{d-1})$. It must be emphasized that the extensions we get do not have the above simple product representation, hence the need for Theorem 3 in §4.

Of course $\theta$ may not be a regular (exit and entrance) boundary point. If, for example, $0$ is exit non-entrance, then $R^0$ cannot be extended beyond $\sigma_0$ continuously. Consequently, in view of our remarks above, there is no diffusion extension of $Z^0$ whatsoever. (The trivial extension, turning the origin into a trap, does not count). If $\theta$ is a non-exit point then $\sigma_0 = \infty$ a.s. on $[R(0) > 0]$ and the question of extension is moot.

The class of skew products for which $0$ is entrance non-exit (and not a trap) for $R$ (i.e. $\sigma_0 = \infty$ a.s. on $[R(0) > 0]$ but $\sigma_0 = 0$ on $[R(0) = 0]$) and the “angular” part $\Theta$ is the spherical Brownian motion (with generator $\frac{1}{2}$ the spherical Laplacian) is precisely the class studied by Ito and McKean [2], §7.16. (See also their §7.15.) This class includes the standard Brownian motion on $\mathbb{R}^d\setminus\{0\}$ ($d \geq 2$) as a member.

Let $K$ be the $BES(d)$ process and $K_t = \int R(s)^{-2}ds$. The problem which Ito and McKean pose and solve is this: when can the origin of $\mathbb{R}^d$ be adjoined, and the skew product completed so as to be a diffusion on all of $\mathbb{R}^d$? Their answer: this can be done if and only if $K(0+) = \infty$ a.s. on $[R(0) = 0]$. They also give an explicit construction of the paths which start from the origin of the completed process. Whether or not a similar explicit construction can be made in our case remains to be seen.

§2. Statement of Main Results

Throughout we assume that $R$ is a conservative nonsingular diffusion on $[0, \infty)$, that $P^0_{\theta}[\sigma_0 = 0] = 1$, that $P^0_{\theta}[\sigma_0 \leq t] = 1$ for all $t > 0$, and that there is no sojourn at $0$. Fixing a scale $s$ and a speed measure $m$ we get from Feller's analytic criteria [2], p. 130, and nonsingularity that $-\infty < s(r) < \infty$ and $m[0, r_s] < \infty$ for all $r \geq 0$. Adding a constant we may suppose $s(0) = 0$. (Recall that $s$ is a strictly increasing continuous function and $m$ charges every open interval.) The local generator is given by $u \mapsto du^+/dm$, $u^+$ being the right hand derivative of $u$ with respect to $s$, and functions in the domain are subject to the boundary condition $u^+(0) = 0$ (no sojourn means $m\{0\} = 0$.) Let us record here one consequence of our assumptions: $\infty$ is a non-exit boundary point, that is, for each $a, \sigma_r \rightarrow \infty$, as $r \rightarrow \infty$, $P^0$-a.s. We are not assuming that $\infty$ is non-entrance, so the resolvents of $R$, $R^0$ and of $Z^0$ need not send $C_0$ to $C_0$.

Let $L(t, r)$ be a local time functional normalized so that a.s. $\int_L(t, r)m\{dr\} = \int_0^1 1_{[a, b]}(R_s)ds$ for all $[a, b]$. Then there is a measure $k$, finite on compact subsets of $(0, \infty)$, such that a.s. $K_t = \int_0^\sigma L(t, r)k\{dr\}$ for $t < \sigma_0$. We will assume that $k$ has no atom at 0 and then define $K_t$ for all $t$ by the above local time integral. With $P^+$-probability 1 ($r > 0$) one of the following possibilities prevails: (a) $K_t$ is finite and
continuous for all \( t \), or else (b) on \( [\sigma _0 < \infty ] \) \( K(\sigma _0 -) = K(\sigma _0) < \infty \) but \( K(t) = \infty \) for \( t > \sigma _0 \), or else (c) on \( [\sigma _0 < \infty ] \) \( K(\sigma _n -) = K(l) = \infty \) for \( t \geq \sigma _n \). (Analytic criteria are developed later.)

The \( \mathbb{S}^{d - 1} \)-valued nonsingular conservative diffusion \( \Theta \) is defined on the same probability space as \( K \) and these processes are assumed to be independent: \( P[K(\cdot) \in A, \Theta(\cdot) \in B | \mathcal{R}_0 = r, \Theta_0 = \theta] = P_K [R(\cdot) \in A] P_{\Theta} [\Theta(\cdot) \in B] \) for all events \( A \) and \( B \) depending only on \( R \) and \( \Theta \) respectively. (The subscripts will be omitted of course.) In addition we assume that the transition semigroup \( (\Gamma _t) \) of \( \Theta \) is smooth enough to possess an invariant probability measure \( \lambda \) with respect to which the transition probability distribution \( P^t [\Theta_t \in \cdot | \Theta_0 = \theta] \) has a symmetric density \( \Gamma _t(\theta, \xi ) = \Gamma _t(\xi, \theta) \) which is continuous in \( (\theta, \xi ) \) for every \( t > 0 \). (Note that nonsingularity means \( P^t[I_{\theta}(\cdot) > 0] > 0 \) for every open set \( A \) of \( \mathbb{S}^{d - 1} \), for all \( \theta \) in \( \mathbb{S}^{d - 1} \), and all \( t > 0 \), so the measure \( \lambda \) must charge every open set and \( \Gamma _t(\cdot, \cdot) > 0 \) for all \( t > 0 \).) From these assumptions we derive that there exists numbers \( \gamma _n = \theta > - \gamma _1 \geq - \gamma _2 \geq \ldots \) the eigenvalues, and continuous eigenfunctions \( e_n = 1, e_1, e_2, \ldots \), such that

\[
\int_{\mathbb{S}^{d - 1}} \Gamma _t(\theta, \xi ) e_n(\xi ) \lambda (d\xi ) = e^{-\gamma _n t} e_n(\theta ),
\]

and

\[
\int_{\mathbb{S}^{d - 1}} e_n(\xi ) \lambda (\xi ) \lambda (d\xi ) = \delta _{nm}
\]

for \( \theta \in \mathbb{S}^{d - 1}, \ t > 0, \ n, m = 0, 1, \ldots \). It can be established that \( \Gamma _t(\theta, \xi ) = \sum _{n=0}^{\infty } e^{-\gamma _n t} c_n(\theta ) e_n(\xi ) \), the convergence of the series being uniform on \( \mathbb{S}^{d - 1} \times \mathbb{S}^{d - 1} \).

Using this and the strong continuity of the semigroup \( (\Gamma _t) \) we can then show that the collection of functions \( \{ e_n \} \) separates points of \( \mathbb{S}^{d - 1} \) and therefore the space of finite linear combinations of the \( e_n \) is dense in \( C(\mathbb{S}^{d - 1}) \) and every finite measure \( \mu \) on \( \mathbb{S}^{d - 1} \) is determined by the numbers \( \langle \mu , e_n \rangle = \int e_n(\theta ) \mu (d\theta ) \), \( n = 0, 1, \ldots \). There are many processes satisfying these assumptions, spherical Brownian motion, \( BM(\mathbb{S}^{d - 1}) \), being only the best known. See [2], p. 275.

Given an extension \( Z \) of \( Z^0 \) the resolvent operators (Green's functions) \( U^a, V^a \) of \( Z \) and \( Z^0 \), resp., defined, as usual, by \( U^a f(z) = \mathbb{E} \int _0^\infty f(Z_t) e^{-at} dt \), etc., are simply related via the strong Markov property by the formula

\[
U^a f(z) = V^a f(z) + E^z e^{-a\tau _G} U^a f(0),
\]

for \( r = |z| \) (Euclidean norm \( z \cdot z \) \( ^{1/2} \)), and \( f \) a bounded or positive Borel function. \( E^z \) is the radial expectation operator. It follows that an extension is essentially determined as soon as \( U^a f(0) \) is specified for sufficiently many functions \( f \).

**Theorem 1.** If \( P(K(\sigma _0) = \infty | \sigma _0 < \infty ] = 1, \ r > 0 \), then there is a unique continuous extension of \( Z^0 \) and its resolvent is given as at the origin by the formula

\[
U^a f(0) = G^a f_0(0), \text{ where } f_n(r) = \int_{\mathbb{S}^{d - 1}} f(r\xi ) \frac{\lambda (d\xi )}{r^{d+1}}, \ f \in bC(\mathbb{R}^d),
\]
and \((G^*)\) is the resolvent for \(R\). Furthermore \(P^*[K(\sigma_0) = \infty | \sigma_3 < \infty] = 0\) or 1 according as the integral \(\int_0^{\sigma_3} s(r)k(\{dr\})\) converges or diverges.

**Theorem 2.** If \(P^*[K(\sigma_0) = \infty | \sigma_3 < \infty] = 0\), then there is an infinite family of continuous extensions. For \(\varphi\) a bounded Borel function on \([0, \infty)\) and \(n = 1, 2, \ldots\), and \(\alpha > 0\), put

\[
E^\alpha \int_0^{\alpha} e^{-\alpha r} \psi(t) dt = \lim_{\alpha \to \infty} \frac{E^\alpha e^{-\alpha \sigma_0}}{1 - E^\alpha e^{-\alpha \sigma_0}}.
\]  

This limit exists and, if \((U^\alpha)\) is the resolvent of a continuous extension \(Z\), there is a uniquely determined probability measure \(\mu\) on \(\Omega(\mathbb{S}^{d-1})\) such that for any "polynomial" \(f = \sum f_n e_n\) (finite sum) with \(f_n \in C\{[0, \infty), f_0(0) = 0\) for \(n \neq 0\) we have

\[
U^\alpha f(0) = G^\alpha f_0(0) + \sum_{n \geq 1} D_n^\alpha (f_n) \langle \mu, e_n \rangle
\]

where \(\langle \mu, e_n \rangle = \int e_n(0) d\mu\). Conversely given a probability measure \(\mu\) on \(\Omega(\mathbb{S}^{d-1})\) there is a unique continuous extension \(Z\) such that (2.5) holds. The probability measure \(\mu\) in (2.5) is the (conditional on \(Z(0) = 0\)) limit distribution of \(Z(\sigma_0)/e\) as \(e \to 0\); where \(\sigma_0\) is the first exit time of \(Z\) from the ball of radius \(e\).

§3. Limit Computations and Criterion for \(K(\sigma_0) = \infty\)

In this section we review various results from the theory of one-dimensional diffusions. The reader may wish to consult Ito and McKean (1964) Chap. 4 in particular 4.6, and Chap. 5, 5.6.

Fix \(\alpha > 0\) and define functions \(g_1\) and \(g_2\) on \([0, \infty)\) by

\[
\begin{align*}
g_1(r) &= E^r e^{-\alpha r}, r \leq 1; \\
g_2(r) &= (E^r e^{-\alpha r})^{-1}, r \geq 1;
\end{align*}
\]

\[
\begin{align*}
g_2(r) &= (E^r e^{-\alpha r})^{-1}, r \leq 1; \\
g_2(r) &= E^r e^{-\alpha r}, r \geq 1
\end{align*}
\]

where \(E^r\) is expectation for the process \(R\). These functions are positive and continuous on \([0, \infty)\), \(g_1\) is increasing and \(g_2\) is decreasing and together they span the solutions to \(du^+/dm = \alpha u\) on \((0, \infty)\). Because of our assumptions concerning the barrier at \(0\), \(g_1\) must satisfy the reflecting boundary condition \(g_1^+(0) = 0\) while \(-\infty < g_2^+(0) < 0\). (Also \(g_1^+\) and \(g_2^+\) are right continuous at 0.) Furthermore, as \(e \to 0\),

\[
1 - E^\alpha e^{-\alpha \sigma_0} = 1 - g_2(e)/g_2(0) = (- g_2^+(0)/g_2(0))o(s(1) + o(1))
\]

\[
1 - E^\alpha e^{-\alpha \sigma_0} = 1 - g_2(0)/g_1(0) = o(s(1)).
\]

The resolvent operator \(G\) is given by a symmetric integral kernel with respect to \(m\) defined by \(g_1(r)g_2(x)/B_r\) for \(r < x\), where \(B = g_1^+ g_2 - g_2^+ g_1 = - g_2^+(0)g_1(0)\), a constant. Thus

\[
G^\alpha \varphi(0) = - g_2^+(0)^{-1} \int_0^\infty g_2(x) \varphi(x) m\{dx\}.
\]
Now define functions \( g_0, g_2 \) exactly as in (3.1) except that the expectation operators \( E^{*} \) are replaced by \( E^{0}_{\gamma} \). Expectation for the process \( K^{*} = K \) killed at the origin. Since we have continuous paths it is clear that \( g_0 g_2 \). Also the boundary condition for \( g_0 \) is \( g_0(0 +) = \lim_{\gamma \to 0} E^{0}_{\gamma} e^{-\sigma\gamma} = \lim_{\gamma \to 0} E^{*}(e^{-\sigma\gamma}; \sigma \leq \sigma_0) \) is exit for \( R \), but now \( g_0^+(0 +) > 0 \). The symmetric resolvent kernel for \( G^{2}_{\gamma} \) is \( g_0^+(0) g_2(x) / B_0, \ r < x \), where \( B_0 = g_0^+(0) - g_2^+ g_0 = g_0^+(0 +) g_2(0) \). Noting that, as \( \varepsilon \to 0^+ , g_1^2(\varepsilon) = g_0^+(0 +) + e(x) + o(e(s)) \) (by a mean value argument), we see that

\[
G^{2}_{\gamma} \phi(x) = B^{-1}_{\gamma} g_0^+(0) - B^{-1}_{\gamma} g_2(x) \int_{0}^{x} g_1^2(\varepsilon) / x \ d\varepsilon \exists \phi \ dm
\]

\[
= g_2^+ \phi(x) - \int_{0}^{x} g_2^+ \phi(x) \{ \phi(x) = 0 \} d\varepsilon(1 + o(1)) + O(s(x) m(0, x))
\]

But \( m(0, x) = o(1) \), and noting (3.2), (3.4), we arrive at the proof of

**Proposition 1.** If \( g_{2} \phi \) is m-integrable then

\[
G^{*} \phi(0) = \lim_{\gamma \to 0} \frac{G^{2}_{\gamma} \phi(x)}{1 - E^{*} e^{-\sigma\gamma}}.
\]  

(3.5)

Note that \( \int_{0}^{x} g_2 dm = - g_2^+(0) / x \) so (3.5) holds for all bounded Borel functions \( \phi \) at least.

We are now going to prove by a similar but somewhat more complicated analysis:

**Proposition 2.** Put \( \gamma = \gamma_{x} \) in (2.4) and assume \( \gamma > 0 \) (in \( \geq 1 \)). For every bounded Borel \( \phi \) on \( [0, \infty) \) the limit in (2.4) exists. Furthermore \( D^{*}_{\gamma}(s) = 0 \) for every bounded \( \phi \) if and only if \( P^{*} K(s_{0}) = \infty \) \( \sigma_{0} < \infty \) = 1, \( r > 0 \). Finally \( P^{*} K(s_{0}) = \infty \) \( \sigma_{0} < \infty \) = 0 or 1, \( r > 0 \), according as \( \int_{0}^{x} s(x) k_{1}(dx) \) converges or diverges. (Note that \( P^{*} (\sigma_{0} < \infty) > 0 \) as the conditional probability makes sense.)

**Proof.** Define a process \( X \) by \( X(t) = X(t) \) for \( t < s_{0} = \lim_{t \to s_{0}} \gamma X(t) \geq e \); \( X(t) = \infty \), for \( t \geq s_{0} \), where \( e \) is an independent exponentially distributed random variable with mean 1. This is a one dimensional diffusion on \( [0, \infty) \) with the same scale \( s \) and speed \( m \) as \( K \) (see [2], 5.6) but with a killing measure \( \gamma k_{1}(dx) \). Denote by \( E^{*}_{x} \) expectations for the process \( X \) and note that for all \( a \) and \( b \geq 0 \) (not both 0)

\[
E^{*}_{x} e^{-\sigma x} = E^{a} e^{-\sigma x} - \gamma x \[ dx].
\]

On considering separately the three possibilities mentioned in §2 for \( K \) at \( s = s_{0} \) and noting \( K(s_{0}) = K(s_{0} -) = K(s_{0}) \) on \( [X(0) > 0] \) one easily checks that the matching conditions,

\[
E^{0}_{x} e^{-\sigma x} = \lim_{b \to b_{0}} E^{b}_{x} e^{-\sigma x} \quad \text{and} \quad E^{0}_{x} e^{-\sigma x} = \lim_{b \to b_{0}} E^{b}_{x} e^{-\sigma x}
\]

are satisfied, and thus the Feller boundary classification in [2] 4.6 applies without modification.
Case I. \( \int_{0^+} s(x)k\{dx\} = \infty \). Since \( s \) is continuous and vanishes at \( 0 \) this forces \( k(0, x) = \infty \) for \( x > 0 \) and consequently \( 0 \) is neither an exit nor an entrance point for \( X \). Among other things this immediately implies \( E_x e^{-\alpha t} = 0 \) for \( x \geq 0 \), and hence \( K(\sigma_c) = \infty \) P*-a.s. on \( [\sigma_0 < \infty] \). Coming to the resolvent \( H^* \) for \( X \) one has for bounded \( \varphi \)

\[
H^* \varphi(x) = E_x^* \int_0^\infty e^{-\alpha t} \varphi(X_t)dt = E_x^* \int_0^\infty e^{-\alpha t - \gamma \psi(t)} \varphi(R_t)dt \\
= E_x^* \int_0^\infty e^{-\alpha t - \gamma \psi(t)} \varphi(R_t)dt \\
= B^{-1}_x h_2(x) \int_0^x h_1 \varphi dm + B^{-1}_x h_1(x) \int_x^\infty h_2 \varphi dm
\]

where \( h_1, h_2 \) are the increasing/decreasing functions defined as \( g_1, g_2 \) in (3.1) except that \( E_x^* \) and \( X \) replace \( E \) and \( R \), respectively. These functions span the solutions to

\[
\frac{u^+(dx)}{m\{dx\}} - \gamma k\{dx\}u(x) = au(x), \quad x > 0
\]

(more precisely \( \int (a_b) u(y)m\{dy\} = u^+(b) - u^+(a) - \gamma \int (a_b) k\{dy\} \) for intervals \( (a,b) \) and satisfy the non-exit, non-entrance boundary conditions

\[
h_4(0) = 0 = h_1(0), \quad h_3(0) = \infty = - h_2^+(0)
\]

The constant \( B_x \) in (3.6) is the Wronskian \( h_1^+ h_2 - h_1^+ h_2 \), and, because it is constant and \( h_1^+ < 0 \), one has \( h_2(s) = O(h_1^+(s)^{-1}) \) as \( e \to 0 \). Furthermore \( h_1^+ \) is an increasing function as one may check from (3.7). Therefore, \( h_1 \leq h_2^+ \)'s and \( h_2(s) = O(s(s)/h_1(s)) \) as \( e \to 0^+ \). In addition \( h_2^+(s) = o(s(s)) \). Now apply these estimates in (3.6). First \( h_2(s) \int_0^x h_1 \varphi dm = O(h_1^+(s)h_2(s)\{m(s)\}) = O(s(s)m(s)) = o(s(s)) \), since \( m(s) = m[0, s] \to 0 \) as \( e \to 0^+ \). Next, for any fixed \( \delta > \epsilon \),

\[
h_1(\delta) \int_{\epsilon^+}^\infty h_2 \varphi dm = O(h_1(\delta)h_2(\delta)\{m(\delta)\}) + O(h_1(\delta) \int_{\delta^+}^\infty h_2 dm)
\]

(The function \( h_2 \) is \( m \)-integrable on \( [\delta, \infty) \), \( \delta > 0 \), as is easily checked from (3.7), and \( h_1^+ < 0 \).) It follows from this that \( \limsup_{e \to 0^+} |H^* \varphi(s)/s(s) = O(m(\delta)) \) and, letting \( \delta \to 0 \) and noting (3.2), one finally sees that the limit in (2.4) must be 0 in this case.

Case II. \( \int_{0^+} s(x)k\{dx\} < \infty \). This is equivalent to \( \int_{0^+} k(x, 1)s\{dx\} < \infty \) so the origin is an exit point for \( X \). (It will be also an entrance point if and only if \( k(0, x) < \infty \) for \( x > 0 \) but this need not be the case.) It follows that \( K(\sigma_0) < \infty \) a.s. on \( [\sigma_0 < \infty] \), P*-a.s. Let \( h_{o1}, h_{o2} \) be as in (3.1) but now use expectations \( E_{s0}^* \) for the process \( X^0 = X \) killed at 0. Then \( h_{o1}(0) = \infty \) but \( h_{o1}^+(0) > 0 \) and \( h_{o2}(0) < \infty \). It follows
that $h_{++}$, being bounded on $[0, 1]$ and $m$-integrable on $[1, \infty]$, is $m$-integrable on $[0, \infty)$. We may now proceed as in the proof of Proposition 1 to conclude that \(\lim_{e \to 0} H_e \psi(s) = \psi(s)\) exists and is a finite positive multiple of \(\int_0^x h_{02} \psi dm\) where \(H_e \psi(s) = E_s^e \int_0^s e^{-s} \psi(X_s) ds = E_s^e \int_0^s e^{-s} \psi(R_s) ds\). This proves (2.4) for this case. To see that \(D^2_0 \psi = 0\) for some \(\psi\) takes \(\psi = 1\). \(\square\)

**Notes.**

1. The limit (3.5) is also valid if \(G'_{--}\), \(G'_{--}\) are replaced by \(H_{--}\), \(H_{--}\) provided \(\theta\) is both exit and entrance for \(X\) and \(E^\theta e^{-\theta s}\) is replaced by \(E^\theta e^{-\theta s} = E^\theta e^{-\theta s} e^{-\theta s}\).

2. From the calculations in this section, one easily gets the following analytic criteria for the three possibilities for \(X\) at \(t = \sigma_2\), mentioned in \(\S\) 2: (a) holds if and only if \(\int_{\theta^+} s(x) h_{12} dx < \infty\) and \(k(0, 1) < \infty\), (b) holds if and only if \(\int_{\theta^+} s(x) h_{12} dx < \infty\) but \(k(0, 1) = \infty\), and (c) holds if and only if

\[\int_{\theta^+} s(x) h_{12} dx = \infty.\]

The following lemma is needed in the proof of Theorem 3 in the next section, but it seems appropriate here.

**Lemma 1.** Fix \(a > 0\). For any \(\epsilon > 0\) there is a \(c > 0\) such that for any \(a > c, b > a\), there is a continuous function \(\psi \geq 0\) with compact support in \([0, \infty)\) which satisfies \(G^\epsilon \psi(0) < \epsilon\) and \(G^\epsilon \psi(x) \geq 1\) for \(x\) in \([a, b]\).

**Proof.** There is no loss in generality if we set \(a = 1\) and then write \(G^\epsilon\) for \(G'\).

As a preliminary choice for the desired function set \(f(x) = 0\) for \(0 < x < c\) and \(f(x) = p\) for \(x \geq c\), where the numbers \(p\) and \(c\) will be fixed in a moment. Then \(Gf(0) = E^0 \int_0^\infty e^{-x} f(R_s) ds = E^0 e^{-\epsilon x} Gf(c)\), and, in the notation of Proposition 1, for \(x \geq c\),

\[BGf(x) = pg_1(x) \int_0^x g_1 dm + pg_1(x) \int_{\theta^+} g_2 dm = pg_1(x) [g_1^+(x) - g_1^+(c)] + pg_1(x) (-g_2^+(x)) = p[B - g_2(x) g_1^+(c)]\]

where \(B\) is the constant \(g_1^+ g_2 - g_1 g_2^+\). Since \(g_2^+\) decreases and \(g_1^+(c) > 0\) it follows that \(Gf(x) > Gf(c)\) for \(x > c\) and that \(BGf(c) = p(B - g_2(x) g_1^+(c)) = -pg_1(x) g_2^+(c)\). Hence if we set \(p = 2B/g_1(c) g_2^+(c)\), then \(Gf(x) \geq 2\) for \(x \geq c\). On the other hand, \(Gf(0) = 2E^0 e^{-\epsilon x} \) and, because this decreases continuously to 0 as \(c \to \infty\) (\(\infty\) is non-exit), we can make it equal \(\epsilon\) for some \(c\). Note also that \(Gf\) is bounded and continuous. Let \(\varphi\) be positive continuous functions with compact support in \([0, \infty)\) such that \(\varphi_n\) increases to \(f\) pointwise as \(n \to \infty\). Then \(G\varphi_n\) increases to \(Gf\) which, since all these potentials are continuous, implies \(G\varphi_n\) converges to \(Gf\) uniformly on bounded intervals. For any \([a, b]\) with \(b > a > c\), \(G\varphi_n \geq Gf - 1 \equiv 1\) on \([a, b]\), eventually, but \(G\varphi_n(0) \leq Gf(0) = \epsilon\). Choose \(\varphi\) to be one of these \(\varphi_n\) for \(n\) sufficiently large. \(\square\)
§4. Extensions and Their Resolvents

For $f_\gamma = f_\gamma(r)$ a radial function and $f = f_\gamma(\theta)$ a spherical function one easily verifies from the independence of the processes $R$ and $\Theta$

$$V^* f_\gamma f_\gamma(z) = E^* \int_0^\infty d\tau e^{-\tau f_\gamma(R_\tau)} \int f(V_\tau, \theta, \xi)f_\gamma(\xi)\lambda d\sigma, r = |z|, \theta = z/r,$$

where $(V^*)$ is the resolvent for $Z^0$. In particular, if $f_\gamma$ is any one of the eigenfunctions in (2.1), say $f_\gamma = e_\nu$, then

$$V^* f_\gamma f_\gamma(z) = (G^\gamma_\nu f_\gamma(r)) f_\gamma(\theta),$$

(4.1)

where $\gamma = \gamma_\nu$ is the corresponding eigenvalue and $G^\gamma_\nu f_\gamma(r) = E^* \int_0^\infty e^{-\tau f_\gamma(R_\tau)}dt$ as in §3. We set $V^* f_\gamma(0) = 0$ for all $f_\gamma$ (Note: There is no need to distinguish between a radial function and its restriction to $[0, \infty)$, nor between a spherical function and its restriction to the sphere; the context makes it clear as in (4.1)).

**Lemma 2.** If $f \in bC(R^d)$, then $V^* f \in bC(R^d)$ for each $\alpha > 0$.

**Proof.** As noted in §2 the collection of eigenfunctions $\{e_\nu\}$ is linearly dense in $C(S^{d-1})$ so it is straightforward to show that the collection of functions $f_\gamma f_\gamma$ where $f_\gamma$ ranges over the eigenfunctions and $f_\gamma$ ranges over $C_0([0, \infty)$ (but $f_\gamma(0) = 0$ if $f_\gamma$ is non-constant) is linearly dense in $C_0(R^d)$ (and in $C_0(R^d)$) and constitutes a determining class for regular Borel measures. For each $z$ $V^* f(z) = \langle v_\gamma, f \rangle$ for a unique finite regular Borel measure $v_\gamma$ and, since the right hand side of (4.1) is continuous in $z = r\theta, r > 0$, it follows that as $z \to z_\theta, (\ast) \langle v_\gamma, f \rangle \to \langle v_\gamma, f \rangle$ for all $f \in C_0, z_\theta \neq 0$. (Recall that $(G^\gamma)$ is the resolvent of a non-singular one dimensional diffusion, so $G^\gamma f_{\gamma_\nu}$ is continuous on $[0, \infty)$. See [2], §3.6.) But $\|v_\gamma\| = V^* f(z) = (1 - E^{1/2}e^{-\alpha t})/\alpha$ (radial expectation) is continuous in $z$ and this implies that the p.m.'s $v_{\gamma,0}/\|v_{\gamma,0}\|$ converge narrowly as $z \to z_\theta, \theta \neq 0$ to the p.m. $v_{\gamma,0}/\|v_{\gamma,0}\|$, i.e., (\ast) holds for $f \in bC(R^d)$, $z_\theta \neq 0$. Since $\|v_{\gamma,0}\| \to 0$ as $z \to 0$ continuity at 0 is clear. \[\square\]

**Theorem 3.** (i) The radial part $|Z|$ of every extension $Z$ of $Z^0$ is an extension of $R^d$. If $Z$ is continuous then $|Z|$ and $R$ are identical in law; hence

$$U^\alpha \varphi = G^\alpha \varphi \text{ for every bounded radial function } \varphi, \quad (4.2)$$

where $(U^\alpha)$ is the resolvent for $Z$ and $(G^\alpha)$ for $R$.

(ii) Conversely, suppose $(U^\alpha), \alpha > 0$, is a family of positive linear operators on $bC(R^d)$ which satisfies (2.2), (4.2) and the resolvent equation $U^\alpha f = U f = (\beta - \alpha)U^\alpha U^\beta f = bC(R^d)$. (This equation always make sense in view of Lemma 2.) Then there exists a continuous extension of $Z^0$ whose resolvent is $(U^\alpha)$ and it is unique in the sense of law equivalence.

(The reader may want to review our definition of extension.)

**Proof of (i).** Let $\bar{E}$ denote expectations for the given extension $Z$. One sees from (2.2) and (4.1) ($\gamma = 0$) and Lemma 2 that the resolvent for $Z$ preserves the class of bounded continuous radial functions. The right continuity of $\bar{E} \to Z$, and a Laplace
transform argument then shows that \( z + E^z \phi (|Z|) \) is, for every \( t \) and every \( \omega \in bC[0, \infty) \), a bounded function of \( r = |z| \) (it may also be continuous in \( r \) but we do not care). It is now a simple matter to check the strong Markov property of \( |Z| \) from that of \( Z \). That \( |Z|^0 \) agrees in law with \( R^0 \) is clear; one needs only check that they have the same resolvent and this follows because \( Z \) killed at \( \sigma_\alpha \) has resolvent \((V^2)\). If \( Z \) is a continuous extension, then \( |Z| \) is a diffusion on \([0, \infty)\) without sojourn extending \( R^0 \). The theory of 1-dimensional diffusions shows there is only one such extension: \( R \).

Proof of (ii). This is a little tricky because the given family \((U^a)\) need not be \( C_0\)-preserving.

**Step 1.** \((U^a)\) is an honest Feller resolvent on \( bC(R^d) \), which satisfies

\[
\lim_{a \to 0} a U^a f (z) = f (z), \quad \forall f \in bC(R^d), \quad \forall z \in R^d. \tag{4.3}
\]

Clearly, any finite operator satisfying (2.2) and (4.2) maps \( bC(R^d) \) to itself by Lemma 2 and continuity of \( r \mapsto E^r e^{-\sigma_0} \). Next \( U^{+1} = G^{+1} = 1 = 1/x \) so \((U^x)\) is honest which fact, together with positivity, implies \( 0 \leq a \leq 1 \) whenever \( 0 \leq f \leq 1 \). For \( r = |z| > 0 \), \( x E^r e^{-\sigma_0} U^a f (0) \leq E^r e^{-\sigma_0} \| f \| \to 0 \) as \( a \to \infty \), \( \sigma \) holds away from the origin by (2.2) and the fact that (4.3) holds when \( V^1 \) replaces \( U^a \). At \( z = 0 \) we have \( x U^a f (0) \to \sigma \leq \sigma_0 \frac{\tilde{f} (0)}{\tilde{f} (0)} = 0 \), as \( a \to \infty \), where \( \tilde{f} (r) = \max \{ | f (z) - f (0) | : \frac{|z|}{r} \leq 1 \} \).

**Step 2.** Bring in the Ray-Knight compactification \( F \) of \( R^d \) based on \((U^a)\) and the attendant canonical \( F \)-valued Ray process \( Z = (\Omega, F, \mathcal{F}, \tilde{P}, \theta_F, \tilde{Z}) \) and resolvent \((U^a)\) (the old resolvent uniquely extended to \( C(F) \)). A description of this construction suitable for our purposes can be found in Williams [4], pp. 182-198. The space \( F \) is compact and metrizable and \( F^0 \) is embedded continuously in \( F \). (Open sets in \( F^0 \) may not be open in \( F \) but compact subsets of \( F^0 \) go over to compact subsets.) The process \( \tilde{Z} \) is strong Markov and \( \Omega \) is the space of Skorohod paths \( w: [0, \infty) \to F, \) the \( \sigma \)-fields \( \mathcal{F}, F \), are suitable completions of the canonical \( \sigma \)-fields, so in particular hitting times of compact sets are \((F)\) stopping times. Put \( \Delta = F \setminus F^0 \).

**Step 3.** The radial sets \( B(a) = \{ z : |z| < a \} \) are open in \( F \).

Let \( \phi_e (r) = 1 \) for \( 0 \leq r \leq \alpha - e, \phi_e (r) = (\alpha - r)/e \) for \( \alpha - e \leq r \leq \alpha \), and \( \phi_e (r) = 0 \) for \( r \geq \alpha \). Regard \( \phi_e \) as a radial function and put \( u_e = U^1 \phi_e \). Then \( u_e \) being the potential of a continuous function with compact support in \( F^0 \), has a unique extension to a function \( \tilde{u}_e \in C(F) \) (but \( \tilde{u}_e \) may no longer be a potential). Let \( \phi_{\alpha} = 1_{[\alpha, \infty)} \) and \( u_\alpha = U^1 \phi_{\alpha} \). \( u_\alpha \) may not be extendable. By (4.2) and an elementary calculation:

\[
u_e(r) = G^1 \phi_e(r) - 1 - (1 - u_e(a - e))E^r e^{-\sigma_1 (r = |z| \leq a - e)} = E^r e^{-\sigma_\alpha} u_e (a), \quad r \geq a,
\]

and exactly the same formula is valid for \( u_\alpha \) if you put \( e = 0 \) throughout. Since \( r \mapsto E^r e^{-\sigma_\alpha} \) is continuous it follows that \( u_\alpha, u_e \) are continuous, and since \( \phi_e \uparrow \phi_{\alpha} \), as
\(e \downarrow 0, u_\epsilon \uparrow u_0\) and this is uniform on bounded intervals. It is also clear that since the diffusion \(R\) is non-singular, \(u_\epsilon(r) = \int_0^r e^{-t} \phi_t(R)dt < 1\) for all \(r \geq 0, \epsilon \geq 0, \epsilon < a\), so that \(u_\epsilon\) strictly decreases on \([0, a - \epsilon]\). Also \(u_\epsilon\) strictly decreases on \([a, \infty)\), so, as \(r \to \infty\), \(u_\epsilon(r)\) has a limit which, though possibly not 0, must, as is clear from the definition of the Ray topology, be the value of the extended function \(u_\epsilon(z)\) for \(z \in A\).

Let \(D_\epsilon\) be the open (in \(F\)) set \(\{z \in F : u_\epsilon(z) > u_\epsilon(\alpha)\}\). The preceding discussion makes it clear that \(D_\epsilon\) equals \(\{z \in \mathbb{R}^d : u_\epsilon(z) > u_\epsilon(\alpha)\}\), a subset of \(B(\alpha)\), and \(D_\epsilon\) increases to \(\{z : u_\epsilon(z) > u_\epsilon(\alpha)\} = B(\alpha)\), so \(B(\alpha)\) must be open in the Ray topology.

Step 3 has the following consequences: \(\mathbb{R}^d\) is open in \(F, A = F \setminus \mathbb{R}^d\) is compact and nonempty, and, last but not least, every \(f \in bC(\mathbb{R}^d)\) which has a limit as \(|z| \to \infty\), may be regarded as a continuous function on \(F\) simply by setting \(f(\delta) = \lim_{|z| \to \infty} f(z)\)

for every \(\delta \in A\).

**Step 4.** For each \(z\) in \(\mathbb{R}^d, \hat{E}^p [\sigma_A < \infty] = 0\) where \(\sigma_A = \inf \{r; Z_r \in A\}\), i.e., \(Z\) stays in \(\mathbb{R}^d\) if it starts there.

We may take \(z = 0\) since the proof is generic. We first note that \(\hat{P}^p [Z_t \in \mathbb{R}^d] = 1\) for (Lebesgue) almost all \(t\). For \(\{\sigma_n\}\) is a sequence of radial functions in \(C^+_1(\mathbb{R}^d)\), increasing everywhere to 1 and if \(\hat{\phi}_n\) denotes the continuous extension of \(\phi_n\) to \(F\) obtained by setting \(\hat{\phi} = 0\) on \(A\), then \(\lim_{n \to \infty} \int_0^t e^{-t} \hat{P}^p [Z_t \in \mathbb{R}^d]dt = \lim_{n \to \infty} \int_0^t e^{-t} \hat{\phi}_n(Z_t)dt = \lim_{n \to \infty} \int_0^t \hat{G}^p [\sigma_n < \infty] = 1\). The assertion follows since \(\hat{P}^p [Z_t \in \mathbb{R}^d] \leq 1\). Let \(S\) be a countable dense (nonrandom) subset of \([0, \infty)\) such that \(\hat{P}^p [Z_t \in \mathbb{R}^d] = 1\). If \(w\) is a path such that \(\sigma_w(t) < t\) for some \(t < \infty\), then there must be points \(s_n\) in \(S\) such that \(s_n < t\) but \(Z(s_n, w) \in B(c) \cap \mathbb{R}^d\) \((= \{z : n \leq |z| < \infty\})\) for all \(n\) by right continuity and the fact that \(B(c) = \{z : |z| > n\}\) is an open set containing the compact \(A\). Therefore to prove that \(\hat{P}^p [\sigma_A < \infty] = 0\) it suffices to show that: for every \(t\):

\[
\lim_{a \to \infty} \lim_{b \to a} \hat{P}^p [\sigma_{a,b} \leq t] = \lim_{a \to \infty} \hat{P}^p [\sigma_{a,b} \leq t] \text{ for some } b > a = 0
\]

\((*)\)

where \(\sigma_{a,b}\) is the first hitting time of the compact set \(a \leq |z| \leq b\). (By step 3 the interiors \(a < |z| < b\) are open in \(F\) and fill up \(B(a) \cap \mathbb{R}^d\) as \(b \to \infty\).) Suppose \((*)\) is false; then there is a \(b > a > 0\) and a \(t_0 > 0\) such that for any \(c\), however large, there is an \(a > c\) such that \((**)*\) \(\hat{P}^p [\sigma_{a,b} \leq t_0] \geq \delta\) for some \(b > a\) sufficiently large. Put \(\epsilon = \frac{\delta}{\delta + \epsilon}\) in Lemma 1 of §3 (\(\epsilon = 1\)), and let \(c\) be as in the conclusion of Lemma 1. Fix an “interval” so that \((**)\) holds and \(a > c\). Let \(\phi\) be the function in the lemma. Regarding \(\phi\) as a radial function on \(\mathbb{R}^d\) with compact support we may extend \(\phi\) to \(\hat{\phi}\) in \(C(F)\) with \(\hat{\phi} = 0\) on \(A\). Then, writing \(U\) for \(U^1, G\) for \(G^1\), we have

\[e \geq G\hat{\phi}(0) = U\hat{\phi}(0) \geq \hat{E}^p \int_{\sigma_{a,b}} e^{-t} \hat{\phi}(Z_t)dt\]

\[\geq \hat{E}^p [e^{-\sigma_{a,b}} U\hat{\phi}(\sigma_{a,b})|, \sigma_{a,b} \leq t_0] \geq e^{-t_0} \hat{P}^p [\sigma_{a,b} \leq t_0] \geq 2e\]


(The fourth inequality is correct because \( \dot{Z}(\sigma \wedge r) \) lies in \([a, b]\) a.s. on \( \sigma \leq b < \infty \) and on \([a, b]\) \( U \sigma = G \sigma \geq 1 \). We thus have a contradiction and we conclude that (*) is valid.

**Step 5.** The process \(|Z|\) is a \([0, \infty)\)-valued conservative diffusion.

This is proved as in part (i) earlier, but some details are different. First note that for \( \varphi \in C_c([0, \infty), \mathbb{R}^d, r = |z|, \alpha > 0, \)

\[
\left[ E^z \varphi(|Z|) - E^z \varphi(R) \right] e^{-\alpha t} dt = U^z \hat{\varphi}(z) - G^z \varphi(\varphi) = 0
\]

where \( \hat{\varphi} \) is the extension of \( \varphi \) to \( F \) in the usual manner as a radial function. But \( t \mapsto E^z \varphi(|Z|) = E^z \hat{\varphi}(Z) \) is right continuous and \( t \mapsto E^z \varphi(R) \) is continuous, consequently \( t \mapsto E^z \varphi(|Z|) = E^z \hat{\varphi}(Z) \) and is therefore a continuous radial function on \( \mathbb{R}^d \). This fact and the simple Markov property of \( \dot{Z} \) make it simple to check the simple Markov property of \(|Z|\). The function \( \rho(z) = |z|/(1 + |z|) \) has a continuous extension \( \hat{\rho} \) to \( F \), so \( t \mapsto \hat{\rho}(Z) \) is right continuous which in conjunction with Step 4 implies that \( t \mapsto |Z| \) is \([0, \infty)\)-valued right continuous process. The resolvent for \(|Z|\) is \((G^z)\) and each \( G^z \) maps \( C_c(0, \infty) \) to \( bC(0, \infty) \) and it is standard that this implies the strong Markov property for \(|Z|\).

The two processes \( R \) and \(|Z|\) are equivalent and consequently \(|Z|\), which has right continuous paths already, has continuous paths after a trivial modification of the paths of \( \dot{Z} \). It follows from this, or steps 3 and 4, that \(|Z(t - )| \leq \infty \) a.s. on \([\dot{Z}(0), \mathbb{R}^d]\). Clearly \( \dot{Z} \) is conservative.

**Step 6.** \( \dot{Z} \) is a continuous extension of \( Z^0 \). Since \( \sigma_a \) is a stopping time for \( \dot{Z} \) we have

\[
U^z f(z) = U^z \tilde{f}(z) = \tilde{E}^z \int e^{-\alpha t} \tilde{f}(\dot{Z}) dt
\]

for \( z \in \mathbb{R}^d, f \in C_c(\mathbb{R}^d) \) \((\tilde{E} \in C(F) \) is the extension of \( f \)) and \( \alpha > 0 \) where \( \tilde{E}^z \) and \( E^z \) are extension of \( f \) and \( f \). Consequently \( \dot{Z}^0 \) killed at \( \sigma_a \) and \( Z^0 \) are equivalent. But \( \dot{Z}^0 \) has right continuous paths (in topology of \( \mathbb{R}^d \)) up to \( \sigma_a \), and \( \dot{Z}^0 \) has continuous paths. So \( \dot{Z}^0 \) has continuous paths up to \( \sigma_a \). (Actually the right continuity (in \( \mathbb{R}^d \) topology) of \( \dot{Z} \) and \( \dot{Z}^0 \) requires step 5. Step 5 implies that \(|Z(t)| \leq T \) lies in a compact subset for any \( T \).

The fact that \( R \) and \( |Z| \) are equivalent implies that \( \dot{Z} \) has no sojourns at the origin since \( R \) does not. \((\sigma_1, \infty) \) is a radial function.)
For any extension \( Z \) with expectations \( \tilde{E}^* \) and resolvents \( (U^*) \) a strong Markov property calculation gives
\[
U^*f(0) = \tilde{E}^0 \int_0^\infty \exp(-z)f(Z_t)dt + \tilde{E}^0 \exp(-z_0)U^*f(Z(\sigma_e))
\] (4.8)
for bounded \( f \) where \( \sigma_e = \inf \{ t ; |Z_t| \geq e \} \). Combined with (2.2), this yields
\[
U^*f(0) = \frac{\tilde{E}^0 \int_0^\infty \exp(-z)f(Z_t)dt}{\Lambda(e)} + \frac{\tilde{E}^0 \exp(-z_0)U^*f(Z(\sigma_e))}{\Lambda(e)}
\] (4.9)
where \( \Lambda(e) = 1 - \tilde{E}^0 \exp(-z_0)E^{*}e^{-z_0} \).

Proposition 3. For each \( e > 0 \), let \( Z' \) be an extension with probabilities \( \tilde{P} \), expectations \( \tilde{E}^* \), and resolvents \( (U^*) \). (The possibility \( Z' = Z \) for all \( e \) is not excluded.) Suppose that for each \( e > 0 \)
\[
|Z'(\sigma_e)| = e, \quad \tilde{P}^0 \text{-a.s.,}
\] (4.6)
and that as \( e \to 0 \)
\[
1 - \tilde{E}^0 e^{-z_0} = o(1 - g(e)),
\] (4.7)
where \( g(e) = E^{*}e^{-z_0} = \gamma_2(e)/\gamma_0(0) \) in the notation of \( \S 2 \). Then for bounded \( f \),
\[
U^*f(0) = \frac{\tilde{E}^0 V^*f(Z'(\sigma_e))}{1 - g(e)} + o(1), \quad e \to 0 .
\] (4.8)

For fixed \( \alpha > 0 \) the little \( O \) term is uniform in \( f \) satisfying \( |f| \leq 1 \).

Proof. Let \( W_e = \tilde{E}^0 V^*f(Z'(\sigma_e))/1 - g(e) \). Given \( |f| \leq 1 \) it is clear from (4.6) that \( |V^*f(Z'(\sigma_e))| \leq (1 - g(e))/\alpha \), so \( W_e \) is bounded by \( 1/\alpha \). Since \( \Lambda(e) \geq 1 - g(e) \) it is clear that the first term in (4.5) is bounded by \( (1/\alpha)(1 - \tilde{E}^0 e^{-z_0})/(1 - g(e)) \) which goes to 0 as \( e \to 0 \) by (4.7). From (4.6) it is also easy to check that \( \Lambda(e) = (1 - g(e)(1 + o(1)) \), and thus
\[
|U^*f(0) - W_e| = o(1) + \left| \frac{\tilde{E}^0(e^{-z_0} - 1 + 1)V^*f(Z'(\sigma_e))}{(1 - g(e))(1 + o(1))} - W_e \right|
\]
\[
= o(1) + O\left( \frac{1 - \tilde{E}^0 e^{-z_0}}{1 - g(e)} \right) + W_e(1 + o(1))^{-1} - W_e
\]
\[
= o(1), \text{ as } e \to 0 . \quad \square
\]

Notice that (4.6) and (4.7) are automatic if each \( Z' \) is a continuous extension, for then (4.6) is obvious and (4.7) is clear from (3.2) and (3.3). And the observation that \( E^0e^{-z_0} \) coincides with the radial \( E^0e^{-z_0} = g_1(0)^2/g_1(e) \).
§5. \( \epsilon \)-Jump Extensions

These are extensions for which the origin is a holding point and whose jumping-in measure is concentrated on the sphere, \( \mathbb{S}^{d-1} \), of radius \( \epsilon \). Intuitively, the path of a particle governed by the law of such an extension upon reaching the origin waits an independent exponentially distributed time and then jumps out according to the jumping-in measure. At the jumping-in point the particle begins anew moving according to the law of \( Z^0 \) until, perhaps, once again it finds itself at the origin waiting to jump, etc. This whole thing is designed to be strong Markov.

Let \( \delta \) be the common mean of the holding times and let \( \mu \) be the projection of the jumping-in measure on the unit sphere. (Why we specify the jumping in measure in this manner will become clear in the next section.) The existence of an \( \epsilon \)-jump extension with the prescribed parameters \( (\epsilon, \delta, \mu) \) follows almost immediately from Theorem 1 in Meyer [3] as we will now demonstrate.

Define a Markov process \( \tilde{X} \) as follows: \( \tilde{X}(t) = Z^0(t) \) for \( 0 \leq t < \sigma_\delta, \tilde{X}(t) = 0 \) for \( \sigma_\delta \leq t < \sigma_\delta + \epsilon, \tilde{X}(t) = \infty \) (isolated point) for \( t \geq \sigma_\delta + \epsilon \). The random variable \( \epsilon \) is independent of \( Z^0 \) and exponentially distributed: \( P(\epsilon > t) = e^{-t/\delta} \). Naturally the original probability space of \( R \) and \( \xi \) may need to be modified slightly but this is standard. This \( \tilde{X} \) induces a canonical function space version of itself: \( X = (\Omega, \mathcal{F}, \mathbb{F}_t, P^x, \theta_u, X_t) (x \in \mathbb{S}^d \cup \infty) \). Define a rebirth kernel "noyau de renais- sance" by \( N(w, A) = \mu(A|\mathcal{E}w) \) \( w \in \Omega, A \in \mathcal{B}(\mathbb{S}^{d-1}) \), \( \xi(w) > 0, N(w,.) = \delta_\infty \) if \( \xi(w) = 0 \) where \( \xi \) is the lifetime of \( X \) (\( \xi = \sigma_\delta + \epsilon \) in distribution). This kernel satisfies the definition in [3] p. 467. Now we may apply Theorem 1, p. 469, in [3]. The process \( Y \) of that Theorem is the \( \epsilon \)-jump extension.

**Proposition 4.** Let \( Z \) be a given continuous extension with expectations \( \tilde{E}^* \) and probabilities \( \tilde{P}^* \) and resolvent \( (U^*) \). Then there is a family of \( \epsilon \)-jump extensions with \( \delta_\epsilon \), satisfying (5.1) and with resolvents \( (U^*\epsilon) \) such that as \( \epsilon \to 0, U^*\epsilon \to U^* \) strongly on \( bC(R^d) \).

**Proof.** Let \( \mu_\epsilon \) be the \( \tilde{P}^* \)-distribution of \( Z(\sigma_\epsilon) \epsilon \) and choose positive numbers \( \delta_\epsilon \) decreasing with decreasing \( \epsilon \) in such a way that

\[ \delta_\epsilon = o(1), \quad \text{as} \ \epsilon \to 0, \quad (5.1) \]

where \( o \) is the scale function of \( R \). Let \( Z^\epsilon \) be the \( \epsilon \)-jump extension with parameters \( (\epsilon, \delta_\epsilon, \mu_\epsilon) \) and denote by \( P^\epsilon, E^\epsilon \) the corresponding probabilities and expectations. Under \( P^\epsilon \) the first hitting time at \( \epsilon, \sigma_\epsilon \), coincides with the exponentially distributed holding time at 0, and we have

\[ 1 - E^\epsilon_0 e^{-\alpha_\epsilon} = 1 - \frac{1/\delta_\epsilon}{\alpha + 1/\delta_\epsilon} = \frac{\delta_\epsilon}{1 + \alpha \delta_\epsilon} = o(s(\epsilon)) = o(1 - g(\epsilon)) \, , \]

as \( \epsilon \to 0 \) by (5.1) and (3.2). Applying (4.8) of Proposition 3 we obtain for \( f \in bC(R^d) \)

\[ U^\epsilon_0 f(0) = \frac{E^\epsilon_0 V^\epsilon f(Z^\epsilon_0(\sigma_\epsilon))}{1 - g(\epsilon)} + o(1), \quad \text{as} \ \epsilon \to 0 \, . \]
As noted at the end of §4, (4.8) also applies to $Z$; so
\[ U_\varepsilon f(0) = \frac{\mathcal{E}_0 V_\varepsilon^* f(Z_{\sigma_\varepsilon})}{1 - g(\varepsilon)} + o(1). \]

But our choice of jumping-in measure guarantees that the numerators on the right in the above two equations are identical. Hence $|U_\varepsilon f(0) - U_\varepsilon^* f(0)|$ goes to 0 as $\varepsilon \to 0$. But, from (2.2), $|U_\varepsilon f(z) - U_\varepsilon^* f(z)|$ is dominated for all $z$ by $|U_\varepsilon f(0) - U_\varepsilon^* f(0)|$ so $U_\varepsilon^* f \to U_\varepsilon^* f$ uniformly on $\mathbb{R}^d$ as advertised. (Actually, as the reader may have noticed, we have more: namely, $U_\varepsilon^* \to U^*$ in the uniform topology of operators.)

\[ \square \]

§6. Proof of Theorem 1

Existence. For any $f \in bC(\mathbb{R}^d)$ we have the formula
\[ \left\{ \begin{array}{l}
\int_{S^{d-1}} V_\varepsilon f(r\theta) \lambda(d\theta) = G_\varepsilon f_0(r)
\end{array} \right. \tag{6.1} \]
for each $r > 0$ where $f_0$ is the radial function defined in (2.3). (Integrate both sides of the formula $V_\varepsilon f(r\theta) = E^* \left[ e^{-\varepsilon^2 \int 
\right.$ $\mathcal{P}(K, \theta, \xi)f(R, \xi)\lambda(\xi) d\xi \right]$ with respect to the invariant measure $\lambda$.) Let $Z^\varepsilon$ be the $\varepsilon$-jump extension with parameters $(\varepsilon, \delta_\varepsilon, \lambda)$ where $\delta_\varepsilon$ satisfies (5.1). As noted before this implies (4.7) and consequently
\[ \lim_{\varepsilon \to 0} U_\varepsilon^* f(0) = G_\varepsilon f_0(0), \quad \text{for every } f \in bC(\mathbb{R}^d) \tag{6.2} \]
by (4.8) and Proposition 1. Define positive linear operators $U^\varepsilon$ on $bC(\mathbb{R}^d)$ by means of the formula (2.2) where $U_\varepsilon f(0)$ is defined to be $G_\varepsilon f_0(0)$ as at (2.3). Then $|U_\varepsilon f(z) - U_\varepsilon^* f(z)| \leq |G_\varepsilon f_0(0) - U_\varepsilon^* f(0)|$ so $U_\varepsilon \to U^\varepsilon$ strongly as $\varepsilon \to 0$. This convergence is enough to show that $(U^\varepsilon)$ satisfies the resolvent equation since $(U^\varepsilon)$ does. Since $f_0 = f$ when $f$ is a radial function it follows that (4.2) also holds. Part (ii) of Theorem 3 now applies to give us the continuous extension.

Uniqueness. In view of Proposition 4 and the inequality $|U^\varepsilon f - U_\varepsilon^* f| \leq |U^\varepsilon f(0) - U_\varepsilon^* f(0)|$, it suffices to show that the resolvent operators $U_\varepsilon^*$ of any $\varepsilon$-jump extension with $\delta_\varepsilon$ as in (5.1) satisfy (6.2). Suppose first that $f$ is a radial function. Then $G_\varepsilon f_0 = G^* f_0$, so $U_\varepsilon^* f(0) = G_\varepsilon^* f_0(0)/(1 - g(\varepsilon)) + o(1) \to G^* f_0(0)$ by (4.1) and (4.3) and Proposition 1. Next, suppose $f = f_+ f_-$ where $f_-$ is an eigenfunction with eigenvalue $\gamma > 0$, and $f_+$ is a radial function. Then $\langle \mu_{\varepsilon}, f_+ \rangle = 0$ (if $f_+$ is orthogonal to $e_0 \equiv 1$) so $f_0 \to \varepsilon \to 0 \Rightarrow G^* f_0(0)$. Now $U_\varepsilon^* f(0) = G_\varepsilon^* f_+(0)\mu_{\varepsilon, f_+} \gamma/(1 - g(\varepsilon)) + o(1)$, where $\mu_\varepsilon$ is the (projected) jumping in measure. But whatever be the probability measure $\mu_\varepsilon$, the term $\langle \mu_{\varepsilon}, f_- \rangle$ is bounded, consequently $U_\varepsilon^* f(0) \to 0 = G^* f_0(0)$, $\varepsilon \to 0$, by Proposition 2 (the $K(\sigma_\varepsilon) = \infty$ case).

We have now demonstrated that (6.2) must hold for every $f$ in a linearly dense set in $bC(\mathbb{R}^d)$. Since the positive linear functionals occurring on both sides of (6.2) all have the same norm, the limit must hold for all $f$ in $bC(\mathbb{R}^d)$. \( \square \)
Remarks. To get the existence of a continuous extension one may define $U^\varepsilon$ by (2.2) and (2.3) and demonstrate "algebraically" the resolvent equation. Thus the $\varepsilon$-jump extension approximation is not needed. However, this algebraic approach does not work so well in the situation of Theorem 2. Even here it is a little cumbersome.

§7. Proof of Theorem 2

The calculations of the last two sections should make it clear that we will get all possible continuous extensions by finding all possible limiting resolvents of $\varepsilon$-jump extensions whose holding time parameter $\delta_\varepsilon = o(s(\varepsilon))$ as $\varepsilon \to 0$, for such limiting resolvents, and only these, satisfy the hypotheses of Theorem 3 (ii). Furthermore we need only determine the value of the limit by evaluating at the origin on functions of the form occurring in (4.1). The limit (6.2) always holds for radial functions so if it also holds for all the functions in (4.1), then it holds on $bC(R^d)$ since (6.2) for radial functions implies that the representing measures of the functionals $f \to U^\varepsilon f(0)$ are tight.

Let $\{Z^\varepsilon\}$ be any sequence of $\varepsilon$-jump extensions with parameters $(\varepsilon, \delta_\varepsilon, \mu_\varepsilon)$ where $\varepsilon = \varepsilon_k \to 0$ as $k \to \infty$ and $\delta_\varepsilon = o(s(\varepsilon))$. If $f \Rightarrow f_\varepsilon$, $\mu_\varepsilon \to \theta$, then by (4.8), (4.1),

$$
U^\varepsilon f(0) = \{G^\varepsilon f_\varepsilon(\varepsilon)/(1 - g(\varepsilon))\} \mu_\varepsilon + o(1), \quad \varepsilon \to 0.
$$

But by Proposition 2, the $K(\sigma) < \infty$ case, as $\varepsilon \to 0$, $G^\varepsilon f_\varepsilon(\varepsilon)/(1 - g(\varepsilon)) \to D^\varepsilon f_\varepsilon$ and $D^\varepsilon f_\varepsilon > 0$ wherever $f_\varepsilon \geq 0$ and $f_\varepsilon > 0$ on an interval. That $U^\varepsilon$ has a limit if and only if

$$
\lim_{\varepsilon \to 0} \langle \mu_\varepsilon, e_n \rangle = \langle \mu, e_n \rangle \quad \text{exists for every } n \geq 0. \tag{7.1}
$$

Since the $e_n$ are linearly dense in the space $C(S^{d-1})$ and since $S^{d-1}$ is compact, we see that (7.1) obtains if and only if $\mu_\varepsilon$ converges to a probability measure $\mu$ on $\mathcal{B}(S^{d-1})$. Clearly by taking all $\mu_\varepsilon = \mu$, a given probability measure on $\mathcal{B}(S^{d-1})$, we get a limit; thus every such $\mu$ gives rise to a unique continuous extension.

Let $Z$ be a given continuous extension. As in the proof of Proposition 4 let $Z^\varepsilon$ be the $\varepsilon$-jump extension with parameters $(\varepsilon, o(s(\varepsilon)), \mu_\varepsilon)$ where $\mu_\varepsilon$ is the distribution of $Z(\varepsilon)/\varepsilon$. Then $U^\varepsilon f(0)$ converges to $U f(0)$. Hence $\mu_\varepsilon$ converges, i.e., $Z(\varepsilon)/\varepsilon$ has a limiting distribution, as $\varepsilon \to 0^+$. Obviously, the limit $\mu$ is the same as in (2.5).

§8. Miscellaneous Comments

(a) Generators. Ito and McKean, [1], 7.16, state that the local infinitesimal generator of a skew product for which the spherical part is $BM(S^{d-1})$ is given in spherical coordinates by

$$
Au(r, \theta) = \frac{u^+(dr, \theta) + k \frac{1}{2} A_\theta u(r, \theta)}{m(dr)}, \quad r > 0, \quad \theta \in S^{d-1}, \tag{8.1}
$$

where $A$ is the spherical Laplacian. (Naturally the generator of $\Theta$ replaces $\frac{1}{2} A$ in our more general situation.) The operator (8.1) is supposed to be the value of the
\[ \lim_{\beta \to 0} \frac{E^z u(W(T_\beta)) - u(z)}{E^z T_\beta}, \text{ when it exists, where } W \text{ stands either for } Z \text{ or for any of its extensions and } T_\beta \text{ is the first exit time of } W \text{ from a small ball of radius } \delta \text{ centered at } z = r, \ r > 0. \] To single out a particular continuous extension of } Z', \text{ the behavior of functions in the domain of the infinitesimal generator must be specified at the origin. The value of } Au \text{ at the origin can then be determined from (8.1) by the general continuity requirement of } Au. \text{ Thus } Au(0) = \lim_{\beta \to 0} Au(r, \theta) \text{ and this limit must be independent of the direction. In the situation of Theorem 2 if } \mu \text{ is the probability measure in (2.5), then a function } \mu \text{ in the domain of the infinitesimal generator at the origin must satisfy}

\[ \int_{S^d - \{0\}} u(r\theta)\mu(d\theta) - u(0) = Au(0) \int_{S^d - \{0\}} \{s(r) - s(x)\}m(dx)(1 + o(1)), \]

as } \beta \to 0. \text{ This implies } \int_{S^d - \{0\}} u(r\theta)\mu(d\theta) - u(0) = o(s(r)) \text{ as } \beta \to 0 \text{ and this is the local condition at } 0 \text{ which must be satisfied. Since I have no particular use for (8.1) here I will not present a detailed derivation of (8.1) and the domains.}

The operational meaning of (8.1) is similar to that of (3.7). If } s, m, \text{ and } k \text{ are smooth enough then (8.1) simplifies to an ordinary second order differential operator. For an interesting example, take } d = 2, \ K = \mathcal{B}\mathcal{E}\mathcal{S}(\beta + 1), \ \Theta = BM(S^1), \ K_1 = \int \beta R(R)^{-2}dt. \text{ Then } s(r) = r^{1-\beta}/(1 - \beta), \ m(dr) = 2rdr, \text{ and } k(dr) = 2\beta r^{\beta - 2}dr, \text{ and}

\[ Au = \frac{1}{2} \frac{\partial^2 u}{\partial r^2} + \frac{\beta}{r} \frac{\partial u}{\partial r} + \frac{\beta}{2r^2} \frac{\partial^2 u}{\partial \theta^2}. \quad (8.2) \]

For } 0 < \beta < 1, \text{ the origin is an exit entrance barrier for } K \text{ and } \int_{S^d - \{0\}} s \mu k = \infty \text{ so there is a unique continuous extension (which is in fact recurrent since } P^*[\sigma_0 < \infty] = 1). \text{ An interesting feature of this example is that if one expresses } Au \text{ in rectangular coordinates one gets}

\[ Au = \frac{1}{2} (x^2 + y^2)^{-1} \left( (x^2 + \beta y^2) \frac{\partial^2 u}{\partial x^2} + 2(1 - \beta)xy \frac{\partial^2 u}{\partial x \partial y} + (y^2 + \beta x^2) \frac{\partial^2 u}{\partial y^2} \right) \]

for } (x, y) \neq (0, 0). \text{ This is an elliptic operator with bounded coefficients without drift terms, yet the associated process hits the origin with probability one. Notice that (8.2) does not aid the intuition either; the drift term points away from the origin. To see a genuine drift term it is best to write } A \text{ in an invariant form as a Laplace-Beltrami operator plus a vector field: } A = \Delta + B. \text{ In this case one must take, in polar coordinates, } (g_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 2r^2/\beta \end{pmatrix} \text{ (there is no choice in this) and then one sees}

\[ B = -\frac{1 - \beta}{2r} \frac{\partial}{\partial r}. \]

Here is a less smooth example. Let } d = 2 \text{ and let } K \text{ have scale } s(r) = r \text{ and speed measure } m \text{ concentrated on } Q \text{ the set of rational numbers in } (0, \infty). \text{ Suppose that } k \text{ is concentrated on a set } S \text{ which is countable, dense in } (0, \infty), \text{ and disjoint from } Q. \]
Again \( \Theta = BM(S^1) \). In this case \( \int_0^t 1_{\rho(R)}(\tau)\,d\tau = t \) for all \( t > 0 \), a.s., so the natural state space of \( R = |Z^0| \) is \( Q \). But \( t \to \Theta(K_t) \) moves only when \( |Z^0| = 1 \). Yet it does move (and very rapidly near \( \sigma_0 \)) if \( \int_{0^+} xK(\{dx\}) = \infty \). The reader can appreciate that the operator (8.1) is not particularly helpful here.

(b) The case \( d = 1 \). One possible generalization is to drop the requirement that \( \xi \) have continuous paths. This does no harm as long as the other assumptions of §2 are in effect. The concept of continuous extension is valid: An extension \( Z \) is continuous if \( P[Z(u) = 0 \text{ for every } t \text{ such that } Z(t-) = 0] = 1 \) and there is no sojourn at the origin. Again \( Z \) is a continuous extension if and only if \( R \) and \( |Z| \) are equivalent. Example \( d = 1 \). Then \( \Theta \) is a \( \{+1, -1\} \)-valued continuous time Markov chain. There are two eigenvalues. The processes \( Z^0 \) and its extensions do not have continuous paths (though \( |Z^0| \) does up to \( \sigma_0 \)) and the paths jump back and forth across the origin. There is a unique extension if and only if there are infinitely many jumps during \( [0, \sigma_0] \).

Acknowledgement I wish to thank Professor R. Blumenthal for his interest and useful suggestions during the preparation of this paper.

References


Received June 12, 1987; in revised form June 4, 1989